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Analysis of mathematical descriptions of gravitational microlensing

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ANALYSIS OF MATHEMATICAL DESCRIPTIONS OF GRAVITATIONAL MICROLENSING

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Table of Contents

I. Introduction

Gravitational lensing is a phenomenon that can be observed in the universe today. It occurs when a massive body at a certain distance from an observer is aligned with a luminous object that is further from the observer. The light from the distant source can be lensed by the intervening mass before it reaches the observer, and result in images that can be seen by the observer. These images differ from what would be observed if the intervening mass were not present. The extent of the lensing of light varies with the mass of the lensing body, and several types of lensing can be studied. Strong and weak lensing have galaxies or galaxy clusters acting as lenses, while gravitational microlensing is lensing by a stellar-mass object.

This thesis will provide an analysis of some aspects of gravitational microlensing and the quantitative descriptions used to model the astrophysical systems involved. One aspect of the mathematical description of microlensing events that is investigated in this thesis is the evolution of microlensing equations from the formalism of general relativity. The derivation of these equations from those governing general relativity and the bending of light rays in geometric optics is explicitly determined. The relationship between the equations used to model events and the light curves produced is investigated. The effect of degeneracies in the mathematical equations used to describe microlensing events is investigated, through the varying of parameters in the equations and observation of the changes induced in the theoretical light curves generated.

Accurate descriptions of microlensing events are necessary to understand correctly the arrangements of objects in the galaxy, which can be used to investigate the formation and structure of objects on astronomical scales. This information can aid in understanding of processes that occur in the galaxy and in the universe, and how those processes affect the Earth. An improved understanding of the distribution of matter in the universe, particularly dark matter, can aid us in formulating a complete theory to describe the ways in which the universe is changing over time, and why it is doing so.

II. Literature Review

The main focus of this project was to investigate gravitational lensing, a major consequence of the theory of general relativity. To this end a number of publications were consulted and read, to provide the information necessary to understand this phenomenon. This information is summarized below. A short summary of the principles of general relativity that apply to gravitational lensing is also included as additional background information. General relativity is important to the study of gravitational lensing because it determines how the light from a distant source is lensed by an intervening mass. The deflection of photons, or particles of light, is dependent upon the effects of the intervening mass, which affects the calculations of general relativity.

One of the first experimental tests of general relativity was carried out during a solar eclipse in 1919 by an expedition headed by Arthur Eddington. Observations of stars near the Sun were made and compared with those made a few months later, when the Sun was not located near the stars. The stars were found to appear farther from the Sun's location during the eclipse than they normally did. The mass of the Sun deflected the light from these stars and effectively lensed them. This confirmation of Einstein's theory of relativity secured the validity of the theory, and provided a precedent for subsequent observations of the lensing action of massive bodies on the light from stars and other light sources beyond them.

A. Introduction to General Relativity

Gravitational lensing is based on the principles of general relativity. One of the most radical departures of general relativity from Newtonian mechanics is in the treatment of time. In general relativity, time is regarded as a fourth dimension, and space and time are merged to give four-dimensional spacetime. Spacetime diagrams are used to show this relation, typically with the time axis vertical and one or more spatial axes perpendicular to it. These diagrams show the paths (or "worldlines") traveled by objects as they move through spacetime. Points on spacetime diagrams are referred to as events, because they represent a specific point in space and in time. An example of a spacetime diagram is shown below in figure 1, with the coordinates of point P.[1] The vertical time axis is labeled 'ct' because time is scaled by a factor of c, the speed of light, to give the axis length dimensions comparable to those of the distance axis. Figure 2 shows the worldlines of a stationary object (A) and a moving object (B).[1] The lines intersect at point (ct_1, x_0) when the objects are in the same place in spacetime.

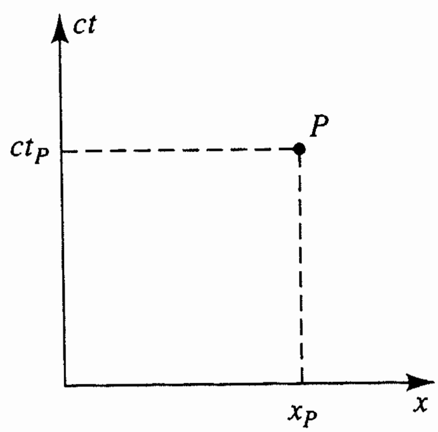

Figure 1. An example of a spacetime diagram with the time axis scaled by the speed of light (*c*) to give appropriate dimensions.

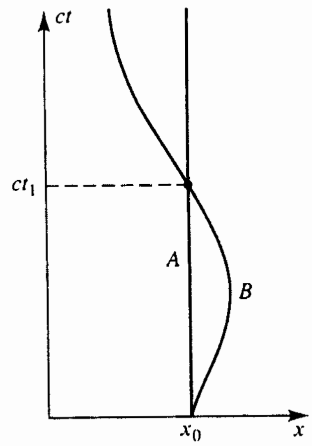

Figure 2. An example of a spacetime diagram with the worldline of a stationary object shown as a vertical line at x_0 and the worldline of a moving object as a curved line.

Time is no longer treated as a steadily changing characteristic of the system in general relativity. The progression of time is different for observers who are moving relative to one another, and the simultaneity of events will be broken for observers in different reference frames.

A reference frame is a set of coordinates chosen to enable easy measurement of a system. Usually a reference frame is chosen that is stationary with respect to an observer, or moving at a constant velocity, so that Newton's laws of motion are applicable and the frame is an inertial reference frame. The observers themselves do not influence the system. In the context of gravitational lensing, the reference frame used is that of an observer on or near the Earth. From this reference frame, the Earth is stationary and the light source and other massive astrophysical bodies move over time. Distances are measured from the Earth and events are considered to happen when they are observed from Earth.

The way in which distances are measured is changed from that of Newtonian physics, to reflect the fact that time is treated as a coordinate variable. A metric including time and the three spatial coordinates is used to measure the distances traveled by particles. The line element of the metric for flat spacetime, with Euclidean spatial coordinates, is given in (1) below.[1] In equation (1) ds^2 is a differential length-squared of a line element in four-dimensional spacetime, c (the speed of light) scales the time change, dt, to put it into distance units, and dx, dy, and dz are the three Cartesian spatial coordinate changes.

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

It is obvious from equation (1), a non-Euclidean Pythagorean equation, that spacetime distances are non-Cartesian, since the differential time factor of the spacetime distance is negative and the others are positive. The distance equation is not a simple four-dimensional Pythagorean equation. If spacetime distances were Cartesian, equation (1) would be identical to equation (2), below.

$$ds^2 = (cdt)^2 + dx^2 + dy^2 + dz^2 \quad (2)$$

Distances on spacetime diagrams can fall into three categories, depending on the values of the time and spatial intervals (dx or Δx, etc.). Figure 3 (below) shows the locations in spacetime of these different types of distances.[2] If the value of ds^2 for the separation between two points or events on a spacetime diagram is positive, the distance is said to be spacelike. If the value of ds^2 is zero, with the magnitude of time component of the distance equal to the magnitude of the spatial component, events on this line have lightlike or "null" separations. Light rays travel on these lines of null separation and form what are called light cones. Within a light cone two events are said to have a timelike separation, when the spacetime distance, ds^2, is negative. The photons of interest in gravitational lensing travel on light cones through spacetime from a source to an observer.

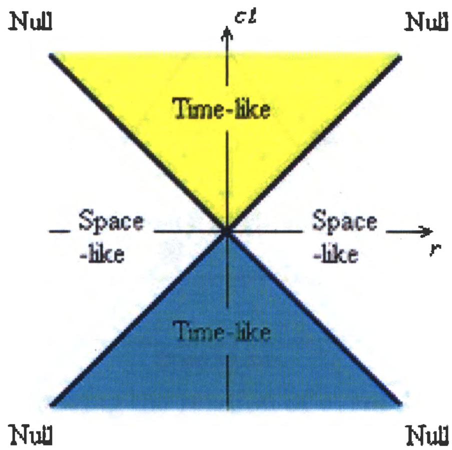

Figure 3. A spacetime diagram showing the different types of separations possible. The colored portions denote timelike separations and the angled lines follow the null separation of photons. Spacelike separation corresponds to the separation of the origin (the cone vertex) with points outside the light cone.

A particle beginning at the vertex in spacetime may travel to a point within the future light cone emerging from that starting point, but may not travel to locations outside of the light cone, since a particle (which has mass) cannot travel faster than the speed of light. In flat spacetime with orthogonal axes, the edges of light cones form lines at 45° angles. A particle cannot travel at an angle greater than 45° from the vertical time axis in flat spacetime, and two events that are spacelike separated cannot communicate with one another. Two events in spacetime that are within a light cone can communicate with each other, and the one farther in the past can influence an event that is in its future. Figure 4[3] (below) shows a light cone in spacetime with 2 spatial coordinates presented. The light cone centered on event A shows the locations of events in the past of event A, which can influence A. Events in the future of A can be influenced by A, but those outside the light cone cannot communicate with A. An event outside the light cone has spacelike separation with respect to A. The spatial distance between two spacelike separated events is greater than their timelike separation, and the two events cannot interact. In order for them to interact information would need to be transferred between the two events at a speed greater than that of light, to cover the necessary distance in the allowed time. This cannot happen in normal spacetime.

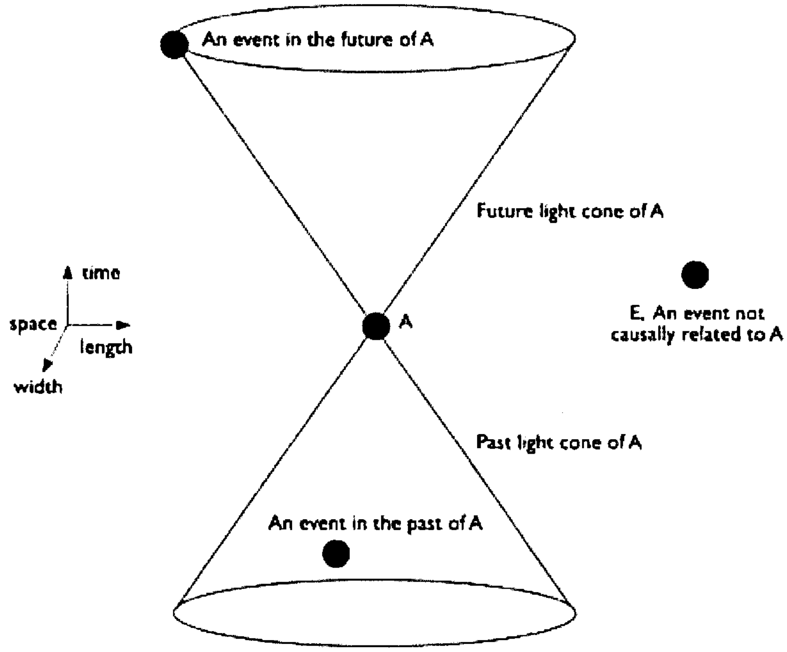

Figure 4. A spacetime diagram showing the relation of event A with events near it.

Flat spacetime is the simplest system to work with, but is only an accurate representation of a vacuum, space without any matter. When massive bodies are included spacetime becomes curved. The simplest form of curved spacetime is that resulting from a spherically symmetric mass. The curvature induced in the local spacetime is also spherically symmetric, which simplifies calculations near the massive body. The gravity of a star, planet, or galaxy warps the spacetime around it in proportion to its mass. The paths taken by particles and light rays follow the curved geometry of spacetime, as the space they travel through is itself curved. The equation for the line element in curved spacetime near a spherically symmetric mass is given in (3) below.[1]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (3)$$

In (3), spherical polar coordinates are used for the spatial components; dr, $d\theta$, and $d\varphi$ represent the standard spherical polar coordinates, with the origin conveniently taken to be the center of the massive body. This line element equation is the Schwarzschild metric for curved spacetime. This metric is spherically symmetric, and reduces to equation (1) as M approaches zero, for the case of flat spacetime.

The curved nature of spacetime near a massive body changes the path of a light ray from its straight line in flat spacetime. The worldline of a photon can be parameterized by λ, with a null vector tangent to the worldline. The parameter λ can be normalized so that the tangent vector is the photon's momentum. The photon's energy and angular momentum can be written in terms of its momentum, and substituted into equation (3). The equation depends on the ratio of the photon's angular momentum to its energy, not on their individual values, due to the ability to normalize the parameter λ in various ways.[1] In the $\theta=\pi/2$ plane, equation (3) can then be solved for $d\varphi/dr$, the change in angle of a photon as it moves closer to a massive body.[1] After integration the angle of deflection can be calculated.

The orbit of a photon approaching a massive body can be calculated and plotted from its deflection angle. If a photon approaches a mass (such as a star) at a small impact parameter, with a small ratio of angular momentum to energy, it may be deflected a great deal and plunge into the mass. This is what often happens near a black hole, which curves the local spacetime around it a great deal with its concentrated mass. If a photon approaches at a large transverse distance, the mass will not affect the photon's path much since the spacetime through which the photon travels is not curved very much. The photon will essentially continue in a straight line. If a photon approaches a star or other astrophysical body at a moderate impact parameter its path can be deflected by an angle, and the photon can continue on that new path. This is what occurs in gravitational lensing systems.

B. Gravitational Lensing of Point Masses

Gravitational lensing occurs when a massive object is located between a light-emitting source and an observer. The gravity of the central object acts as a lens and bends the path of the light rays coming from the source, by bending the spacetime through which the photon travels. This changes the distribution of light rays that reaches the observer, and creates the images of the source that the observer sees. The amount and direction of deflection that a light ray experiences depends on the geometry of the system and the mass of the lensing object. A more massive lens bends spacetime more, and deflects light rays to a greater extent. The lensing object may be a distant galaxy or galaxy cluster, a single star, or any other massive body. The image produced may be a single image of the source which is magnified to a higher luminosity or several images of one source, located around the lensing object. Figure 5 shows a basic representation of the situation necessary for gravitational lensing to occur.[4]

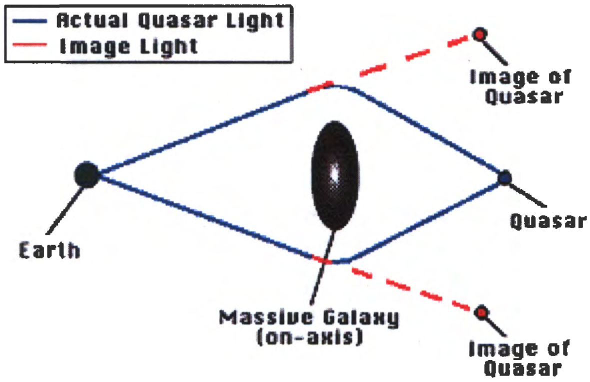

Figure 5. Alignment needed for gravitational lensing to occur in an astrophysical system.

Gravitational lensing can occur on a variety of scales. Strong gravitational lensing of a source by a cluster of galaxies can lead to separations on the order of a few arcseconds between images from a source (as measured by an observer on Earth).[5] (In astronomy, arcminutes (') and arcseconds (") are used to measure angles in the sky. There are 60 arcminutes in a degree and 60 arcseconds in an arcminute.) The mass of a lensing galaxy cluster is about 10^{15} $M_\odot$ (solar masses), and it is hundreds of megaparsecs away (one parsec (pc) is about 3.26 light years).[6] Strong gravitational lensing leads to multiple images of a source. Weak gravitational lensing is done by small galaxy clusters and single galaxies. A galaxy has mass of about 10^{12} $M_\odot$.[5] The separation between images is smaller for weak lensing. A typical galaxy is at least 100 kiloparsecs (kpc) away from Earth.[6] A single star can act as a lens to deflect the path of light rays in microlensing. A star may be 1-100 kpc away, depending on its location in a galaxy. Since the mass of a star is much less than that of a galaxy, the images created are very close together. The separations on the order of 10^{-6} arcseconds[5] are not resolvable with current telescopes, which can only resolve about 0.1".[7] Microlensing can still be observed, by observing the brightness variation of a source when a lensing mass passes in front of it. A planet passing in front of a distant light source may cause nanolensing of light, but the small mass of a planet (10^{-6} $M_\odot$) makes this difficult to observe.[6] Nanolensing events may be an efficient way to search for Earth-like exoplanets when more advanced lensing searches become possible.

Gravitational lensing can occur at any wavelength of light. Light is deflected over the entire range of wavelengths that the source emits, making gravitational lensing an achromatic phenomenon. The source object in gravitational lensing may be any luminous object that is visible in some wavelength range. For strong and weak lensing this is often a galaxy or quasar. In microlensing the source may be a point-like quasar or another star.

There are several ways to model the mass distribution of the lensing mass; the simplest is to consider a point mass. The luminous source may also be modeled as a point for simplicity.

The geometry of the system is diagramed in figure 6 below.[1] Only one of the images produced by gravitational lensing is shown in figure 6; the other image would appear on the left side of the observer-lens line optical axis.

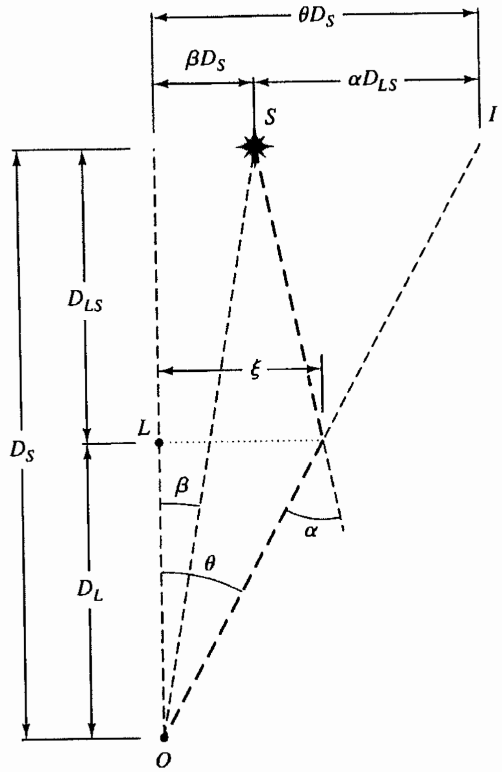

Figure 6. Geometry of a gravitational lensing system, with angular and distance measurements shown.

All the measurements in gravitational lensing are taken in the observer's reference frame. Because the distances to the lens and source are very large in relation to the angles, the distances of the source and image from the optical axis in the source plane may be approximated by the small angle approximation so that[1]

$$\theta D_S = \beta D_S + \alpha D_{LS}. \tag{4}$$

Equation (4) is known as the lens equation. The deflection angle of light due to the effects of gravity is[1]

$$\alpha = \frac{4GM}{c^2 b}, \tag{5}$$

where b is the impact parameter, or the closest the light ray comes to the lens in the lens-Earth-source plane. G is the gravitational constant (6.67×10^{-11} $m^3kg^{-1}s^{-2}$) and M is the mass of the lens. The value of b is approximately that of ξ in the small angle approximation,[1] so equation (4) can be written as

$$\theta D_S = \beta D_S + \frac{4GM}{c^2 \xi} D_{LS}. \tag{6}$$

The value of ξ can be approximated as θ_{DL},[1] so that the lens equation can be written as

$$\theta = \beta + \frac{4GM D_{LS}}{c^2 \theta D_L D_S}. \tag{7}$$

The Einstein angle can be defined as

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}} \tag{8}$$

and the lens equation can be rewritten as[1]

$$\theta^2 = \theta\beta + \theta_E^2. \tag{9}$$

Equation (9) can be solved with the use of the quadratic formula to give two solutions, θ_+ and θ_-. These correspond to the angular distances of the images from the optical axis. The angles are given by equation (10).[1]

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) \tag{10}$$

The two images formed are located on opposite sides of the lens, with the image located at the larger angle farther away from the lens and outside the Einstein angle. The image located at the negative angle, θ_-, is inside the Einstein angle. This image is inverted with respect to the source, while the image at θ_+ is not. The image locations can be seen in figure 7.[1] The image at θ_+ is at the top of figure 7 and the image at θ_- is at the bottom.

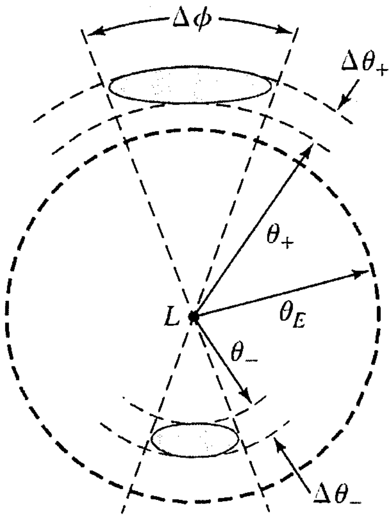

Figure 7. Image locations of light lensed from the source, relative to the lens position.

The Einstein angle is characteristic of the lensing mass and distances involved in a system. It is the angle at which the image appears if the observer, lens, and source are all aligned on a single axis. In this case the value of β is zero, and the two solutions to the lens equation are equal to θ_E, as can be easily seen from equation (9). The image of the source is a ring around the lensing mass, called an Einstein Ring. The aligned geometry necessary to produce an Einstein ring is rare, but several examples of near-complete Einstein rings have been found in lensing surveys.[8] Figure 8 shows several examples from the Sloan Lens ACS (SLACS) Survey.[8]

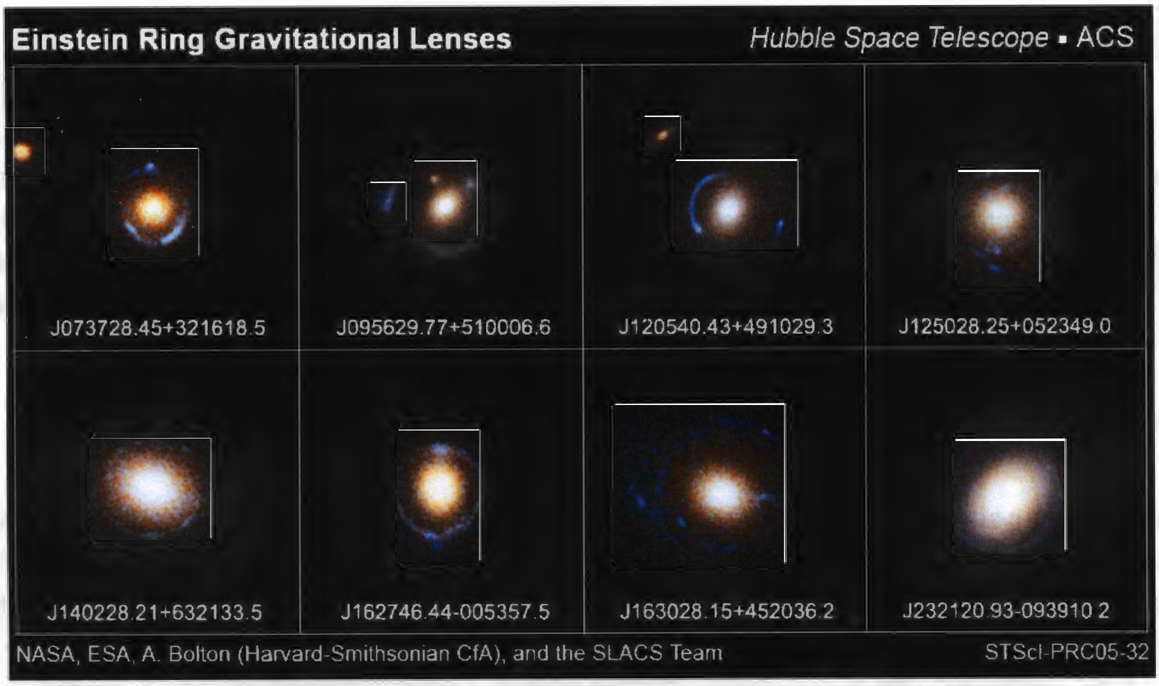

Figure 8. Several lensing systems with very good alignment, showing arcs that nearly form Einstein rings.

C. Complex Mass Distributions

Many lensing mass distributions, such as galaxies and galaxy clusters, are not modeled well with a point-like mass. For these lenses more complex models are needed. The total deflection of a light ray is the sum of the deflections caused by all parts of the lensing mass distribution. Mass distributions that do not have large extensions in the direction of the optical axis in comparison with the distances between the observer, lens, and source may be modeled as their projections onto the lens plane, perpendicular to the optical axis at the lens location. The simplest extended mass model is the circularly symmetric one in the lens plane.

One type of circularly symmetric model is the singular isothermal sphere. This is often used as a simple approximation for lensing galaxies. For a spherical mass distribution the deflection angle of light rays depends on the total mass enclosed within a circle with a radius equal to the impact parameter.[9] The mass density distribution, ρ, is[7]

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}. \tag{11}$$

In equation (11) σ_v is the velocity dispersion of the stars composing the lensing galaxy; r is the radius enclosed. The deflection angle of light rays due to this mass distribution is[7]

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2} \,. \tag{12}$$

The deflection is directed toward the center of the lensing mass. In the more general case where the mass distribution model is not spherically symmetric, the deflection angle is a two-dimensional vector quantity. If the source is far from the center of the lensing mass distribution, it is only weakly lensed and one image is formed. If the source is within the Einstein radius of the singular, spherically symmetric lens, 2 images are formed. They are located at[7]

$$\theta_{\pm} = \beta \pm \theta_E \,. \tag{13}$$

The source, lens, and two images are located on a line in the lens plane. There is a third image formed at $\theta=0$, which has zero flux and thus is not visible.[9] If the singularity at the galaxy lens center ($r=0$) is replaced with a core region of finite density, the image at $\theta=0$ has finite flux and is visible. This corresponds to a nonsingular isothermal sphere model of the lensing mass distribution. The mass density profile is said to have the singularity at $r=0$ "smoothed" to a finite number.

Several measurements are used to describe the characteristics and effects of the lensing mass. A scalar potential can be used to describe the lensing potential. The convergence and shear experienced by a light ray may then be defined in terms of linear combinations of the second derivatives of this potential.[9] The convergence is a measure of the isotropic focusing of light rays, and leads to magnification of the source. It changes the size of an image without distorting it. Shear introduces anisotropy to the image, and distorts its shape. A circular source acted on by convergence and shear would have a larger, elliptical image.

The determination of the image locations depends on the mapping of light rays from their locations in the source plane to their locations in the image plane, where they are observed. Several important locations in these planes define characteristics of the observed images. Caustics are lines or points in the source plane along which the amplification of the image diverges.[7] A source located at a caustic is mapped to the corresponding critical curve or critical line in the image plane. Along these lines the images are infinitely magnified.[7] Caustic lines separate regions of different image multiplicities. If a source is outside all the caustics of a lensing mass distribution (far from the center), only one image of the source is produced. In this case the source is weakly lensed. For a source positioned near the center of a circularly symmetric lensing mass distribution, three images are produced. This corresponds to the strong lensing regime. If the source is positioned incrementally farther from the center, two of the images move closer together then merge and vanish.[9] The location in the source plane at which the images merge is a caustic.

For a non-singular lens, the number of images produced is odd, though images near the center may be obscured by a luminous lensing mass. This is known as Burke's Theorem.[7] Starting far away from the center of the lensing mass distribution, each time the source crosses a caustic two additional images are produced. The images have opposite parities, so one image is inverted.[7] For a spherically symmetric lens the inner, tangential caustic is a point at the center of the source plane. A source located near this point produces two tangentially elongated arcs near the outer critical curve and a faint image near the center, in the lens plane.[9] The tangential arcs occur with one inside the critical curve and the other outside it. For a non-spherically symmetric lens, the tangential caustic is a curve instead of a point. The outer circular caustic is known as the radial caustic because a source located on it produces a pair of radially elongated images on the inner critical curve in the image plane and one tangentially elongated image outside the outer, tangential critical curve. The radius of the circle formed by the radial caustic is less than the Einstein radius of the mass distribution.[7] The outer caustic in the source plane corresponds to the inner critical curve in the lens plane. Figure 9, below, shows the caustics and critical lines of a non-singular, spherically symmetric lens.[9] A complex mass distribution can have caustics that are not simple circles, but diamond-like astroids or other shapes with cusps and lips.[7] The location of the arcs is related to the size of the core radius of the mass distribution. If the core radius goes to zero the model approaches that of the singular point mass; the radial critical curve becomes a point and the third image is lost, giving the two images for a point mass, as seen above.[7]

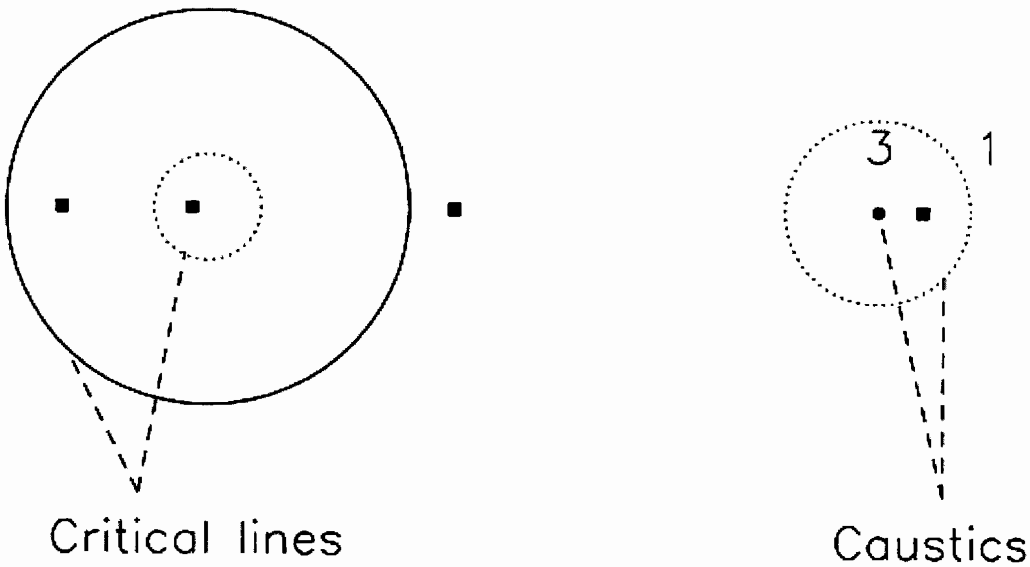

Figure 9. In the right panel the square represents a source located within the outer, circular caustic. The numbers three and one denote the number of images formed for a source in that region. The image locations are shown with squares in the left panel. Since the source is located in the region within one caustic, three images are formed.

Many lensing mass distributions are not accurately modeled by a spherically symmetric model. Elliptical models are a common way to take into account the lack of symmetry in galaxies and galaxy clusters. The elliptical potential may be considered directly with elliptical isopotential lines, or as a spherically symmetric lens with external shear applied to it.[9] The tangential caustic for an elliptical mass distribution is an astroid shape, with 4 cusps. The radial caustic, which is circular for a spherically symmetric mass, is distorted to an ellipse. Images formed in the region between the two resulting critical curves are inverted with respect to the source.[7] Since many galaxies are in clusters, modeling a galaxy's mass as a symmetric mass distribution and incorporating the external shear from the mass around it can be an accurate way to create a model that conforms to observational data.

Complex mass distributions can lead to complex lensing image configurations, with multiple images in various locations. One example of a strong lens is SDSS J142015.85+601914.8 from the SLACS survey, shown in figures 10 and 11 below. The lensed images are four arcs forming a partial ring around the lensing galaxy. The SLACS data used a singular isothermal ellipsoid as the model for the lensing galaxy.[10] The image beneath the lens and to the right is significantly tangentially elongated, indicating that the source is likely located close to the tangential caustic. The fairly symmetric arrangement of the two images nearest the bottom, elongated image indicate that the source is close to a cusp of the astroid-shaped caustic. The number of images indicates that the source is probably within the caustics, close to the center of the lensing mass. Since the model used for the lensing galaxy is singular, Burke's theorem does not apply to the image configuration. The central image expected from a nonsingular model is likely obscured by the luminous lens, and the remaining four images are observed.

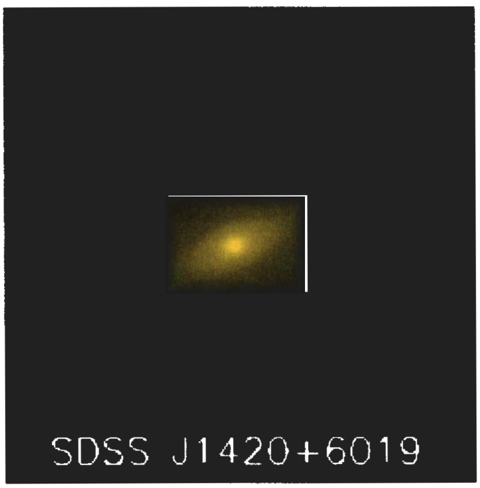

Figure 10. The lensing galaxy is shown in yellow with the lensed images around it in blue.[8]

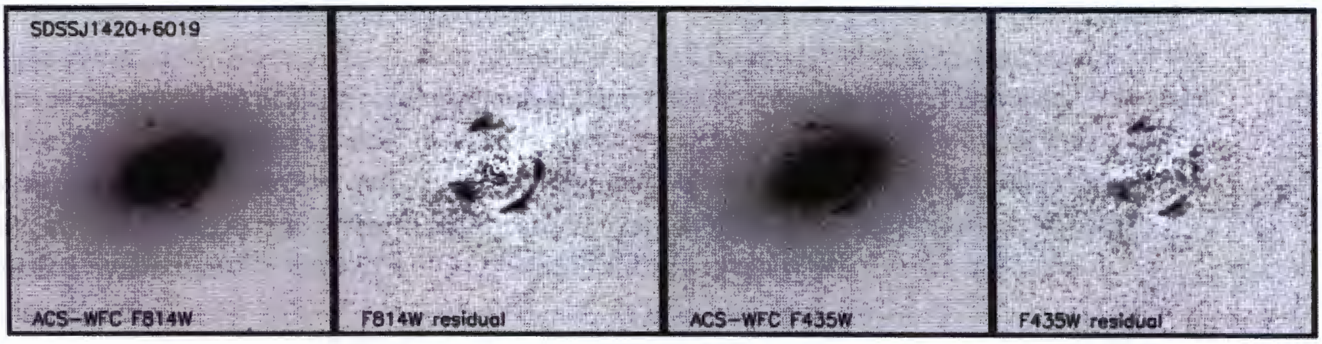

Figure 11. The leftmost panel and third panel from the left show the lensing system as imaged in two different wavelength regions by the Hubble Space Telescope Advanced Camera for Surveys. The second panel from the left shows the residual image after the lens-galaxy model has been subtracted from the leftmost image. The lensed images are clearly visible as four dark arcs. The rightmost panel shows the residual image from the third panel, after the subtraction of the lens-galaxy model.[10]

D. Gravitational Microlensing

For most microlensing events, the source and lens are small objects that can be approximated as point masses. Since the lensing mass is small in microlensing events, the images of the source produced are closely spaced. They are not separable with current Earth-based observations or those taken from orbit. The amplification of the light from a source can be measured to determine the parameters of a microlensing event. The amplification measured in microlensing is the sum of the amplifications of the images produced. This amplification can be written as[11]

$$A = \frac{u^2+2}{u(u^2+4)^{1/2}} \,. \tag{14}$$

In equation (14) u is the projected lens-source separation in the lens plane, in units of the Einstein radius. As u goes to infinity, the amplification goes to 1, which corresponds to a system with no gravitational lensing. Equation (15) shows the dependence of u on time.[12]

$$u(t) = \sqrt{u_0^2 + \frac{(t-t_0)^2}{t_E^2}} \tag{15}$$

In equation (15) t_0 is the time at which the source and lens are closest. The Einstein timescale is t_E. This is the time it takes the source to travel across the Einstein radius of the lens, and is used to scale the time in much the same way in which the Einstein radius is used to scale the distance separating the lens and source. The value of u is minimized when the source is closest to the lens, at a distance of the impact parameter u_0, at time t_0. Light curves are often plotted with time values centered on t_0, showing the number of days before and after this time and the brightness variation of the source as time progresses. Figure 12, below, shows a theoretical light curve for 2 values of the impact parameter u_0.[13] The larger value of u_0 corresponds to the lower amplification peak, since the source remains farther from the lens in this case. It is clear from the figure that the value of the impact parameter makes a large difference in the observed magnification of the event, and very good alignment is needed to observe high magnification events. The time axis is given as time from the time of the peak in amplification (t_0), and scaled by the Einstein timescale. The magnification is given relative to the unlensed magnification.

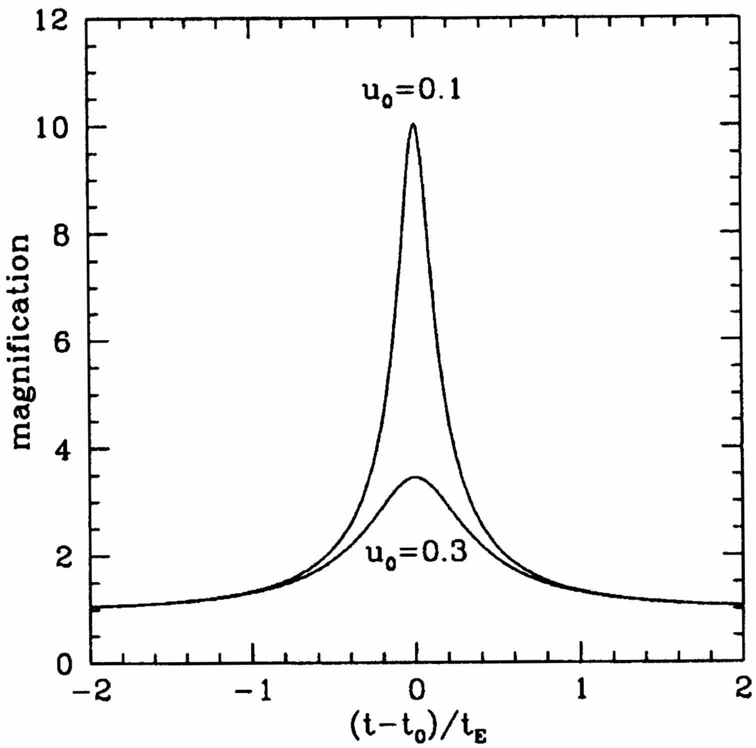

Figure 12. An example of the magnification difference in a microlensing event due to differences in the impact parameter.

The detection of microlensing events requires very specific conditions. Although the images produced are not separated enough to be resolved, the brightness variation in a source may be observed. The light curve of a source can be monitored, showing the brightness variation as time passes. In order to be a microlensing event, the light curve must show a symmetric light increase in time. As the lensing mass moves relative to the source, the luminosity increases until the lensing mass is at its minimum distance to the source, then decreases as the lensing mass moves away. Microlensing by a binary star system can result in an asymmetric light curve due to the orbiting of the two masses around each other, and two peaks may be observed.[14] If the microlensing event duration is more than a few months, the parallax motion of the Earth relative to the source and lens can make the light curve asymmetric.[14] In order to be a microlensing event the brightness variation of a source must only occur once. Numerous observations of a source at regular time intervals are needed to detect microlensing. This makes it a time-consuming subject of study, but one that has been pursued by several devoted microlensing surveys and projects.

E. Nonstandard Microlensing Conditions

The approximation of the lens and source as point masses is not accurate in some lensing systems. If the impact parameter is less than the Einstein radius of the lensing mass, the finite size of the source affects the light curve considerably, and its effect must be taken into account. In this case the amplification is the ratio of the area of the image of the source to the unlensed source area. The maximum area of the image is that of the ring formed when the source and lens are perfectly aligned, and is the area of the outer circle minus that of the inner one. The amplification in this case is[11]

$$A = \frac{\pi r_1^2 - \pi r_2^2}{\pi r_0^2} = \left(1 + 4\frac{R_0^2}{r_0^2}\right)^{1/2}. \tag{16}$$

In equation (16) r_1 is the outer radius of the ring and r_2 is the inner radius. The source's radius projected in the lens plane is r_0. R_0 corresponds to the Einstein radius of the lens. In the case of perfect alignment with an impact parameter of zero, the amplification is calculated using equation (16), which accounts for the finite size of the source and prevents the amplification from diverging. The finite size of the source must also be taken into account in other situations where the use of equation (14) would cause the amplification to diverge, such as when a source crosses a caustic in a binary lensing system.

The amplification of the source in a microlensing event can be related to its observed magnitude. The magnitudes of astronomical bodies are given as positive and negative numbers based on a logarithmic scale of brightness, with smaller numbers corresponding to brighter objects. For example, the Sun has a magnitude of -26.7, the full Moon -12.6, and the faintest star visible with the naked eye 6.5.[15] The magnitude of a source is given by equation (17).[13]

$$m = 2.5 \log_{10} F \tag{17}$$

In equation (17) m is the magnitude of the object and F is the flux of the object. The magnitude change in a microlensing event is the object's unlensed magnitude minus the observed magnitude during the event. The magnitude change of a lensed object is[14]

$$\Delta m = 2.5 \log_{10} A\ . \tag{18}$$

Since the object gets brighter during the lensing event, the magnitude decreases and the magnitude change is negative. The brightening of a star with a magnitude of 11.4 to a magnitude of 7.5 corresponds to an amplification of about 40. A source positioned at the Einstein radius would experience an amplification of 1.34, which corresponds to a brightening by 0.32 magnitudes and is easily observable.[9] Many astrophysical objects have different magnitudes in different wavelength regions, due to the emission of different types of light. Data from different wavelength regions are often used together to plot the most accurate light curves possible for microlensing events.

Many microlensing events have been recorded by various survey and observation groups, and light curves can be fit to the photometric data obtained. Computer-based algorithms search through measurements of the brightness of millions of stars nightly, and alerts for additional observation are issued when a star exhibits a brightening that is significantly larger than the noise level of the measurements. Worldwide networks of telescopes monitor suspected microlensing events and gather data, which is then fit to calculated light curves and used to determine the parameters of the lensing system. The Optical Gravitational Lensing Experiment (OGLE) has photometric data and light curves for the microlensing events detected during the second phase of the project, from 1997 to 1999. Examples of a light curve are given below in figures 13 and 14 for the microlensing event BUL_SC3 577547, observed by OGLE in 1998.[16]

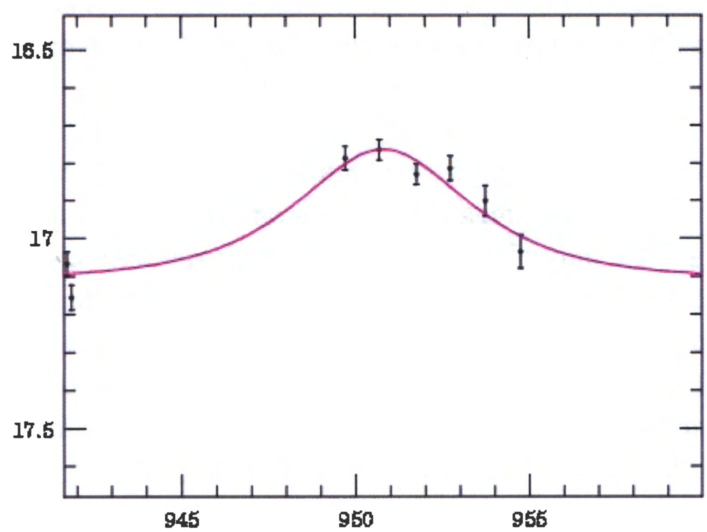

Figure 13. The light curve of microlensing event BUL_SC3 577547 near the magnification peak. The violet curve is the calculated light curve fit to the black data points. The vertical axis is the magnitude of the star. The horizontal axis is time in days after HJD 2450000. (HJD refers to Heliocentric Julian Date, and is a calculated time that takes into account the travel time of light and the motion of the Earth around the Sun.)

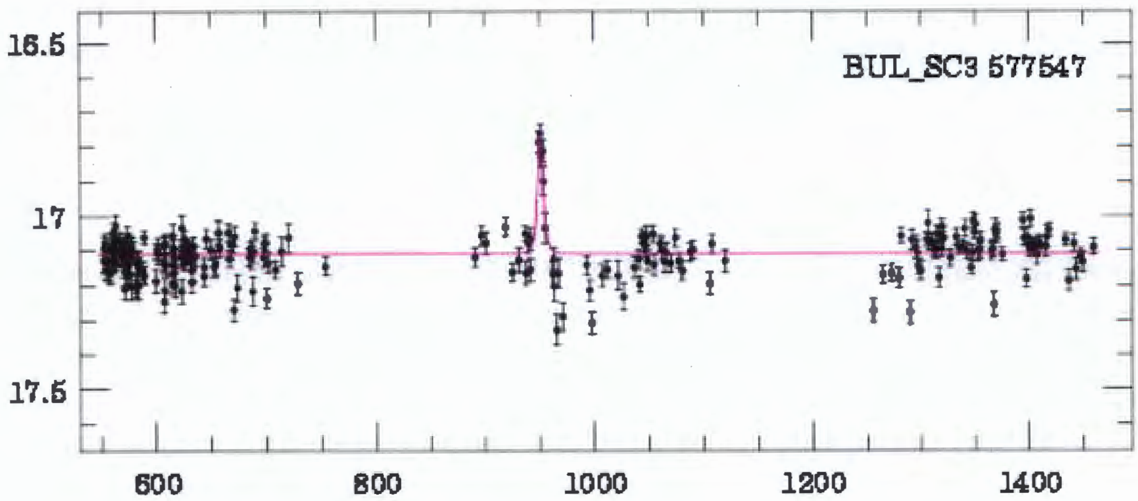

Figure 14. The full span of the light curve for BUL_SC3 577547. The large gaps in data collection correspond to the time between observing seasons. The axes are the same as in figure 13.

The microlensing event BUL_SC3 577547 was observed in May of 1998 by OGLE; the time of the maximum was HJD 2450950.800 ± 0.553.[16] The star has a magnitude of 17.109 ± 0.003 when not lensed.[16] The minimum magnitude can be seen from the light curve to be about 16.75, which corresponds to an amplification of 1.37 ± 0.05.[16] The Einstein crossing time is 3.06 ± 0.92 days,[16] which makes this event a relatively short one. A typical, average Einstein crossing time is about 19 days for microlensing events.[17]

A single lensing mass acting on the paths of light rays from a source produces the simplest microlensing curve, with a symmetric shape. In real systems, there are several situations that can lead to deviation from these simple light curves. If the lensing mass is a binary system, with two stars or a star and a planet, the combined effects of the two masses must be evaluated. In a close binary system of point masses caustics and their associated critical curves appear. If the binary masses are widely separated, their combined gravitational effect may not be enough to create a caustic system. If the source crosses a caustic, the light curve is changed significantly. Often two or more peaks will appear, and they will be sharp due to the dramatic increase in amplification near the caustic. If the source crosses a closed caustic twice (moves into the area enclosed by it and then out), there will be a set of two sharp peaks due to the increase in amplification when it crosses the caustic.

The flux measured in a microlensing event may be due to the lensed source, the lensing mass, and background stars. Many microlensing events are observed in crowded areas of the sky, so the chance of light from other stars blending with the light from the lensed source is fairly high. Since only the source is being lensed, only the light from the source should change with time. The constant flux from blended light can be subtracted from the observed flux to give the time-varying light from the lensed source. This allows the isolation of light from the lensed source, even in areas of the sky which are crowded with other stars, where lensing is most likely to occur.

III. Research Methods

The deflection of light by massive astrophysical bodies according to general relativity provides the basis for the occurrence of gravitational lensing, while the equations of geometric optics describe how the intensity of a light source changes when that light is lensed. The relative intensity increase due to lensing of a source was analyzed and the microlensing equation given in literature references was derived from it. The origin of the equation describing the separation of a source from a lens was also investigated.

Theoretical light curves generated with the use of MATLAB software were analyzed in the investigation of degeneracies in microlensing equations. Some parameters that affect the light curve include lens mass, lens distance from observer, source distance from observer, and impact parameter. The combination of these parameters in the equations of gravitational microlensing determines the unique properties of each light curve. Because of this there may be degeneracies in the equations. Variances in the plots generated were related to the parameters used in their generation to observe the effects that changes in parameters have on the light curves produced. Since theoretical light curves are often fit to those observed in order to determine the values of various parameters, changing parameters can affect the calculated values of the characteristics of a system.

IV. Results and Discussion

A. Origin of the Microlensing Equation

The equation for the relative intensity of a source that is gravitationally lensed in comparison with its unlensed brightness is given by geometric optics as[1]

$$\frac{I_+ + I_-}{I_*} = \frac{1}{2}\left[\frac{\beta}{\left(\beta^2+4\theta_E^2\right)^{1/2}} + \frac{\left(\beta^2+4\theta_E^2\right)^{1/2}}{\beta}\right].$$

This equation can be manipulated to reproduce the microlensing equation given by equation (14) above. A factor of θ_E can be taken from the terms to give

$$\frac{I_{+}+I_{-}}{I_{*}} = \frac{1}{2}\left[\frac{\frac{\beta}{\theta_E}}{\left(\left(\frac{\beta}{\theta_E}\right)^2+4\left(\frac{\theta_E}{\theta_E}\right)^2\right)^{1/2}} + \frac{\left(\left(\frac{\beta}{\theta_E}\right)^2+4\left(\frac{\theta_E}{\theta_E}\right)^2\right)^{1/2}}{\frac{\beta}{\theta_E}}\right].$$

This equation can be written in terms of $x = \frac{\beta}{\theta_E}$ to give

$$= \frac{1}{2}\left[\frac{x}{(x^2+4)^{1/2}} + \frac{(x^2+4)^{1/2}}{x}\right].$$

The terms within the bracket can be combined by factoring out the common denominator and simplified to give the equations below.

$$= \frac{1}{2}\left[\frac{x^2 + (x^2 + 4)}{x(x^2 + 4)^{1/2}}\right]$$

$$= \frac{1}{2}\left[\frac{2x^2 + 4}{x(x^2 + 4)^{1/2}}\right]$$

The factor of ½ can be absorbed by the terms in the numerator to give

$$= \frac{x^2+2}{x(x^2+4)^{1/2}},$$

which can be rewritten with the substitution of $u = x = \frac{\beta}{\theta_E}$ as

$$A = \frac{I_{+}+I_{-}}{I_{*}} = \frac{u^2+2}{u(u^2+4)^{1/2}}.$$

This is the equation for the magnification of a source during a microlensing event, equation (14) above. This explains the origin of the microlensing equation that is the basis of the light curve models used in conjunction with observations of events to determine the properties of the systems involved. The microlensing equation follows directly from the relative intensity of the lensed image to its unlensed counterpart, which is determined by the principles of geometric optics.

The microlensing equation gives the amplification of the light as a function of the variable *u(t)*. The value of *u* is equal to the distance from the lens to the source, from the projection in the lens plane. This distance is a function of time, and is minimized when the lens is at its closest approach to the source, at its impact parameter, u_0. This distance is measured as a two-dimensional Pythagorean distance, with u_0 the distance in one dimension and the distance in the second related to the velocity of the source relative to the lens. This velocity is measured as the time before or after the instance of closest approach divided by the time it takes to cross the Einstein angle. The addition of these two perpendicular distances gives the distance between the source and lens, u_0, as is shown below.

$$u(t) = \sqrt{u_0^2 + \frac{(t - t_0)^2}{t_E^2}}$$

B. Impact of Degeneracy of Parameters

The amplification of a source can be observed as it is microlensed, and the maximum amplification can be determined from the light curve produced. The maximum amplification corresponds to the amplification when the source is at its minimum distance from the lens, the impact parameter. This means that the maximum amplification can provide information about the minimum separation in terms of the Einstein angle, but it cannot be used to determine the Einstein angle itself. The information about the physical parameters of the system that can be gained from microlensing observations is limited.

The Einstein angle is dependent upon a number of the physical parameters of the system, as can be seen in the equation (8), reproduced below.

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$$

This measurement is degenerate; there are multiple combinations of the physical parameters contained within the definition of the Einstein angle that could produce the same result. For example, if the distance from the observer to the source is doubled and the fraction of this distance that is the distance to the lens (for example, half of the distance) is kept the same, the distances to the lens and source would each be doubled as well. This would result in an Einstein angle that is a factor of $\sqrt{2/(2 \cdot 2)} = \sqrt{1/2}$ as large as it would be without the doubling of the distances. If the lens mass is twice as large, this would result in an Einstein angle equal to the initial value. If the source passes by the lens at the same minimum separation, the maximum amplification would be the same for the microlensing event.

The light curves for events with similar impact parameters and Einstein crossing times would look very similar to one another. Figure 15, below, shows calculated light curves for four microlensing events with very similar parameters. The lensing mass, distance from the lens to the source, and distance from the observer to the lens were varied with the distance from the observer to the lens and the Einstein crossing time held constant. Through the manipulation of the parameters, it is possible to create light curves with nearly the same peak magnifications, as seen by the magnitude of the peaks on figures 15 and 16, below. Even for a lensing mass that was twice that of event 1, event 4 had a peak magnification differing by only 0.0085, with a proportional change in the distances involved in the system. This is well within the error of most fitted light curve plots, and shows the degeneracy of microlensing observations. For an observed time duration and maximum amplification, the lensing mass could be increased by a factor of two, with a decrease in the distance between the lens and source, with the source held at the same distance. This shows the limited amount of information that can be derived from an observation of a single microlensing event. Events with nonstandard conditions can give more information about the physical characteristics of the system, which can be used to characterize them further.

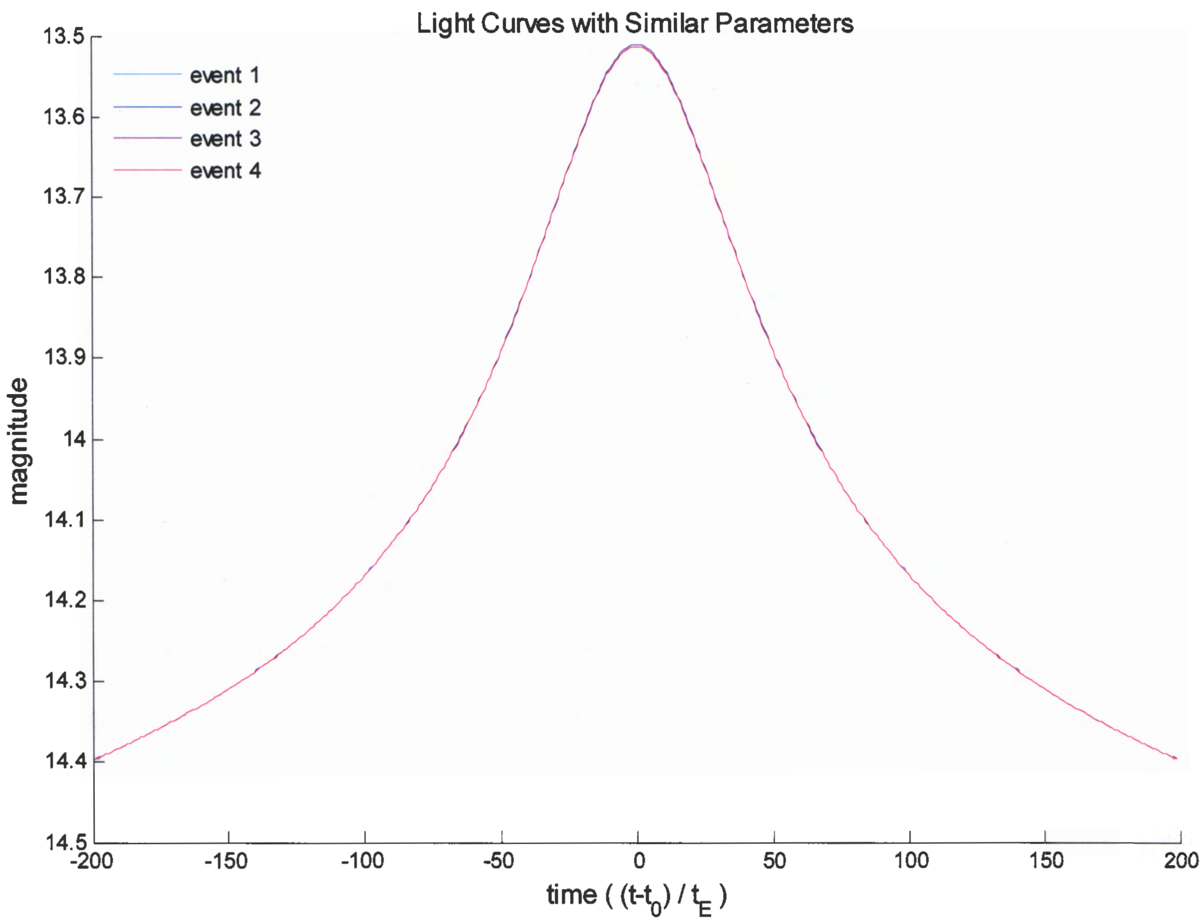

Figure 15. Four light curves for events with very similar parameters. The curves lie almost on top of one another. See figure 16 below for a magnified view of the peak region.

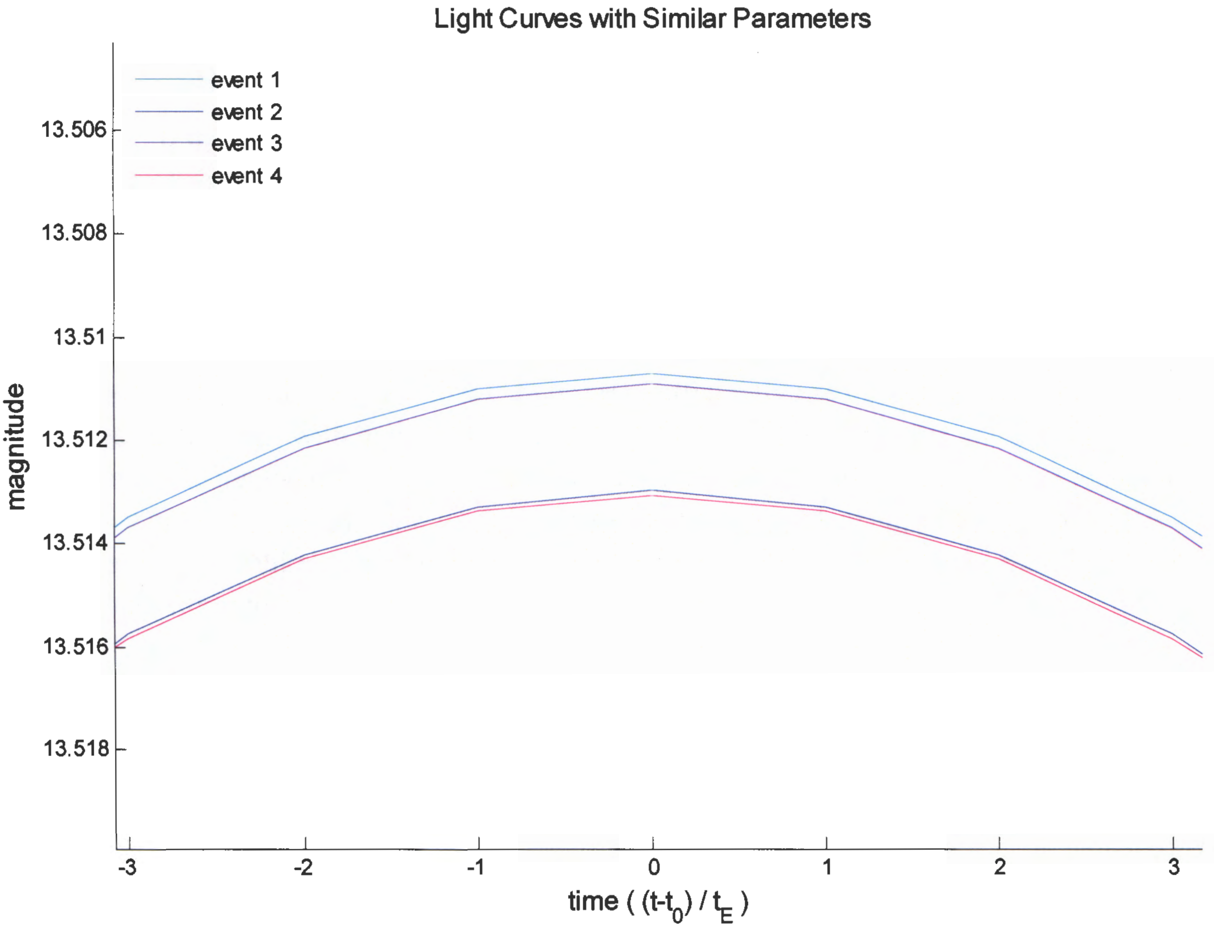

Figure 16. Magnified view of magnitude peak for the events of figure 15. The maximum magnitude is very similar for the events plotted.

C. Possibility of Gravitational Lensing Near the Earth

As was shown by the gravitational lensing of light from background stars near the Sun during the 1919 eclipse observed by Eddington and his team, gravitational lensing can be observed in the region near the Earth. It also showed that a mass as (relatively) small as that of the Sun can have an impact on the light from these sources, with reasonable distances separating the lens, source, and observer. In terms of the Einstein radius for this system, the distance from the lens to the observer was quite small and the distance from the source to the lens was nearly equal to that from the source to the observer. These factors contribute to increasing the Einstein radius and thus increasing the effect of the lensing mass on the light rays passing it. The Sun is the most massive body near the Earth, but it is difficult to observe stars near it under normal circumstances due to its high luminosity. It is theoretically possible to observe lensing of light by the other planets in the solar system, but not highly probable.

Although the observations of gravitational microlensing by the Sun show that lensing events can be seen close to the Earth, they are not very likely to be observed. The probability of observing gravitational lensing is highly dependent on the optical depth in a region. Optical depth is a measure of the fraction of the sky that is covered by the cross-section of the objects in question. If there are more objects in an area, the optical depth of the area is increased. This means that gravitational microlensing is more likely to be observed in areas with a large number of stars. Because of this, many studies are conducted in the areas of the Large Magellanic Cloud (LMC) and the Small Magellanic Cloud (SMC), which are dwarf galaxies close to the Milky Way, in the Local Group. Because they are so close, there are a large number of luminous sources at various distances from them that could potentially be lensed by the stars within the LMC and the SMC. In order to efficiently use observing time with telescopes, the regions with high optical densities are often observed in the search for gravitational lensing events. Objects near Earth in space have the potential to act as lenses for objects that are farther away. Although it is possible to observe lensing by nearby stars and other masses, it is unlikely because there are simply not many masses near the Earth that could lens the light from distant stars.

D. Importance of Gravitational Lensing Measurements

The study of gravitational lensing events can provide a variety of information. Gravitational lenses can act as cosmic telescopes, and allow the observation of objects that would otherwise be too far away or faint to see. Unlike man-made telescopes, it is not possible to aim these telescopes, so the information they can provide is limited.[9] Images resulting from gravitational lensing correspond to light rays that have taken different paths to reach the observer. The light rays have different path lengths, and thus have different travel times. If the source of the light rays varies in time, the variations will be detected at different times, corresponding to the time delay for each path. This time delay has a geometric component due to the curving of spacetime and a gravitational component.[7] If the source is far from the observer, the universe expands while the light rays are traveling. Measurement of the time delay for various images can provide information about the expansion rate of the universe, and constrain the value of the Hubble constant and other cosmological constants.[9]

Measurement of the distances of the images from the center of a lensing mass distribution and the distance to the lens can give a measurement of the lens mass in strong gravitational lensing. The redshifts of the lensing galaxy or cluster and background source can be measured from their spectra, and used to determine their distances from Earth. The determination of the mass of the lens can give information about its composition. Many lensing systems have calculated masses that are much larger than what is suggested by the luminosity of the lens; this suggests that they have a significant portion of their mass in dark matter.[9] The determination of lensing masses can give an estimate of the amount of dark matter in the universe.

Dark matter composes approximately 25% of the universe, according to calculations. Because it is difficult to observe this matter, it is difficult to determine of what it is composed. It may be mainly massive compact halo objects, such brown dwarfs of concentrations of heavy elements. These objects could act as lenses in microlensing events. The dark matter composing a portion of the mass of a galaxy could also be mainly in the form of exotic, weakly interacting massive particles, such as neutrinos. Studying these particles may provide information about the processes undergone by astrophysical bodies in the universe, and the effects of these processes on the Earth.

Some objects appear close together in the sky but do not interact with one another, such as when they are in a similar location but one is much farther from the Earth than the other. Gravitational lensing observations can be used to determine if objects that appear near one another in the sky are actually related to one another physically. Multiple images formed in strong gravitational lensing may be mistaken for unrelated objects, or the reverse. In order to determine if a collection of images represents gravitational lensing of one source, several characteristics are used. Multiple images of a single source retain the spectroscopic properties of the source, even though their shapes and brightnesses may be altered. This means that the images must have the same color and redshift. The images must be fairly close to one another in the sky; the largest separation ever measured between images of one source was 22".5 in 2006.[18] There should also be a massive object between images at a smaller redshift than the images, which is capable of deflecting the light from the source. The smaller redshift corresponds to the object being located closer to the Earth than the source, and thus being able to bend the light from the source.

Many variable stars have cyclic brightness variations. A change in the apparent magnitude of an observed star may be mistaken for a microlensing event quite easily. The possibility that the source is a variable star must be eliminated in order to determine if an event is microlensing. The brightness variation must occur at all emitted wavelengths of light, since gravitational lensing occurs at all wavelengths. This ensures that the event is not an emission from the source of a certain type of light. The brightness variation in a microlensing event typically takes place over a time period of a few days to a few years.[9] Periodic observations of a source may reveal regular fluctuations in its magnitude, which indicates that the source is a variable star of some type. Microlensing surveys have discovered thousands of previously unknown variable stars, and provided a great deal of information about these stars.

Observations of gravitational lensing can also provide information about the optical depth of the region observed. If a large number of gravitational lensing events are observed in a region, it indicates that there is a high optical depth in that region, and a relatively large number of massive objects. Lensing masses can be dark, but the effect of their mass and the gravity associated with that mass on photons passing them can be measured. In this way gravitational lensing observations can give information about the distribution of mass in a region.

Most of the systematic surveys done so far have been in the strong and weak lensing regimes, although several more dedicated surveys are planned for microlensing and nanolensing.[19] With improved technology and the ability to detect lensing caused by small masses, gravitational lensing may be an effective means of search for exoplanets. Several exoplanets have already been detected via gravitational lensing. Microlensing can detect planets of smaller mass than many more commonly used techniques, such as radial velocity shifts. Microlensing can also detect exoplanets orbiting farther from their stars than other techniques. This can allow the study of the population of cooler planets and their formation far from their stars.

Gravitational lensing is one of few techniques that depend solely on the mass of the object for detection, not the luminosity. Because of this it is possible to detect small dark objects, including exoplanets which do not orbit host stars.[20] The detection of MACHOs (massive astrophysical compact halo objects) such as black holes, neutron stars, or brown dwarf stars in the galactic halo is an area of great interest because of the information about the structure and mass distribution of the galaxy that can be gained. Microlensing provides a way to study the distribution of dark massive objects in the galaxy and beyond.

V. Summary and Conclusions

Gravitational lensing can occur when astrophysical objects are aligned. If a luminous source is located far from an observer and a massive body is located near the line connecting the source and observer, gravitational lensing can occur. The mass of the intervening object bends the local spacetime near it. The light photons traveling toward the observer from the source must pass through this bent spacetime and are deflected, from the observer's perspective, as they follow their null paths in spacetime. The massive body acts as a lens in changing the path of the light rays that come near it.

When a galaxy or galaxy cluster acts as the lensing mass in a gravitational lensing system, the light rays that reach the observer can be viewed as separate images. Images of gravitationally lensed sources such as distant galaxies and quasars from many sky surveys show the effect these masses have on their light rays. The distribution of the mass in a galaxy or galaxy cluster may not be symmetric, and the various mass concentrations may create several images of the source that is being lensed, in different positions around the lens. If the source and lens are exactly aligned from the observer's perspective, the image will be an Einstein ring around the lens, separated from the lens by the Einstein angle. This configuration is very unlikely to occur, due to the precise alignment needed.

The Einstein angle is characteristic of the lensing system. It is dependent on the mass of the lens and the distances from the source to the lens, lens to observer, and source to observer. It is an especially useful concept in the study of gravitational microlensing events. The amplification of the source in a microlensing event depends on how close the source comes to the lens, in terms of the apparent distance between the two. A source which reaches the Einstein angle of a lens at its closest point is amplified by a factor of 1.34. This increase in brightness can be measured by observations with telescopes, and the lensing effect can be observed.

Since the mass of the lens is relatively small in microlensing events, the images of the source formed are very close together and cannot be resolved by current telescopes. Instead of measuring the image positions, the increase in brightness of the source is measured in a microlensing event. This time-dependent change is plotted as a light curve from which some characteristics of the lensing system can be determined. The influence of various parameters (such as the lens mass, distance between the lens and the source, distance between the lens and the observer, and the distance between the source and observer) on the equations used to describe microlensing events means that there is degeneracy in the light curves observed. Systems which have significantly different physical parameters may produce very similar light curves, which are fit to theoretical light curves in an attempt to characterize the system involved. The information given by observations of a single microlensing is limited, but can be used to learn more about the universe and its components.

Because gravitational lensing depends only on the mass of the lens, observations of lensing events can be used to determine the distribution of mass in the universe. If a galaxy or galaxy cluster acts as a lens in a system, the lens mass determined from calculations of the image locations can be compared with the observed luminosity to determine the amount and location of dark mater within the galaxy. A dark massive object may also act as a lens in a microlensing event; observations of the location of these objects provide information about the structure of the Milky Way and the objects near it. Not much is known about dark matter or its relationship to astrophysical objects and processes, and a more extensive understanding of the composition and distribution of this dark matter in the universe can answer some of the questions that exist on the subject today.

Gravitational lensing observations can provide information about the expansion rate of the universe, if multiple images are observable. The time delay for these images as the light rays composing them are lensed is not equal, and the difference observed with a variable source can be used to constrain the rate of expansion based upon how distant the source is. This may provide a better sense of fundamental constants governing the universe and how it changes over time.

Gravitational microlensing is a technique that can detect objects other than the lenses themselves. The intensive surveys conducted in order to find microlensing events have uncovered many variable stars, and provided a great deal of information about these astrophysical objects. Extrasolar planets may also be detected by observations of microlensing events. Exoplanets of small mass, orbiting cool, dim stars may be detected this way. Any massive object can act as a lens, provided it is aligned with a luminous source. This means that there is a potential for unbound exoplanets to be detected. The search for exoplanets is a scientific pursuit of great interest to many, as there may be a possibility of extraterrestrial life on some Earth-like worlds.

Gravitational lensing observations can provide information about the region near the Earth, as well as the region containing distant galaxies. Dark masses can be detected based on their interactions with the luminous sources behind them. The optical depth of a region of the sky can be analyzed by determining the frequency of lensing events, although a good understanding of the detection rate of events is needed to obtain an accurate optical depth measurement. If a survey of a region of the sky does not detect all the lensing events that occur, the optical depth of the region which is calculated based on the number of events observed will be less than the actual value.

Although the scope of this thesis is limited, there is a great deal of further research to be done in the study of gravitational lensing. It is a phenomenon that has wide-ranging implications. Observations can provide information about the nature of the universe itself, as well as the objects within it. Studies of gravitational lensing can lead to a more thorough understanding of the structures that impact the Earth and the environment in which we live. An increase in the number of survey and follow-up teams that take data on microlensing events would improve the detection rate and allow for more statistically-driven analysis. It would also almost assuredly result in the detection of more unusual events, which could provide a more thorough understanding of the processes involved and the best methods for determining the characteristics of the systems involved in the events themselves. The study of gravitational lensing is a fairly new one, and it will certainly lead to new discoveries and insights in the years to come.

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