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Developing crochet patterns for surfaces of non-constant curvature

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DEVELOPING CROCHET PATTERNS FOR SURFACES OF NON-CONSTANT CURVATURE

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Date

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Abstract

The purpose of this project is to develop an algorithm to create crochet patterns for a variety of surfaces. I start with surfaces of constant curvature: the Euclidean surface and the sphere. Then, I generate patterns for surfaces of revolution by calculating the change in circumference for each row of stitches. My methods suggest an approach to crochet more surfaces such as surfaces whose cross section is not a circle. This research demonstrates how crochet can act as a discrete model of differential geometry. Producing these patterns allows for further research into the surfaces themselves by providing accurate models as well as continues the study of the relationship between crochet and mathematics. By studying this relationship I can increase the amount of understanding of mathematics (a subject often found difficult) for those who understand crafts, such as crochet, by describing mathematics in terms they can better understand. This is a very important part of researching mathematics; not only researching advanced topics in mathematics but also how to teach and demonstrate mathematics to non-mathematical people.

1 Introduction

This project came about after reading the book written by Daina Taimina titled *Crocheting Adventures with Hyperbolic Planes*. This book outlines the history of crochet, defines the hyperbolic surface, and most importantly for this project, shows how to create a crochet pattern for the hyperbolic surface. After reading this book, I wondered how to crochet other surfaces, specifically a sphere. By using the example from *Crocheting Adventures with Hyperbolic Planes* it was possible to develop a pattern for a sphere as well as other surfaces. (4)

I started working on this project with the idea that I would work with the three basic surfaces of constant curvature (Euclidean surface, sphere, and hyperbolic surface) to crate surfaces of non-constant curvature. I started with reading the Taimina book and studying the process she used to develop the pattern in order to repeat it for the Euclidean and spherical surfaces. Since the hyperbolic surface has constant negative curvature, the Euclidean surface has constant 0 curvature, and the spherical surface has constant positive curvature I believed that I would be able to construct patterns for surfaces of non-constant curvature based on the patterns for these surfaces.

In order to begin developing the patterns for the Euclidean and spherical surfaces I had to understand what made them different from other surfaces, such as a torus that is shapes like a donut with a single hole in the middle. To understand this, I read about spherical geometry and curvature in *Experiencing Geometry: Euclidean and Non-Euclidean with History* the Third Edition by David W. Henderson and Daina Taimina. This book talked about what geometry looks like on a sphere, including the geodesics of spherical geometry and triangles on a sphere. I also learned about curvature from this book, especially extrinsic curvature or Gaussian curvature. (3)

I also learned about curvature from *Modern Differential Geometry of Curves and Surfaces* by Alfred Gray. This textbook also gave a very good description of extrinsic curvature that helped me to learn how to calculate curvature that was to be very important to me process. (2)

I proceeded to develop the patterns for the Euclidean and spherical surfaces based on what I had learned from *Crocheting Adventures with Hyperbolic Planes*, *Experiencing Geometry: Euclidean and Non-Euclidean with History* and *Modern Differential Geometry of Curves and Surfaces* by Alfred

Gray.

The next step to developing the patterns for surfaces of non-constant curvature did not go as well. I attempted to develop the pattern for a paraboloid based on the fact that a paraboloid has positive curvature, but not constant curvature. I was unable to figure out how to take parts of the patterns for different spheres (each of which has different curvature based on the radius of the sphere) since I was unable to figure out how to combine the different spherical patterns together in a way that resulted in a paraboloid.

After being unable to develop accurate patterns using curvature I turned to using arc length. I determined that one stitch equals one unit of arc length. The paraboloid pattern was much easier to develop using arc length after I recalled that the paraboloid is a surface of revolution and can be treated as circles rotating about an axis. The arc length gives the radius of the circle that is each row. In order to calculate the radius I used a mathematical program called Wolfram Mathematica. I had never used this program before and so used *The Student's Introduction to Mathematica: A Handbook for Precalculus, Calculus, and Linear Algebra* the second edition by Bruce F. Torrence and Eve A. Torrence to learn what the syntax is and what kinds of calculations Wolfram Mathematica can do. (5)

After determining to use arc length and how to generate the patterns using arc length I was able to generate patterns for the paraboloid, hyperboloid (of 1 and 2 sheets), and ellipsoid. These surfaces are all surfaces of revolution. Lastly, I experimented with rotating the surfaces about the axis in a square rather than a circle, finding that the curvature remains the same whether or not the change in stitches is evenly distributed or concentrated at four corners, as with the square rotations.

2 Crochet

To crochet these surfaces I have been crocheting each row as a circle. Each row will be one time around, making a complete circle. This means that each row could also be referred to as a "round." In Figure 1 below the circle could be a cross section of any of the patterns. Each block in Figure 1 represents a stitch. The number of stitches will change from one row to another as the surfaces change. This means the circumference of the circle that makes up that row (since I am crocheting in rounds) will change from one row to another.

Figure 1: Cross section with blocks for stitches (11)

Another way to think of the patterns is to think of each stitch as a step an ant may take if it were walking across the entire surface, as described in Gaussian curvature. So, if the surface changes in curvature or size from one row to another, the number of steps the ant would take (or the number of stitches) will also change. Figure 2 below can be thought of as the image of a cross section of a pattern with each stitch marked out as an ant.

Figure 2: Cross section with ants for stitches (7)

3 Surfaces of Constant Curvature

3.1 Curvature

The best way to think of curvature for this project is to think of it as the best-fit sphere. The best-fit sphere is the sphere that has the same curvature value as the surface at a given region. So, it is possible to think of curvature as the sphere that best fits in that section of the surface because they have the same curvature. Curvature can be measured in Gaussian Curvature which will be discussed later in this report. For example, the Gaussian curvature of a sphere is calculated by taking $\frac{1}{R^2}$.

A Euclidean surface is what is usually thought of when one starts thinking about geometry as it is what is taught in grade school. Euclidean surfaces are geometrically flat and are what we usually work with when working with geometry. Euclidean geometry is used on a Euclidean surface. An example of a Euclidean surface that may not seem obvious is a cylinder. The cylinder is a Euclidean surface because it can be unrolled onto a flat plane without tearing.

A Euclidean surface has zero curvature, see Figure 3 for an example of a Euclidean surface. A spherical surface is one that appears to curve outward like on an egg or an apple and has positive curvature, see Figure 4 for an example of a spherical surface. A hyperbolic surface is one that appears to curve inward like on coral or kale leaves and has negative curvature, see Figure 5 for an example of a hyperbolic surface.

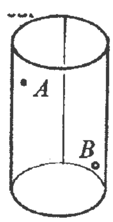


Figure 3: Cylinder with zero curvature (9)

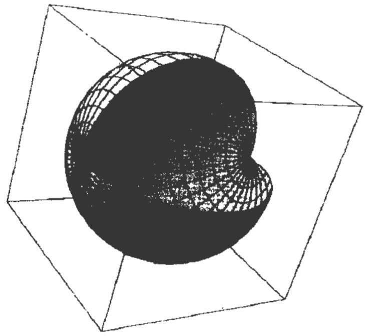


Figure 4: Sphere with positive curvature (8)

Figure 5: Crocheted hyperbolic surface with negative curvature (12)

3.2 Gaussian Curvature

Gaussian curvature is the intrinsic curvature of a surface; this is the curvature a bug on the surface would be able to calculate. Gaussian curvature is defined as $k_1 \times k_2$ where k_1 and k_2 are the principal curvatures. The principal curvatures are the curvature in two different directions from one point on the surface. Figure 3.2 is an example of the principal curvatures marked off on a surface. However, there are many ways to calculate Gaussian curvature; I will talk more on this later.

Mean curvature is the average of k_1 and k_2; however, I will use Gaussian curvature for the following reason. To determine the correct way to measure curvature. I will calculate the curvature of a cylinder. A cylinder is a Euclidean surface, and so, by the design of this project, it must have constant 0 curvature. Note that the one of the principal curvatures is 0 and the other is

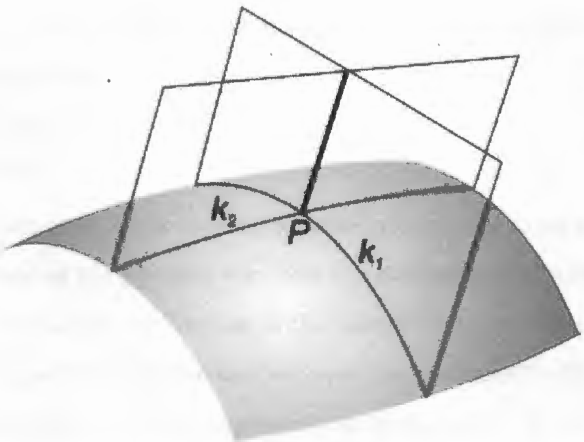


Figure 6: Example of principal curvatures (10)

some positive number. The Mean curvature, given by $H = \frac{k_1+k_2}{2}$, will be a positive number. The Gaussian curvature of a cylinder, given by $K = k_1 \times k_2$, be 0 since one of the principal curvatures is 0. So, in order for a cylinder, a Euclidean surface, to have constant 0 curvature, I will use Gaussian curvature.

3.3 Euclidean Surface

A Euclidean surface, as discussed above, has curvature 0. So, the basic Euclidean surface will be crocheted by crocheting stitches in circles of increasing radii.

For this pattern, I started with the size of a stitch, represented by l. I calculated my stitch size by crocheting a patch of 10 stitches in 10 rows, measuring the size of the patch in inches and dividing by 100 stitches; I found my stitch size to be 0.75 inch. In order to have an accurate pattern, it is important to measure your stitch size as the size of a stitch changes drastically depending on the yarn used, the size of the crochet needle, and the crocheter.

To crochet the circumference of the circle, it is necessary to know the radius of the circle. The radius of the row, represented by r, will increase by one stitch size since each row increases the distance from the center by one stitch. For example row 1 has radius 0.75 inch (1×0.75) and row 4 has radius of 3 inches (4×0.75). Then, it is possible to calculate the circumference of the circle for each row by using $C = 2\pi r$. The next step is to calculate the circumference in stitches. This is done by dividing the circumference in inches by the size of one stitch.

$C(inches) = 2\pi r(inches)$

$C(stitch) = 2\pi \frac{r(inches)}{l}$

Then it is simple to calculate the increase or decrease from one row to another by subtracting the circumference (in stitches) of the previous row from the circumference (in stitches) of the current row. It is interesting to note that the increase is the same for all rows, 6 stitches. This is because when calculating the circumference in stitches, the equation becomes $C = 2\pi n$ with n representing the row number. This means that the circumference is increasing by 2π from one row to another and when this is rounded, as I had to do since stitches must be whole numbers, 2π rounds to 6.

See Figure 7 below for the pattern for the basic Euclidean surface. See Figure 8 below for a picture of the resulting crochet surface.

stitch size	0.75	inches		
line	R	C (inches)	C (stitches)	Inrease/Decrease
1	0.75	5	6	6
2	1.5	9	13	6
3	2.25	14	19	6
4	3	19	25	6
5	3.75	24	31	6
6	4.5	28	38	6
7	5.25	33	44	6
8	6	38	50	6
9	6.75	42	57	6
10	7.5	47	63	6
11	8.25	52	69	6
12	9	57	75	6
13	9.75	61	82	6
14	10.5	66	88	6
15	11.25	71	94	6
16	12	75	101	6
17	12.75	80	107	6

Figure 7: Euclidean surface pattern

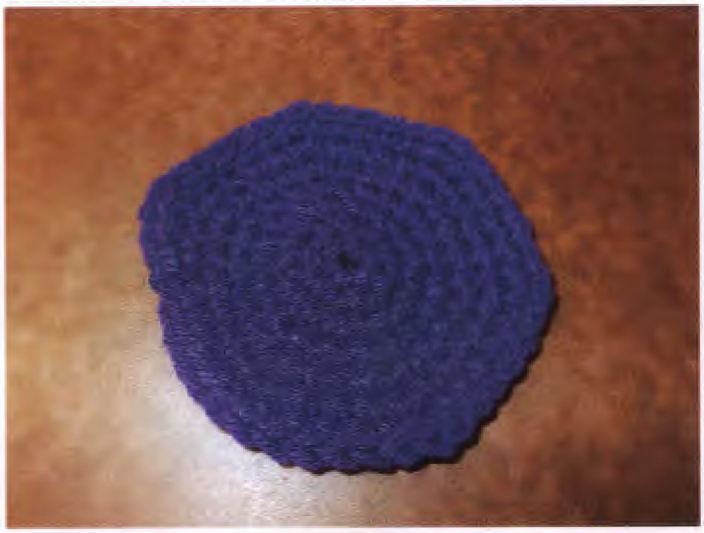


Figure 8: Crocheted Euclidean surface

3.4 Sphere

3.4.1 Pattern for Sphere

In order to create a reliable program for a sphere pattern, I started by looking at the work of Daina Taimina in *Crocheting Adventures with Hyperbolic Planes.* In her book, Taimina outlines how to evaluate the radius of a circle at each row in the pattern and how to add stitches to the pattern so the ratio of the circumference of the circle in the current row to the previous row is consistent with the constant negative curvature of the hyperbolic plane. A hyperbolic surface I crocheted using Taimina's book is in Figure 9 below.

To begin generating a pattern for a sphere, I started with a given radius, R (in inches), of the sphere and the size of my stitch, l (in inches). The stitches are assumed to be all the same size. While this is not true, it will be assumed that any differences between stitch sizes will be negligible.

The circumference of a circle on a sphere has the formula $C = 2\pi R\sin(\theta)$. θ is the angle from the center of the sphere of the circle on the sphere, this can be seen in Figure 10. Also from Figure 10, you can see that the radius of the circle on the sphere has radius $R\sin\theta$. This is from forming the right triangle with angle θ and hypotenuse R and then using trigonometry to solve for the radius

Figure 9: My crocheted hyperbolic surface based on Taimina's book

of the circle. This means that the radius in the formula for the circumference of a circle ($C = 2\pi r$) can be substituted with $R\sin\theta$ to result in $C = 2\pi R\sin(\theta)$.

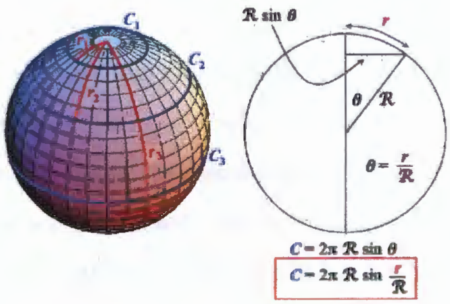


Figure 10: Circumference of a circle on a sphere (13)

It is necessary to know how many radians a stitch equals because C depends on θ, which is in radians. To do this, I will need to define what a great circle is. A great circle is a circle on a sphere that divides the sphere into two even hemispheres. For example, the Equator on Earth is a great circle since it divides the Earth into the Northern and Southern Hemispheres. A prime meridian would also be a great circle on Earth. A great circle is important because it is possible to relate the circumference of a circle to elements of the sphere when dealing with a great circle. The circumference of a great circle on a sphere is $C = 2\pi R$ since the radius of the great circle equals the radius of the sphere, R.

In order to find how many radians a stitch equals, I will use the equation of circumference of a great circle. We know that a circle equals 2π radians around, so it is possible to relate radians to stitch size as follows.

$2\pi(\text{radians}) = \frac{2\pi R}{l}\text{stitches}$

$1(\text{radians}) = \frac{R}{l}\text{stitches}$

$\frac{l}{R}(\text{radians}) = 1(\text{stitch})$

Now, note that θ increases each row by the size of one stitch in radians. θ is equal to the row number, n, multiplied by the size of a stitch in radians: $\theta = nl$.

So,

$$\begin{aligned} C &= 2\pi R\sin(\theta)\text{stitches} \\ &= 2\pi R\sin(nl)\text{stitches} \end{aligned}$$

This is the equation to use to determine the number of stitches in each row of the pattern.

An example of the pattern for a sphere of 6 inches with crochet stitch size of 0.75 inch is below in Figure 11. For an example of a sphere crocheted using this pattern see Figure 12 below.

Below in Figure 13 you can see a crocheted sphere with radius 6 inches (in blue) and with radius 2 inches (green). You can see the difference in the size of the spheres, as the spheres should be.

3.4.2 Calculating Error

After creating this pattern I verified that the pattern maintains the characteristics of the surface. I verified the spherical pattern by measuring out a circle on the surface and counting the number of stitches that form the area of that circle. I then compared that count to the calculated area of that circle, given that radius, and then calculated the error.

For example, on the 6 inch radius sphere I have crocheted based on my pattern (see Figure 12

R	**6**	**inches**		
stitch size	**0.75**	**inches**		
stitch size	0.125	radians		
C euclid	38	inches		
C euclid	50	stitches		
Total Rows	25			
line	theta	C (r) (inches)	C (r) (stitches)	Inrease/Decrease
1	0.125	5	6	6
2	0.25	9	12	6
3	0.375	14	18	6
4	0.5	18	24	6
5	0.625	22	29	5
6	0.75	26	34	5
7	0.875	29	39	4
8	1	32	42	4
9	1.125	34	45	3
10	1.25	36	48	2
11	1.375	37	49	2
12	1.5	38	50	1
13	1.625	38	50	0
14	1.75	37	49	-1
15	1.875	36	48	-2
16	2	34	46	-2
17	2.125	32	43	-3
18	2.25	29	39	-4
19	2.375	26	35	-4
20	2.5	23	30	-5
21	2.625	19	25	-5
22	2.75	14	19	-6
23	2.875	10	13	-6
24	3	5	7	-6
25	3.125	1	1	-6

Figure 11: Sphere pattern example

above), a circle of radius 6 stitches on the sphere has an error of 1.8 %. I counted 111 stitches in the area of the circle when there should have been 113 stitches. This error is very small when considering that the pattern must be rounded to whole numbers for each row because it is not possible to crochet a fraction of a stitch. This helps to justify that the patterns I have produced follow the curvature and geodesic that the surface has and to validate my results.

3.4.3 Gaussian Curvature of a Sphere

The Gaussian curvature of a sphere is $K = \frac{1}{R^2}$. This is not calculated by the product of the two principal curvatures, but it is possible to calculate Gaussian curvature by $K = \frac{F_{xx} \times F_{yy} - (F_{xy})^2}{(1+(F_x)^2+(F_y)^2)^2}$ if the surface is given by a formula $z = f(x, y)$. F_x and F_y are single derivatives and F_{xx}, F_{yy}, and

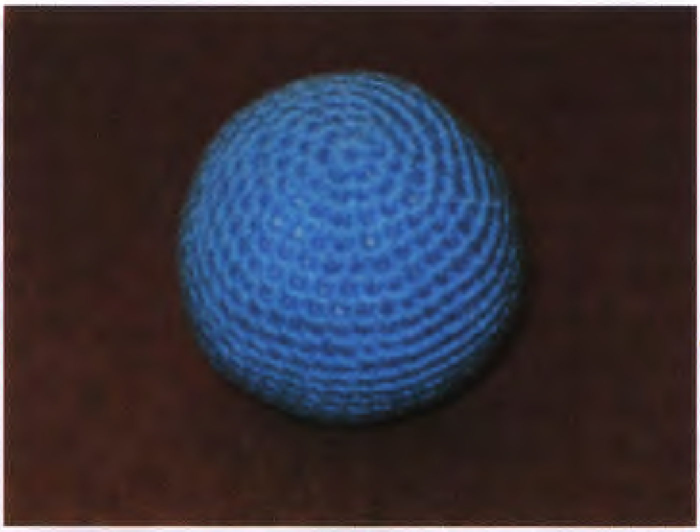


Figure 12: Crocheted sphere of radius 6 inches

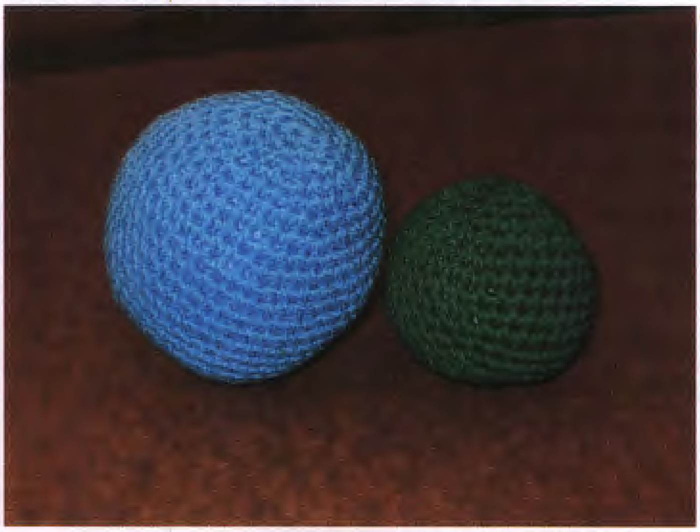


Figure 13: Spheres of radii 6 inches and 2 inches

F_{xy} are second derivatives of the function $z = f(x, y)$ of the surface. The equation for a sphere is $R^2 = z^2 + x^2 + y^2$ so $f(x, y) = z = \sqrt{R^2 + x^2 + y^2}$ with R being the radius of the sphere. Below are the calculations to determine $K = \frac{1}{R^2}$.

Here I will list the derivatives of $f(x, y)$:

$F_x = -x(R^2 + x^2 + y^2)^{-1/2}$

$F_{xx} = -(R^2 + x^2 + y^2)^{-1/2} + -x^2(R^2 + x^2 + y^2)^{-3/2}$

$F_{xy} = -xy(R^2 + x^2 + y^2)^{-3/2}$

$F_y = -y(R^2 + x^2 + y^2)^{-1/2}$

$F_{yy} = -(R^2 + x^2 + y^2)\ ^{1/2} + -y^2(R^2 + x^2 + y^2)\ ^{3/2}$

For the sake of readability I am going to use $w = R^2 + x^2 + y^2$. So,

$$\begin{aligned}
K &= \frac{(-w^{-1/2} + -x^2w^{-3/2}) \times (-w^{-1/2} + -y^2w^{-3/2}) - (-xyw^{-3/2})^2}{(1 + (-xw^{-1/2})^2) + (-yw^{-1/2})^2))^2} \\
&= \frac{w^{-1} + y^2w^{-2} + x^2w^{-2}}{1 + 2x^2w^{-1} + 2y^2w^{-1} + x^4w^{-2} + 2x^2y^2w^{-2} + y^4w^{-2}} \\
&= \frac{w^{-1} + y^2w^{-2} + x^2w^{-2}}{1 + 2x^2w^{-1} + 2y^2w^{-1} + x^4w^{-2} + 2x^2y^2w^{-2} + y^4w^{-2}} \\
&= \frac{w^{-1}(1 + y^2w^{-1} + x^2w^{-1}}{1 + 2x^2w^{-1} + 2y^2w^{-1} + x^4w^{-2} + 2x^2y^2w^{-2} + y^4w^{-2}} \\
&= \frac{1 + y^2w^{-1} + x^2w^{-1}}{w + 2x^2 + 2y^2 + x^4w^{-1} + 2x^2y^2w^{-1} + y^4w^{-1}}
\end{aligned}$$

At this time I will replace w for $R^2 + x^2 + y^2$ once again.

$$\begin{aligned}
K &= \frac{1 + \frac{y^2}{R^2+x^2+y^2} + \frac{x^2}{R^2+x^2+y^2}}{R^2 - x^2 - y^2 + 2x^2 + 2y^2 + \frac{x^4}{R^2-x^2-y^2} + \frac{2x^2y^2}{R^2-x^2-y^2} + \frac{y^4}{R^2-x^2-y^2}} \\
&= \frac{\frac{R^2}{R^2+x^2+y^2}}{\frac{(R^2+x^2+y^2)(R^2-x^2-y^2)+x^4+2x^2y^2+y^4}{R^2-x^2-y^2}} \\
&= \frac{\frac{R^2}{R^2-x^2-y^2}}{R^2 - x^2 - y^2} \\
&= \frac{R^2}{R^2 - x^2 - y^2} \times \frac{R^2 - x^2 - y^2}{R^4} \\
&= \frac{1}{R^2}
\end{aligned}$$

That concludes showing that the Gaussian curvature of a sphere is indeed calculated by $\frac{1}{R^2}$.

This will be used below to compare the positive curvature of a sphere to other surfaces with positive curvature, like a paraboloid.

4 Attempting Surfaces of Non-Constant Curvature

Originally when I started working on this project I had believed that it would be possible to construct patterns for surfaces of non-constant curvature by using parts of patterns for the three surfaces of constant curvature. I started with working on a paraboloid. I started by calculating the curvature of a paraboloid.

For a paraboloid, the surface is give by $z = x^2 + y^2$. So, $f(x, y) = x^2 + y^2$.

Then,

$f_x = 2x$

$f_y = 2y$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 0$

And so,

$K = \frac{F_{xx} \times F_{yy} - (F_{xy})^2}{(1+(F_x)^2+(F_y)^2)^2}$ with F_{xx}, F_{yy}

$K = \frac{2\times 2 - 0^2}{(1+(2x)^2+(2y)^2)^2}$

$K = \frac{4}{(1+4x^2+4y^2)^2}$

This means that each point on the paraboloid has positive curvature and different for each point. This means that technically, each point should be related to a different sphere pattern since the sphere has constant positive curvature. It is possible to get different curvatures on a sphere by changing the radius of the sphere.

When I tried to work on pattern for the paraboloid, I had trouble determining how many stitches on the paraboloid should come from the same sphere pattern. I also had trouble in determining what part of the pattern for the sphere to use for the paraboloid. While it is true that the sphere has constant curvature, the pattern for the sphere changes from one row to another.

The incorrect pattern for a paraboloid can be seen in Figure 14. The result of my attempt to use curvature to crochet a paraboloid can be seen in Figure 15. A correct crocheted paraboloid can be seen in Figure 24.

z	x	y	K - curvature	Radius of Closest Sphere	number of stitches in each row	Increase
1	0	1	0.16	2.5	16	
2	0	1.414213562	0.049382716	4.5	28	13
3	0	1.732050808	0.023668639	6.5	41	13
4	0	2	0.01384083	8.5	53	13
5	0	2.236067977	0.009070295	10.5	66	13
6	2	1.414213562	0.0064	12.5	79	13
7	0	2.645751311	0.004756243	14.5	91	13
8	0	2.828427125	0.003673095	16.5	104	13
9	0	3	0.002921841	18.5	116	13
10	0	3.16227766	0.002379536	20.5	129	13
11	0	3.31662479	0.001975309	22.5	141	13
12	0	3.464101615	0.001665973	24.5	154	13

Figure 14: Incorrect paraboloid pattern

5 Surfaces of Revolution

I will now develop crochet patterns for surfaces of revolution. This means that the surfaces are those created by rotating a curve about an axis, in this case the x or y axis.

For example, the paraboloid is created by rotating a parabola about an axis on an x and y graph. Different surfaces can be obtained by rotating a curve about a different axis. An example of this is the two types of hyperboloids: one created by rotating a hyperbola about the x-axis and one created rotating a hyperbola about the y-axis. See Figure 16 below for a hyperboloid of one sheet. See Figure 17 below for a hyperboloid of two sheets.

5.1 Arc Length

Arc length is defined as the length of the curve. Arc length is used when the distance needed is not a straight line, such as on the surfaces I will be crocheting. Figure 18 below is of a section of

Figure 15: Incorrect crocheted paraboloid

Figure 16: Hyperboloid of 1 Sheet

a circle (a curve) that has marked on it three arc lengths. Arc length is represented as s and is calculated by $s = \int_0^x \sqrt{1 + (y'(t))^2}dt$.

This formula can be found by looking at Figure 5.1 below: draw a right triangle as in the image, by the Pythagorean Theorem, $\partial s^2 = \partial x^2 + \partial y^2$. So then $\frac{\partial s}{\partial x} = \sqrt{1 + \frac{\partial y^2}{\partial x^2}}$. However, ∂s is only an estimation of the curve, it is not the exact measure of the curve. So, take infinitely small $\partial s, \partial x,$ and ∂y by taking the limit of the equation as ∂s (and thus ∂x and ∂y) go to 0. To find the length of the curve, add up infinitely many of these infinitesimally small triangles; this means to

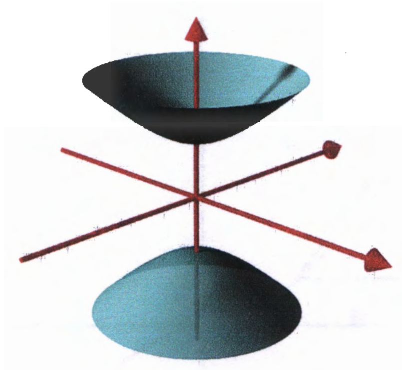


Figure 17: Hyperboloid of 2 Sheets

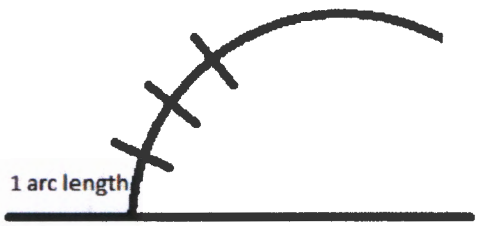


Figure 18: Example of arc length

find the length of the curve, take the integral. The result is $s = \int_0^x \sqrt{1 + (\frac{\partial y}{\partial x})^2}dt$.

5.2 The General Idea

The basic idea to formulate the patterns for surfaces of revolution is to use arc length to calculate x or y value that serves as the radius of the row. Then, calculate the change in circumference from one row to the next to find the increase or decrease of stitches. The following sections will describe this process in greater detail.

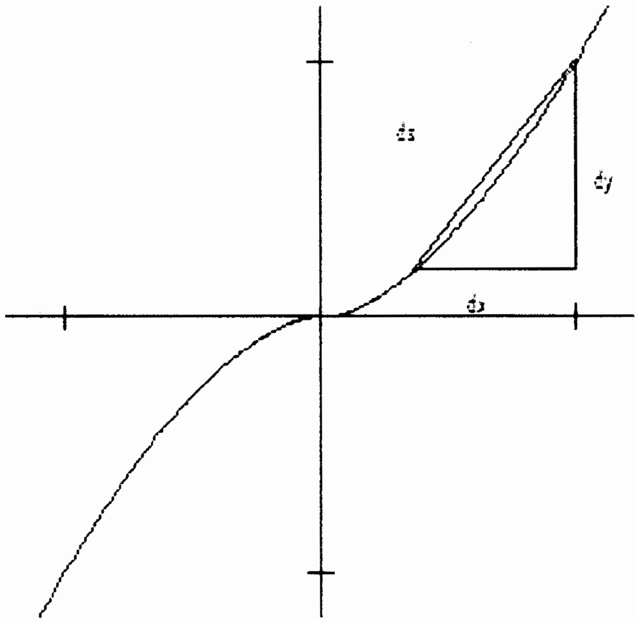


Figure 19: Right triangle used to derive arc length formula (6)

5.2.1 Using Wolfram Mathematica to Calculate Arc Length and x or y values

To calculate arc length using Wolfram Mathematica I started by inputting as $f(t)$ the "inside" of the integral in the arc length formula. This way $f(t) = \sqrt{1 + y'(t))^2}$ with $y'(t)^2$ for the specific curve I was dealing with. An example of the command in Wolfram Mathematica is `f[t_] :=` $\sqrt{1 + (4t^2)/b^4}$. This equation is used for a paraboloid.

I then defined the arc length formula as the integration of $f(t)$. This was best done by utilizing the integration abilities of Wolfram Mathematica. I defined this equation as $s(t)$. The command in Wolfram Mathematica to do this is `s[t_] := NIntegrate[f[u], u, 0, t]`.

I made sure to use the numeric integration command rather than the integration command so I could calculate the inverse and then specific values of the inverse later in the process. Otherwise, Wolfram Mathematica will solve the integral and the inverse symbolically.

Then I used the interpolation abilities of Wolfram Mathematica to interpolate `f(t)`, from above, from 0 to some value, usually 10, with small steps. The interpolation function is defined as `ss[x]`. The Wolfram Mathematica command is `ss = Interpolation[Table[x, s[x], x, 0, 10, 0.5]]`.

This interpolation helps with graphing the function, which is helpful in visualizing what is going on with the arc length. If the arc length is negative when graphed, there is a problem. I graphed each `ss[x]` to make sure the values for the function made sense with what I was expecting. The Wolfram Mathematica command is `Plot[ss[x], x, 0, 10]`.

The next step to using Wolfram Mathematica to calculate the x or y value of the arc length is to define the inverse function of `ss[x]`. Since `ss[x]` is the arc length, the inverse of this function will give the x or y value that produces that arc length. To define the inverse function in Wolfram Mathematica I used `inverse = InverseFunction[ss]`.

Then, it is possible to calculate the needed values for the different arc lengths. The way to do this is to have Wolfram Alpha calculate the inverse function with a specific value by inputting `inverse[2]` or `inverse[6]`.

It is possible to double check that this x value is correct by calculating `s(x)` with x being the value from the inverse function. This can be done by inputting `s[%]` or typing the number resulting from evaluating the inverse function where the `%` is: `s[1.97997]` for example.

The `%` command in Wolfram Mathematica means the result from the preceding input, not matter where is was on the page. Another way to think of this, is that the most recent output can be represented as `%`. That means if I were calculate `inverse[2]`, `inverse[3]`, and then input `s[%]`, the result should be 3 (or close to it). See the Appendix for an example of how accurate the `inverse[ ]` and `s[%]` calculations are for the hyperboloid.

The x or y value that has been calculated using Wolfram Mathematica is the radius that will be used to calculate the circumference of the circle of the rotation of the curve at that arc length.

5.2.2 Circumference of the Circle at x or y values

Since these surfaces are surfaces of revolution about an axis, they can be thought of as each point of the curve in two dimensions as making a circle about the axis of rotation. So, it is possible to compute the circumference of the circle at an x or y value. The x or y value that is being rotated will be the radius of the circle at that point.

The x or y value has been calculated above, using Wolfram Mathematica, so now it is possible to calculate the circumference at that point.

$C(x) = 2\pi x$

5.2.3 Calculating the Change in Circumference

The last step in developing the crochet pattern is to determine how many stitches to increase or decrease by from one row to the next. The best way to do this is to find the change in circumference.

$C(x) = 2\pi x$

$\frac{\partial C}{\partial x} = 2\pi \frac{\partial x}{\partial n}$

The change of arc length with respect to x is represented as $\frac{\partial n}{\partial x}$. This is necessary to calculate the change in circumference later.

$s = \int_0^x \sqrt{1 + (f'(t))^2 dt}$

$\frac{\partial n}{\partial x} = \sqrt{1 + (\frac{\partial y}{\partial x})^2}$

Since we calculated $\frac{\partial n}{\partial x}$ we can calculate $\frac{\partial x}{\partial n}$ by using the Inverse Function Theorem. The theorem states that if a function is continuously differentiable and non-zero derivative then the inverse of the derivative is $\frac{1}{\frac{\partial y}{\partial x}}$. So, by the Inverse Function Theorem it is possible to take the reciprocal to get $\frac{\partial n}{\partial x}$ and $\frac{\partial x}{\partial n} = \frac{1}{\sqrt{1+4x^2}}$. (1)

So, $\frac{\partial x}{\partial n} = \frac{1}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}$.

$\frac{\partial C}{\partial x} = 2\pi \frac{1}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}$

5.3 Paraboloid

A paraboloid is formed by rotating a parabola around an axis. An image of a paraboloid can be seen below in Figure 20. A parabola has the equation $y = ax^2 + bx + c$. The parabola I will be using $a = 1, b = 0$ and $c = 0$. So, this paraboloid is created by rotating the graph the parabola represented by $y = x^2$ about the y axis.

A paraboloid can also be made by rotating $y^2 = x$ about the x axis. $y^2 = x$ has the same shape as $y = x^2$ however, it is centered along the x axis rather than the y axis. Compare Figure 21 to

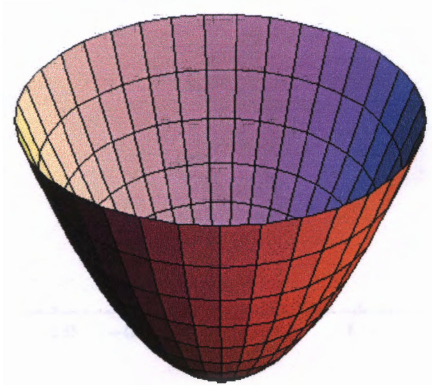


Figure 20: Paraboloid (14)

Figure 22 to see how the two parabolas have the same shape, just located in different sections of the x and y graph. Working with $y = x^2$ is enough since my crocheted surfaces do not depend on where the stitches fall on the graph as long as the shape of the surface is the same.

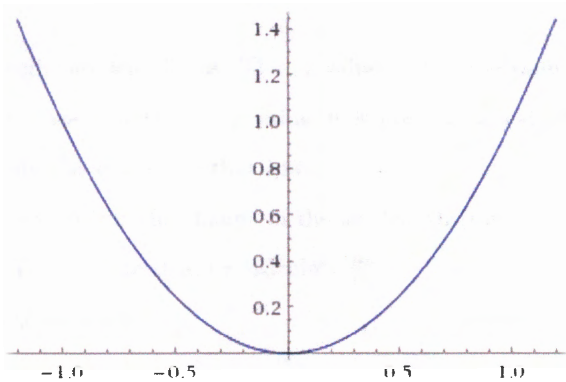


Figure 21: Graph of $y = x^2$

To determine the pattern for a paraboloid I will be using arc length as described above. Since $f(x) = x^2$, then the parameterization of the function is $y(t) = t^2$ and $y' = 2t$. Then, the arc length is $s = \int_0^x \sqrt{1 + (y'(t))^2} dt$

$s = \int_0^x \sqrt{1 + (2t)^2} dt$

$s = \int_0^x \sqrt{1 + 4t^2} dt$

Note that this will be used to calculate circumference as this equation, when solved for x, tells

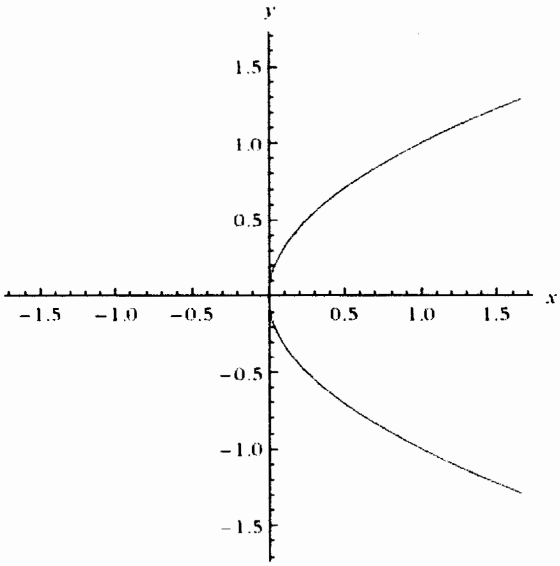


Figure 22: Graph of $y^2 = x$

us what x value the specific arc length has. That x value is then the radius of the circumference of the circle that forms that row. With that x value, it is possible to calculate the circumference of the row and so the number of stitches in that row.

What we wish to work with is the change in the arc length (represented as n) with respect to the change in x value. To calculate this, we calculate $\frac{\partial n}{\partial x}$.

$\frac{\partial n}{\partial x} = s'(t) = \frac{\partial}{\partial x}(\int_0^x \sqrt{1+4t^2}dt)$

By the Fundamental Theorem of Calculus, $\frac{\partial n}{\partial x} = \sqrt{1+4x^2}$.

Later I will use $\frac{\partial x}{\partial n}$ to calculate the change in circumference. So, by the Inverse Function Theorem it is possible to take the reciprocal to get $\frac{\partial n}{\partial x}$ and $\frac{\partial x}{\partial n} = \frac{1}{\sqrt{1+4x^2}}$.

In order to know how many stitches to increase from one row to another, it is necessary to calculate the change in circumference with respect to the row, represented as $\frac{\partial C}{\partial n}$.

Circumference is defined by $C = 2\pi r$ with r being the radius of the circle. Since the radius of the circle that is the row is equal to the x value found from arc length, substitute x in for r to get $C(x) = 2\pi x$.

Then, to find the number of stitches to increase or decrease by from one row to another, calculate $\frac{\partial C}{\partial n}$. In order to do this, the radius is equal to the change in radius with respect to the change in arc length or row.

$\frac{\partial C}{\partial n} = 2\pi\frac{\partial x}{\partial n}$

$\frac{\partial C}{\partial n} = 2\pi\frac{1}{\sqrt{1+4x^2}}$

The above formula to find the change in circumference depends on the x value of the arc length. To find the x value, I need to solve for x in the equation for arc length.

$s = \int_0^x \sqrt{1+(y'(t))^2}dt$

$s = \int_0^x \sqrt{1+4t^2}dt$

To see the calculations for the x values used for this pattern see the Appendix.

See Figure 23 below to see a table with all the values calculated. Note that Circumference and $\frac{\partial C}{\partial n}$ have been rounded to the nearest whole number since both numbers are an amount of stitches and stitches can only be whole numbers. See Figure 24 to see an image of a crocheted paraboloid from the pattern I generated.

row	arc length	x value of arc lenth	C	dC/dn
1	1	0.763927	5	3
2	2	1.2144	8	2
3	3	1.55405	10	2
4	4	1.83688	12	2
5	5	2.08401	13	1
6	6	2.3061	14	1
7	7	2.50942	16	1
8	8	2.69801	17	1
9	9	2.87463	18	1
10	10	3.0413	19	1

Figure 23: Paraboloid pattern

5.4 Ellipsoid

The ellipsoids I will be dealing with are created by rotating an ellipse about the x or y axis; this means that an ellipse on a 2 dimensional graph would be rotated out of the page, back into the page to end where it began while being centered about an axis. An ellipse is defined as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Figure 24: Crocheted paraboloid

with $a, b \in \mathbb{R}$.

The first step is to calculate $\frac{\partial y}{\partial x}$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2 - \frac{b^2x^2}{a^2}$

$y = \sqrt{^2 - \frac{b^2x^2}{a^2}}$

$\frac{\partial y}{\partial x} = \frac{-b^2}{a^2\sqrt{b^2 - \frac{b^2x^2}{a^2}}}$

The next step is to plug this into the arc length equation.

$s = \int_0^x \sqrt{1 + (\frac{-b^2}{a^2\sqrt{b^2 - \frac{b^2t^2}{a^2}}})^2 dt}$

$s = \int_0^x \sqrt{1 + \frac{b^4}{a^4(b^2 - \frac{b^2t^2}{a^2})}} dt$

$s = \int_0^x \sqrt{1 + \frac{b^2}{a^4 - a^2t^2}} dt$

Then, I use this equation to calculate the x value that gives each arc length, such as in the Appendix.

Then, in order to calculate $\frac{\partial C}{\partial n}$, I determined $\frac{\partial x}{\partial n}$.

$s = \int_0^x \sqrt{1 + \frac{b^2}{a^4 - a^2t^2}} dt$

$\frac{\partial n}{\partial x} = \sqrt{1 + \frac{b^2}{a^4 - a^2x^2}}$

$\frac{\partial x}{\partial n} = \frac{1}{sqrt1 + \frac{b^2}{a^4 - a^2x^2}}$

Then, it is possible to calculate $\frac{\partial C}{\partial n}$.

$C = 2\pi r$

$\frac{\partial C}{\partial n} = 2\pi \frac{1}{\sqrt{1 + \frac{b^2}{a^4 - a^2x^2}}}$

With this, it is possible to calculate the increase in stitches from one row to another. It is important to realize that the equation for determining the x value that gives a certain arc length is undefined for decreasing arc length. However, since the ellipse is symmetric about the y axis, the crochet pattern will be the same for the positive x values as the negative x values. So, the change in stitch numbers is repeated from the first half of the pattern in reverse.

5.4.1 Ellipsoid with $a = 6$ and $b = 4$

An ellipse with $a = 6$ and $b = 4$ would have the equation $\frac{x^2}{36} + \frac{y^2}{16} = 1$ and would appear as in Figure 25 below.

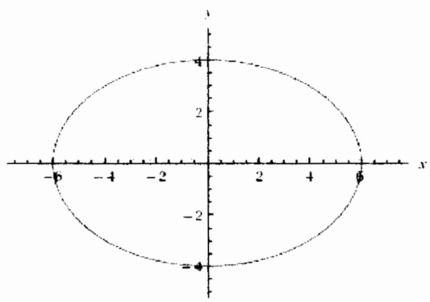


Figure 25: Ellipsoid with $a = 2$ and $b = 6$

In order to determine the formula for arc length, $\frac{\partial y}{\partial x}$ must be calculated.

$\frac{\partial y}{\partial x} = \frac{-b^2}{a^2\sqrt{b^2 - \frac{b^2x^2}{a^2}}}$

$\frac{\partial y}{\partial x} = \frac{-16}{36\sqrt{16 - \frac{16x^2}{36}}}$

Arc length can then be determined.

$s = \int_0^x \sqrt{1 + (\frac{-16}{36\sqrt{16 - \frac{16t^2}{36}}})^2 dt}$

$s = \int_0^x \sqrt{1 + \frac{16}{1296-36t^2}} dt$

Using Wolfram Mathematica I used the same sequence of ideas and formulas as described previously to determine the x values to given an arc length, see the Appendix.

In the Appendix, I was unable to use `inverse[6]` when trying to calculate the x value for arc length of 6. This is because Wolfram Mathematica was unable to calculate the value since the function the integral does not act in a way that Wolfram Mathematica can handle at 6. So I used the l function to estimate the most accurate x value to two decimal places.

The next step was to use those x values to calculate the circumference of the circle of rotation at that point, using the equation $C = 2\pi x$. Then, also using the x values. I was able to calculate $\frac{\partial C}{\partial n}$.

$\frac{\partial C}{\partial n} = 2\pi \frac{1}{\sqrt{1+\frac{b^2}{a^4-a^2x^2}}}$

$\frac{\partial C}{\partial n} = 2\pi \frac{1}{\sqrt{1+\frac{16}{1296-16x^2}}}$

Note that since the ellipse of $\frac{x^2}{36} + \frac{y^2}{16} = 1$ is symmetric about the y axis, the ellipsoid is also symmetric about the y axis. This means that when the x value is equal to a (or as close to it as is reasonable) the pattern is halfway done and the second half of the pattern decreases the same number of stitches each row as the corresponding increase row. In general, when a surface is symmetric about an axis and requires increasing and decreasing it is necessary to repeat this process. It was not necessary to do this for the hyperboloid of 2 sheets and paraboloid because there was no need to repeat the pattern while decreasing like for the sphere and the ellipsoid.

In this example when the arc length (or row number) is 6, the x value is 5.92. This is as close to 6 as is reasonable to expect judging from the way the x values increase just short of 1 with an increase of 1 arc length. So, row 7 of the pattern will decrease the same number as row 6 increased. The pattern repeats the increases in reverse order as decreases. This is because the ellipsoid pattern is the same whether the starting x value is positive or negative on the graph; the two halves of the ellipsoid increase the same amount when moving toward the center independent of the starting side. See Figure 26 below for the entire ellipsoid pattern with $a = 6$ and $b = 4$. See Figure 27 below for the crocheted ellipsoid from this pattern.

a	6		b	4	
n	x value from Mathematica	calculated y value	C	dC/dn	Total Stitches
1	0.99	3.945174267	6	6	6
2	1.98	3.775923728	12	6	12
3	2.97989	3.471807444	19	6	17
4	3.97063	2.998821798	25	5	23
5	4.9567	2.253996462	31	5	28
6	5.92	0.6510163	37	2	30
7			31	-2	28
8			25	-5	23
9			19	-5	17
10			12	-6	12
11			6	-6	6
12			0	-6	0

Figure 26: Pattern for ellipsoid with $a = 6$ and $b = 4$

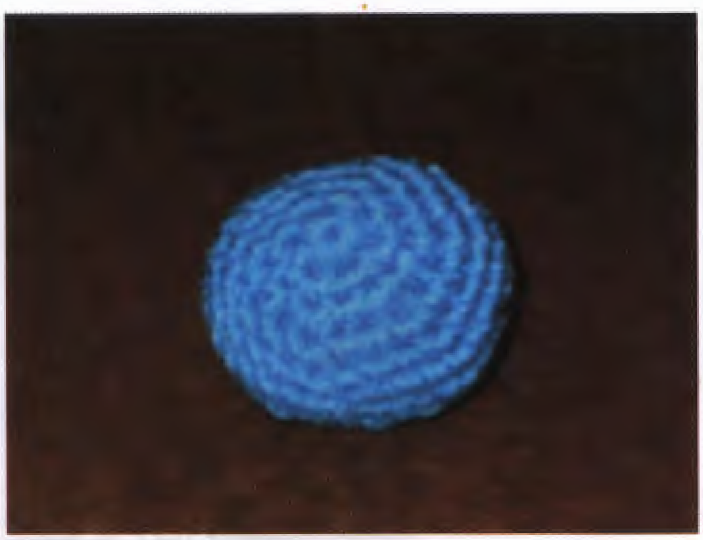


Figure 27: Crocheted ellipsoid with $a = 6$ and $b = 4$

5.5 Hyperboloid

The hyperboloid is created by rotating a hyperbola about the x or y axis; this means that a hyperbola on a 2 dimensional graph would be rotated out of the page, back into the page to end where it began while being centered about an axis. There are two different hyperboloids that can be obtained: a hyperboloid of one sheet (Figure 28) and a hyperboloid of two sheets (Figure 29). I will cover each in different subsections below.

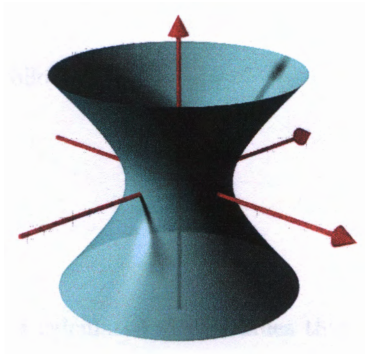


Figure 28: Hyperboloid of 1 Sheet

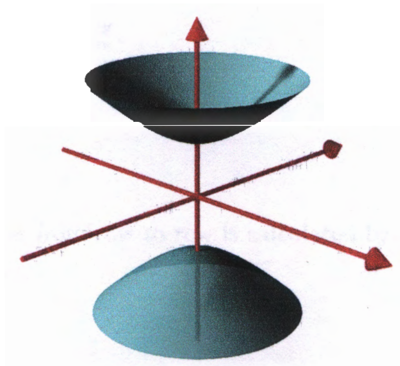


Figure 29: Hyperboloid of 2 Sheets

5.6 Hyperboloid of Two Sheets

The hyperboloid of two sheets is created by rotating a hyperboloid of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ about the y axis. I will be using $a = b = 1$. The same hyperboloid can be obtained from $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ rotated about the x value and if the x and y values are switched in the following calculations. Note that this hyperboloid is crocheted the same way, but does look different when graphed.

So, it is necessary to determine $\frac{dy}{dx}$ and to use Wolfram Mathematica to determine the x values of certain arc lengths.

$y^2 - x^2 = 1$

$y = \sqrt{1 + x^2}$

$\frac{\partial y}{\partial x} = \frac{x}{\sqrt{x^2+1}}$

So, then the arc length is as follows.

$s = \int_0^x \sqrt{1 + \frac{\partial y}{\partial x}} dt$

$s = \int_0^x \sqrt{1 + (f'(t))^2} dt$

$s = \int_0^x \sqrt{1 + \frac{t^2}{t^2-1}} dt$

$s = \int_0^x \sqrt{\frac{2t^2-1}{t^2-1}} dt$

Using Wolfram Mathematica I calculated the x values that give each consecutive arc length, this can be seen in the Appendix. Using that x value I was able to solve for the y value using the equation $y = \sqrt{1 + x^2}$. Then, I was able to calculate the circumference of the circles at those arc lengths. The next step is to calculate $\frac{\partial y}{\partial n}$.

$s = \int_0^x \sqrt{\frac{2t^2-1}{t^2-1}} dt$

$\frac{\partial n}{\partial y} = \sqrt{\frac{2y^2-1}{y^2-1}}$

So, $\frac{\partial y}{\partial n} = \sqrt{\frac{y^2-1}{2y^2-1}}$

Then, the increase in stitches from row to row is calculated by the following.

$C = 2\pi r$

$C(y) = 2\pi y$

$\frac{\partial C}{\partial n} = 2\pi \frac{\partial y}{\partial n}$

$\frac{\partial C}{\partial n} = 2\pi \sqrt{\frac{y^2-1}{2y^2-1}}$

For a complete look at the different calculations see Figure 30 below and for an image of the crocheted hyperboloid of two sheets from this pattern see Figure 31 below.

5.6.1 Hyperboloid of One Sheet

A hyperboloid of one sheet can be thought of as the inverse of the hyperboloid of two sheets: the hyperboloid of one sheet covers the parts of the graph that the hyperboloid of two sheets misses. Since the hyperboloid of two sheets is the rotation of $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ about the x axis, the hyperboloid of one sheet is the rotation of $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ about the y axis. In order to crochet a hyperboloid of

n	x or r	C	dC/dn	total Stitches
1	0.918028	6		6
2	1.70359	11	4	10
3	2.44735	15	4	14
4	3.17547	20	4	18
5	3.896166	24	4	23
6	4.61277	29	4	27
7	5.3269	33	4	31
8	6.03939	38	4	36
9	6.75077	42	4	40
10	7.46134	47	4	45

Figure 30: Excel pattern for hyperboloid of 2 sheets

Figure 31: Crocheted hyperboloid of 2 sheets

one sheet it is possible to use the x values already calculated from generating the pattern for the hyperboloid of two sheets to find the y values that will serve as the radius of the circles that make up the pattern.

I will once again use $a = b = 1$. Since $y^2 - x^2 = 1$ it is possible to solve for $y = \sqrt{x^2 - 1}$. So, using the x values previously calculated, calculate the corresponding y values. However, there are two x values that give a particular y value. This means that -6.03939 and 6.03939 will give the y value of 5.956025 and similarly for all other y values. This means that the pattern will increase until the graph reaches y axis.

For convenience, I will start with $x = -7.46134$ and continue to $x = 7.46134$. The graph continues on infinitely along the x axis, but I will only be focusing on the x values between -7.46134 and 7.46134. This means that the x values will increase throughout the pattern and $y = 1$ will be the middle of the pattern.

After calculating each y value from the known x values, it is possible to calculate the circumference for each circle. The circumference, when rounded, tells how many stitches there are total in that row. From this information it is possible to calculate the change in stitches from one row to another by subtracting the circumference of the previous row from the current circumference.

Figure 32 is the pattern produced by this process for the hyperboloid of $a = b = 1$. Figure 5.6.1 is an image of the crocheted hyperboloid of one sheet.

n	x or r	y value	C	change in circumference
1	-7.46134	7.394024	46	
2	-6.75077	6.676294	42	-5
3	-6.03939	5.956025	37	-5
4	-5.3269	5.232195	33	-5
5	-4.61277	4.503071	28	-5
6	-3.89617	3.765649	24	-5
7	-3.17547	3.013903	19	-5
8	-2.44735	2.233724	14	-5
9	-1.70359	1.37921	9	-5
10	0.918028	1	6	-2
11	1.70359	1.37921	9	2
12	2.44735	2.233724	14	5
13	3.17547	3.013903	19	5
14	3.896166	3.765649	24	5
15	4.61277	4.503071	28	5
16	5.3269	5.232195	33	5
17	6.03939	5.956025	37	5
18	6.75077	6.676294	42	5
19	7.46134	7.394024	46	5

Figure 32: Excel pattern for hyperboloid of 1 sheet

Figure 33: Crocheted hyperboloid of 1 sheet

5.7 Square Rotations

Up to this point I have been dealing with surfaces of rotation about an axis where the rotation has always been a circle. Now, I will discuss if the rotation has a square cross section. It is possible to crochet a square by doing three single crochets in the stitches that are the corners of the square.

5.8 Two Sheet Hyperboloid Square Rotation

In Figure 34 below the circle labeled "Regular Hyperboloid Cross Section" demonstrates how the radius is considered when the hyperbola is being rotated about a circle to form the hyperboloid. The square demonstrates how the radius is considered when the hyperbola is being rotated about a square. Note that there are 8 r's about the whole square.

To calculate the total number of stitches the first step is to note that the radius is the same as was calculated before using Wolfram Mathematica. Then, as seen in the above figure, multiply r by 8 to find the number of stitches before figuring the extra stitches for the corners of the square. To form the corners of the square, three single crochets are stitched in one stitch, this is the corner stitch. This results in an increase of two stitches for each corner for each row. Add 6 stitches to

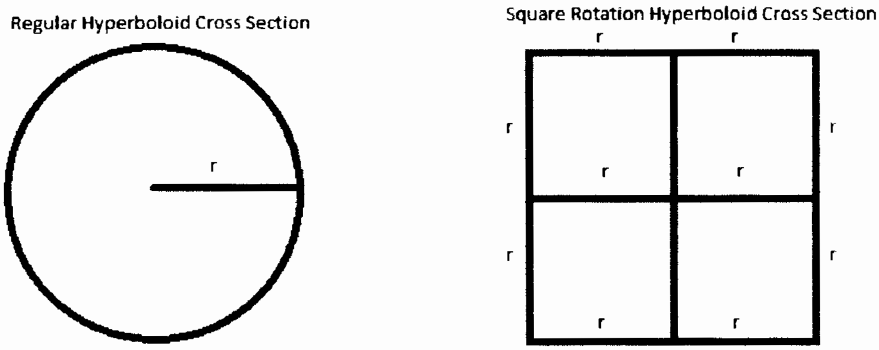


Figure 34: Square cross section

the previously calculated number of stitches from the radius. Lastly, it is possible to calculate the increase from one row to another. See Figure 35 below for the pattern of rotation a paraboloid about a square. See Figure 36 for an image of the side view of a hyperboloid of square rotation, note how the shape is the same as a hyperboloid of circular rotation. See Figure 37 for an image of a hyperboloid of square rotation from the top, note how there are 4 sides and 4 corners to form a square.

n	x or r	Stitches with out corners	Total Stitches	Increase
1	0.918028	7	15	
2	1.70359	14	22	6
3	2.44735	20	28	6
4	3.17547	25	33	6
5	3.896166	31	39	6
6	4.61277	37	45	6
7	5.3269	43	51	6
8	6.03939	48	56	6
9	6.75077	54	62	6
10	7.46134	60	68	6

Figure 35: Hyperboloid square rotation pattern

Figure 36: Side view of square rotation of hyperboloid

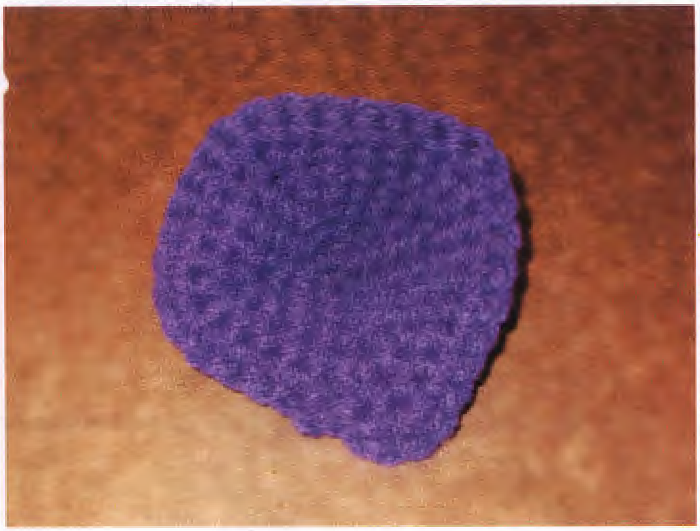


Figure 37: Top view of square rotation of hyperboloid

6 Conclusion

In conclusion, I have used certain mathematical principles (such as circumference, arc length, and surfaces of revolution), to develop crochet patterns for the Euclidean surface, the spherical surface, the parabolic surface, the hyperboloid surface, and the ellipsoid surface. These surfaces are surfaces of constant curvature or of revolution. I used the idea of crocheting in circles to use the circumference of that circle to determine the number of stitches in each row. Doing this research has increased the amount of research done into relating crochet and mathematics, specifically discrete geometry. By researching how to crochet mathematical surfaces I am increasing the ability for people, especially crocheters and crafters, to understand mathematics and learn more about geometry and surfaces.

References

Texts

[1] T. A. Garrity, *All the mathematica you missed: But need to know for graduate school*, Cambridge University Press, New York, 2001.

[2] A. Gray, *Modern differential geometry of curves and surfaces*, CRC Press, Inc., Boca Raton, FL 1993.

[3] D. W. Henderson & D. Taimina, *Experiencing geometry: Euclidean and non-Euclidean with History*, Pearson Prentice Hall, Upper Saddle, NJ, 2005.

[4] D. Taimina *Crocheting adventures with hyperbolic planes*, A K Peters, Ltd., Natick, MA, 2009.

[5] B. F. Torrence & E. A. Torrence, *The student's introduction to mathematica: A handbook for precalculus, calculus, and linear algebra second edition*, Cambridge University Press, Cambridge, 2009.

Images

[6] 48v, *File:ArclengthSegment.png*, 2006, Url: http://en.wikipedia.org/wiki/File:ArclengthSegment.png.

[7] Anonymous, *Ant Border Circle*, Url: http://www.wpclipart.com/page_frames/animal/ant/ant_border_circle.png.html.

[8] A. Gray, *Modern differential geometry of curves and surfaces*, CRC Press, Inc., Boca Raton, FL 1993.

[9] D. W. Henderson & D. Taimina, *Experiencing geometry: Euclidean and non-Euclidean with History*, Pearson Prentice Hall, Upper Saddle, NJ, 2005.

[10] R.J. Lisle, N. Toimil, J. Aller, N. Bobillo-Ares, F. Bastida, The hinge lines of non-cylindrical folds, *Journal of Structural Geology*, Volume 32, Issue 2, February 2010, Pages 166-171.

[11] OCAL, *Dotted Circle clip art*, 2007, Url: http://www.clker.com/clipart-7865.html.

[12] D. Taimina, *Crocheting adventures with hyperbolic planes*, A K Peters, Ltd., Natick, MA, 2009.

[13] Whittle, *Circumference of Circle on Curved (Sphere) Surface*, 2013. Url: http://www.astro.virginia.edu/class/whittle/astr553/Topic16/t16_circumfe rence.html.

[14] Zcrayfish, *Paraboloid*, July 2005, Url: http://commons.wikimedia.org/wiki /File:ParaboloidOfRevolution.png.

A Summary of Notations

θ Angle of arc length on a sphere

$\frac{\partial n}{\partial x}$ Derivative of arc length with respect to x

$\frac{\partial x}{\partial n}$ Derivative of x with respect to row number

$\frac{\partial C}{\partial n}$ Derivative of circumference with respect to row number

F_x Frst derivative of F(x,y) with respect to x

F_y First derivative of F(x,y) with respect to y

F_{xx} Second derivative of F(x,y) with respect to x

F_{yy} Second derivative of F(x,y) with respect to y

a Real number that is used to form an ellipse, denominator of $\frac{x^2}{a^2}$

b Real number that is used to form an ellipse, denominator of $\frac{x^2}{b^2}$

C Circumference of circle

k_1 Principal curvature

k_2 Principal curvature

l Stitch size

n Row number, equal to arc length

R Radius of sphere

r Radius of circle

T Total number of stitches

s Arc length

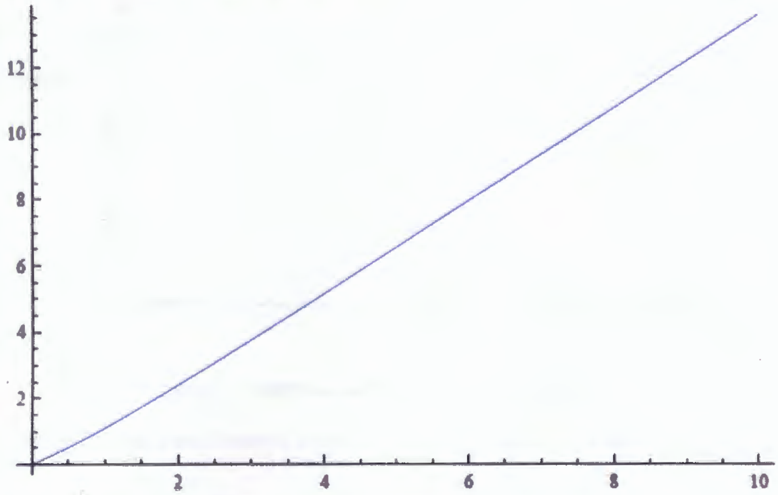
w Representation for $R^2 + x^2 + y^2$

B Wolfram Mathematica Code

Mathematica code to show the accuaracy of computations and use of % in reference to previous inputs.

$$f[t\_] := \sqrt{\frac{2\,t^2+1}{t^2+1}}$$

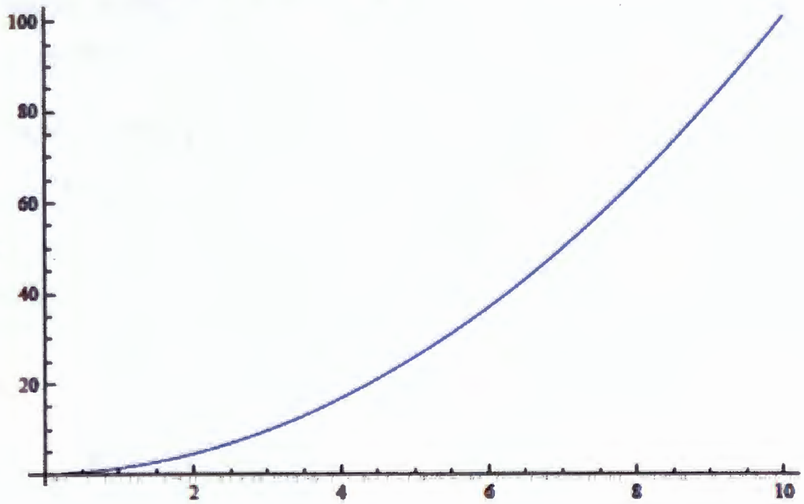
```
s[t_] := NIntegrate[ f[u], {u, 0, t}]

ss = Interpolation[Table[{x, s[x]}, {x, 0, 10, 0.1}]]

InterpolatingFunction[{{0., 10.}}, <>]

Plot[ss[x], {x, 0, 10}]
```




```
inverse = InverseFunction[ss]

InverseFunction[InterpolatingFunction[{{0., 10.}}, <>]]

inverse[1]

0.918028

s[0.918028]

1.

inverse[2]

1.70359

s[%]

2.
```

Mathematica code for paraboloid.

```
In[1]:= f[t_] := Sqrt[4 t^2 + 1]

In[2]:= s[t_] := NIntegrate[ f[u], {u, 0, t}]

In[3]:= ss = Interpolation[Table[{x, s[x]}, {x, 0, 10, 0.1}]]

Out[3]= InterpolatingFunction[{{0., 10.}}, <>]

In[4]:= Plot[ss[x], {x, 0, 10}]

Out[4]=
```




```
In[5]:= inverse = InverseFunction[ss]

Out[5]= InverseFunction[InterpolatingFunction[{{0., 10.}}, <>]]

In[6]:= inverse[1]

Out[6]= 0.763924

In[7]:= inverse[2]

Out[7]= 1.2144

In[8]:= inverse[3]

Out[8]= 1.55405

In[9]:= inverse[4]

Out[9]= 1.83688

In[10]:= inverse[5]

Out[10]= 2.08401
```

```
In[11]:= inverse[6]
Out[11]= 2.3061

In[12]:= inverse[7]
Out[12]= 2.50942

In[13]:= inverse[8]
Out[13]= 2.69801

In[14]:= inverse[9]
Out[14]= 2.87463

In[15]:= inverse[10]
Out[15]= 3.0413
```

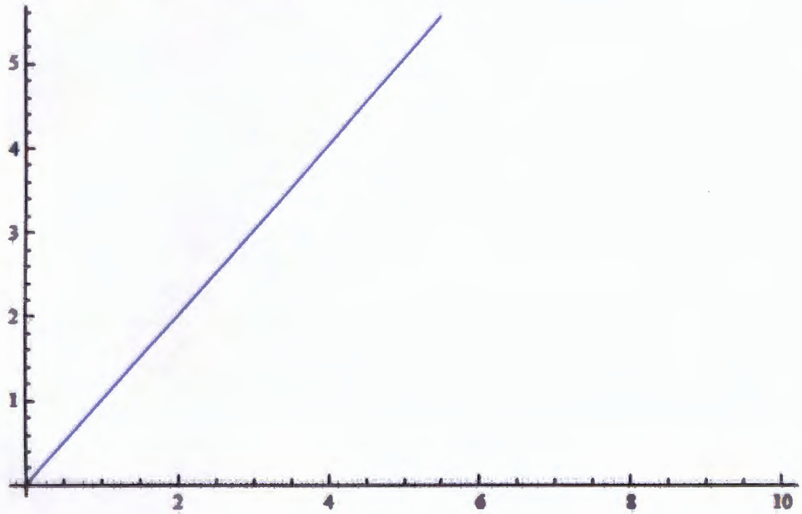
Mathematica code for ellipsoid calculations.

```
In[1]:= a = 6
Out[1]= 6

In[2]:= b = 4
Out[2]= 4
```

In[3]:= $f[t\_] := \sqrt{1 + \frac{b^2}{a^4 - a^2 t^2}}$

```
In[4]:= s[t_] := NIntegrate[ f[u], {u, 0, t}]

In[5]:= ss = Interpolation[Table[{x, s[x]}, {x, 0, 10, 0.5}]]
Out[5]= InterpolatingFunction[{{0., 10.}}, <>]

In[6]:= Plot[ss[x], {x, 0, 10}]
```

Out[6]=




```
In[7]:= inverse = InverseFunction[ss]
Out[7]= InverseFunction[InterpolatingFunction[{{0., 10.}}, <>]]
```

```
In[8]:= inverse[1]
Out[8]= 0.993827

In[9]:= inverse[2]
Out[9]= 1.98729

In[10]:= inverse[3]
Out[10]= 2.97989

In[11]:= inverse[4]
Out[11]= 3.97063

In[12]:= inverse[5]
Out[12]= 4.9567

In[14]:= s[5.92]
Out[14]= 6.01061
```

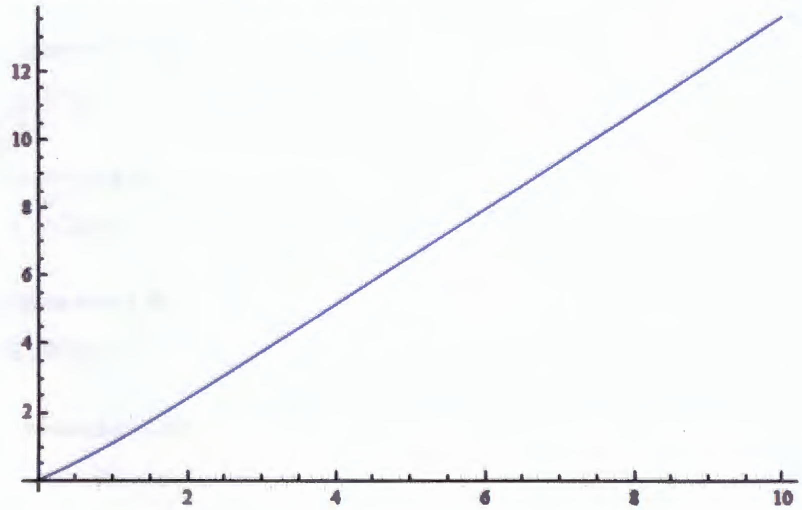
Mathematica code for hyperboloid of 2 sheets calculations.

In[15]:=
```
f[t_] := Sqrt[(2 t^2 + 1)/(t^2 + 1)]
```

$$f[t\_] := \sqrt{\frac{2 t^2 + 1}{t^2 + 1}}$$

```
In[16]:= s[t_] := NIntegrate[ f[u], {u, 0, t}]

In[17]:= ss = Interpolation[Table[{x, s[x]}, {x, 0, 10, 0.1}]]

Out[17]= InterpolatingFunction[{{0., 10.}}, <>]

In[18]:= Plot[ss[x], {x, 0, 10}]
```

Out[18]=




```
In[19]:= inverse = InverseFunction[ss]

Out[19]= InverseFunction[InterpolatingFunction[{{0., 10.}}, <>]]

In[20]:= inverse[1]

Out[20]= 0.918028

In[21]:= inverse[2]

Out[21]= 1.70359
```

```
In[22]:= inverse[3]
Out[22]= 2.44735

In[23]:= inverse[4]
Out[23]= 3.17547

In[24]:= inverse[5]
Out[24]= 3.89617

In[25]:= inverse[6]
Out[25]= 4.61277

In[26]:= inverse[7]
Out[26]= 5.3269

In[27]:= inverse[8]
Out[27]= 6.03939

In[28]:= inverse[9]
Out[28]= 6.75077

In[29]:= inverse[10]
Out[29]= 7.46134

In[30]:= inverse[11]
Out[30]= 8.17132
```