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# Properties of left-separated spaces and their unions

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PROPERTIES OF LEFT-SEPARATED SPACES AND THEIR UNIONS

An Abstract of a Thesis  
Submitted  
in Partial Fulfillment  
of the Requirement for the Degree  
Master of Arts

Eric Scheidecker  
University of Northern Iowa  
December 2017

## ABSTRACT

Left-separated spaces are topological spaces which can be well ordered such that every initial segment is closed. In this paper, we examine what topological properties imply left-separation, and under what circumstances left-separation is preserved by unions. We also introduce several known theorems regarding elementary submodels as they are one of the primary tools that we use. We prove that for a topological space  $X$ ;

1. If  $X$  has a point-countable base, then  $X$  is left-separated if and only if  $X$  has closed intersection with any elementary submodel  $M$  such that  $X \in M$ .
2. If every elementary submodel  $M$  with  $X \in M$  and  $|M| < \lambda$  has closed intersection with  $X$ , then  $X$  has a left-separated subspace of size  $\lambda$  whose initial segments are closed in  $X$ .
3. If  $X$  is locally countable and metalindelöf, then  $X$  is left-separated.
4. If  $X$  is neat with  $|X| = \kappa^+$  such that  $X$  is left-separated in order type  $\kappa^+ \cdot \alpha$  with  $\alpha < \kappa$ , then  $d(X) < \kappa^+$ , or  $X$  is the union of less than  $\kappa^+$  many nowhere dense sets.
5. If  $X$  is left-separated in order type  $\kappa$  and  $Y$  is a topological space that is left-separated in order type  $\omega_1$  such that  $X \cup Y$  is locally countable, then  $X \cup Y$  is left-separated in order type less than or equal to  $\kappa \cdot 2$ .

We finish with several open questions that outline the general direction of our future work.

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This Study by: Eric Scheidecker

Entitled: PROPERTIES OF LEFT-SEPARATED SPACES AND THEIR UNIONS

Has been approved as meeting the thesis requirement for the  
Degree of Master of Arts.

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Date

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Dr. Patrick Pease, Interim Dean, Graduate College

*To Clark, Devin, Kaiyla, Shane, and Sienna.*

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## CHAPTER 1

### INTRODUCTION

In 1967, András Hajnal and István Juhász [1] introduced the concepts of left and right-separation. Informally stated, a topological space  $X$  is left (right) separated if it can be well ordered such that all initial segments in the well ordering are closed (open). The focus of this paper is primarily on left-separated spaces.

In general, it is non-trivial to prove that a space is or is not left-separated as we either must find a well ordering that witnesses left-separation or prove that under any well ordering, there is an initial segment that is not closed. It is known that the real numbers under the Euclidean topology are not left-separated, but proving this requires leaning hard on countable density and other very strong properties of the real numbers that we do not usually have for general topological spaces. While the real numbers are rarely the least complicated non-trivial example, they do provide an interesting problem to tackle as we develop tools.

We begin in Chapter 2 by defining our fundamental concepts such as left-separation and elementary submodels. We also prove results about elementary submodels that provide us with a foundation for our proofs and also illustrate the techniques we use in the following chapters.

With elementary submodels established as useful tools, we move on to Chapter 3 where we look at what topological properties imply that a space is left-separated, while in Chapter 4 we examine unions of left-separated spaces and what properties preserve left-separation across unions.

In Chapter 5 we pose several of our open questions and outline the direction of future work in this area.

Whenever possible we use standard notation and terminology. For a standard reference, we refer the reader to [7] for topology and [2] for set theory. All spaces in this paper are assumed to be Hausdorff.

CHAPTER 2  
BASIC DEFINITIONS AND ELEMENTARY SUBMODELS

One of the primary tools in our exploration of left-separated spaces are elementary submodels. The next several definitions lay the groundwork for understanding the logic behind models in general.

**Definition 2.1.** *A sentence in first order logic is a formula with no free variables (That is, every variable must be associated with a quantifier) that can be defined solely in the language of first order logic.*

**Definition 2.2.** *A first order theory is a set of axioms in first order logic. A theory can also be used to denote a mathematical universe; a set of axioms together with every theorem that follows from these axioms.*

**Definition 2.3.** *A set  $M$  is a model of a first order theory  $\mathcal{T}$  if for any  $\Phi \in \mathcal{T}$ ,  $\Phi$  is true in  $M$ . If  $N \subset M$  is a model of  $\mathcal{T}$ , we call  $N$  a submodel of  $M$ .*

**Definition 2.4.** *A submodel  $M$  of  $\mathcal{T}$  is elementary if  $\Phi \in \mathcal{T}$  if and only if  $\Phi$  is true in  $M$ .*

Implicit in this definition of elementary submodels is the stipulation that  $\Phi$  is true in  $M$  when we restrict the quantifiers of  $\Phi$  to  $M$ . In the background,  $M$  is built up recursively to (if necessary) add in a witness to match the truth value of each statement that can be made using only the elements of  $M$ . To illustrate, consider a countable elementary submodel  $M$ . Since  $M$  is countable, it can not contain  $\omega_1$  as a subset, so there is some  $\delta \in \omega_1$  such that  $M \cap \omega_1 = \delta$ . That is,

$\delta \subset M$  but  $\delta \notin M$ . As far as  $M$  can see,  $\delta$  is uncountable since  $\delta \notin M$  means that we never added a witness to the statement that there is a function from  $\omega$  onto  $\delta$ . Thankfully, even though  $M$  does not have enough information to be right about the cardinality of  $\delta$ , it is still right about everything it can actually see. An unintuitive consequence of this is that if we have  $A \in M$  and  $\delta \in A$ , then  $A$  is actually uncountable. This can be seen as an immediate result of Corollary 2.6 below.

Everything we concern ourselves with in this paper is set within the mathematical universe of the ZFC axioms, so when we say  $M$  is an elementary submodel, we are really saying that  $M$  is an elementary submodel of a model of a "big enough" piece of the universe of ZFC. By "big enough" we mean big enough to model everything that is true about the set that we are interested in. The proof of the existence of such an elementary submodel is given by a generalized version of the Downward Löwenheim-Skolem theorem which originally proved that if a first order theory has an infinite model, then there must exist a countable model of the theory.

The essential idea that we use is that for any topological space  $X$  and any cardinal  $\kappa$ , we can find an elementary submodel  $M$  such that  $|M| = \kappa$  and any first order logical statement  $\Phi$  is true about  $X$  if and only if  $\Phi$  is true in  $M$ . The following theorem highlights a useful property of elementary submodels.

**Theorem 2.5.** *Let  $M$  be an elementary submodel and  $\kappa$  be a cardinal such that  $\kappa \in M$  and  $\kappa \subset M$ . Then for all  $A \in M$  such that  $|A| \leq \kappa$ , we have that  $A \subset M$ .*

Proof.

Let  $M$  be an elementary submodel and let  $A$  and  $\kappa$  be as in the hypothesis. Since  $|A| \leq \kappa$ , there exists a function from  $\kappa$  onto  $A$ . Since  $M$  is elementary, a function with the same properties must exist in  $M$ . Thus we let  $f \in M$  such that

$f : \kappa \xrightarrow{\text{onto}} A$ . Let  $a \in A$  and  $\alpha \in \kappa$  such that  $f(\alpha) = a$ . Since  $\kappa \subset M$ ,  $\alpha \in M$  and so  $f(\alpha) = a \in M$ . Thus  $A \subset M$ .

□

One consequence of working within the universe of ZFC is that every elementary submodel must contain  $\omega$  as an element and a subset; this is because every elementary submodel must model the axioms of ZFC which establish the ordinals up to  $\omega$ .

**Corollary 2.6.** *Let  $M$  be a countable elementary submodel. Then for all  $A \in M$  such that  $|A| \leq \omega$ , we have that  $A \subset M$ .*

Now that we have established the basics of elementary submodels, we can begin exploring their uses in relation to left-separated spaces. First, we give a formal definition of left-separation.

**Definition 2.7.** *A space  $X$  is left-separated in order type  $\kappa$  if there exists a well ordering  $X = \{x_\alpha | \alpha < \kappa\}$  such that any initial segment  $\{x_\beta | \beta < \alpha\}$  in this ordering is a closed subset of  $X$ . If  $\kappa$  is the least such ordinal witnessing the left-separation of  $X$ , we say that  $\kappa$  is the left-separation order type of  $X$ , denoted by  $\text{ord}_\ell(X) = \kappa$*

The definition of right-separated spaces can be obtained by replacing left with right and closed with open in the preceding definition.

The following theorem is a known result with the proof below given by L. Soukup.

**Theorem 2.8.** *Let  $X$  be a topological space. Suppose that for each elementary submodel  $M$  such that  $X \in M$ ,  $X \cap M$  is a closed subset of  $X$ . Then  $X$  is left-separated. That is,*

$$\left[ \forall M \left( X \in M \rightarrow X \cap M \stackrel{\text{closed}}{\subset} X \right) \right] \Rightarrow (X \text{ is left-separated})$$

We note that the proof of this theorem requires the use of a Davies-tree. Ideally, we would cover  $X$  with an increasing continuous chain of elementary submodels, so that the union at each limit step would be an elementary submodel. We want to use countable elementary submodels to build an ordering on  $X$ , but nothing larger than  $\omega_1$  can be covered by an increasing continuous chain of countable elementary submodels. We bridge this gap with a Davies-tree. The general idea behind Davies-trees is that we can cover any arbitrarily large space with an increasing continuous chain of elementary submodels of strictly smaller size, and then cover each elementary submodel in that chain with an increasing continuous chain of elementary submodels of strictly smaller size, and so on and so forth until we have a tree of increasing continuous chains of countable elementary submodels that we can use to cover initial segments of  $X$  with only finitely many elementary submodels. For a further explanation of Davies-trees and their uses, we direct the reader to a paper written by Dániel and Lajos Soukup [4].

Proof.

Let  $X$  be a space. Suppose that for every elementary submodel  $M$ , we have that if  $X \in M$ , then  $X \cap M$  is a closed subspace of  $X$ . Let  $(M_\alpha)_{\alpha < \kappa}$  be a family of countable elementary submodels and for each  $\alpha < \kappa$ , let  $(N_{\alpha_i})_{i < n_\alpha}$  be a finite collection of elementary submodels where  $n_\alpha \in \omega$ , so that

1. for each  $\alpha < \kappa$ ,  $X \in M_\alpha$
2.  $X \subset \bigcup_{\alpha < \kappa} M_\alpha$
3. for each  $\alpha < \kappa$ ,  $\bigcup_{\beta < \alpha} M_\beta = \bigcup_{i \in n_\alpha} N_{\alpha_i}$ .

We proceed by recursion on  $\alpha < \kappa$ . Fix  $\alpha < \kappa$ . Suppose  $X \cap M_\gamma$  has been well ordered for each  $\gamma < \alpha$  such that all initial segments of  $X \cap \bigcup_{\gamma < \alpha} M_\gamma$  are

closed. We first notice that

$$X \cap \bigcup_{\gamma < \alpha} M_\gamma = X \cap \bigcup_{i \in n_\alpha} N_{\alpha_i} = \bigcup_{i \in n_\alpha} (X \cap N_{\alpha_i}).$$

As this is the finite union of closed sets we have that  $X \cap \bigcup_{\gamma < \alpha} M_\gamma$  is closed. The new points to be considered are

$$(X \cap \bigcup_{\gamma \leq \alpha} M_\gamma) \setminus (\bigcup_{\gamma < \alpha} M_\gamma).$$

Notice that this is a countable set and can be written as an  $\omega$ -sequence. We then append this to the end of the previously ordered points. We only need to show that all initial segments of  $X \cap \bigcup_{\gamma < \alpha} M_\gamma$  are closed with this order. As we have appended an  $\omega$ -sequence, we need only show that

$$X \cap \bigcup_{\gamma < \alpha} M_\gamma$$

is closed.

Let  $\alpha < \kappa$ . As before, notice that

$$X \cap \bigcup_{\gamma < \alpha} M_\gamma = X \cap \bigcup_{i \in n_\alpha} N_{\alpha_i} = \bigcup_{i \in n_\alpha} (X \cap N_{\alpha_i}).$$

Further, this is a finite union of closed sets by our hypothesis and thus

$$X \cap \bigcup_{\gamma < \alpha} M_\gamma$$

is closed. Therefore,  $X$  is left-separated.

□

For general left-separated spaces, the converse of Theorem 2.8 does not necessarily hold. However, the converse is true for left-separated spaces with point-countable bases.

**Definition 2.9.** *A space  $X$  has a point- $\kappa$  base if there is a base  $\mathcal{B}$  of  $X$  so that for every  $x \in X$ ,  $|\{B \in \mathcal{B} | x \in B\}| = \kappa$ . When  $\kappa = \omega$ ,  $X$  is said to have a point-countable base.*

The following theorem is due to Scheidecker and Stanley [3].

**Theorem 2.10.** *Let  $X$  be a topological space with a point-countable base. If  $X$  is left-separated, then for every countable elementary submodel  $M$  such that  $X \in M$  we have that  $X \cap M$  is a closed subset of  $X$ . That is, whenever  $X$  has a point-countable base, we have*

$$\left[ \forall M \left( X \in M \rightarrow X \cap M \stackrel{\text{closed}}{\subset} X \right) \right] \iff (X \text{ is left-separated})$$

Proof.

Let  $X$  be a left-separated topological space with a point-countable base. Let  $M$  be a countable elementary submodel such that  $X \in M$ . Let  $\text{ord}_\ell(X) = \kappa \in M$  and let  $f : \kappa \rightarrow X$  witness that  $X$  is left-separated with  $f \in M$ . Let  $x \in \overline{X \cap M}$  and let  $\mathcal{B} \in M$  be a point-countable base for  $X$ . Let  $\alpha \in \kappa$  such that  $f(\alpha) = x$ . Notice that  $x \in f([\alpha, \kappa])$  is an open subset of  $X$  since  $f$  witnesses the left-separation of  $X$  and thus maps cofinal segments to open sets. Let  $\hat{B} \in \mathcal{B}$  such that  $x \in \hat{B} \subset f([\alpha, \kappa])$ . Since  $x \in \overline{X \cap M}$ ,  $\hat{B} \cap (X \cap M)$  is non-empty. Let  $y \in \hat{B} \cap (X \cap M)$ . Since  $y \in M$  we have  $\{B \in \mathcal{B} | y \in B\} \in M$ , and since  $|\{B \in \mathcal{B} | y \in B\}| = \omega$ , by Corollary 2.6 we have that  $\{B \in \mathcal{B} | y \in B\} \subset M$ . Thus  $\hat{B} \in M$ , so we have  $x = f(\min f^{-1}(\hat{B})) \in M$ . Thus  $X \cap M$  is closed in  $X$ .



□

This proof works for spaces with point-countable bases as every elementary submodel contains  $\omega$  as an element and a subset. Unfortunately, this is not guaranteed for larger cardinal numbers. In the simplest case, if  $\kappa = \omega_1$ , then any countable elementary submodel can not contain  $\omega_1$  as a subset. Further complicating things is that restricting ourselves to elementary submodels that are at least size  $\kappa$  is not enough to guarantee that  $\kappa$  is a subset of  $M$ , so Theorem 2.5 is not applicable and we can not guarantee that a point- $\kappa$  base  $\mathcal{B} \in M$  is also a subset of  $M$ . The same proof that we used for spaces with point-countable bases works for spaces with point- $\kappa$  bases if we add additional conditions. Scheidecker and Stanley [3] provide the following generalization.

**Theorem 2.11.** *Let  $X$  be a topological space such that  $|X| > \kappa$  and  $X$  has a point- $\kappa$  base. If  $X$  is left-separated, then for every elementary submodel  $M$  such that  $X, \kappa \in M$  and  $\kappa \subset M$ , we have that  $X \cap M$  is a closed subset of  $X$ . That is, whenever  $X$  has a point- $\kappa$  base, we have*

$$(X \text{ is left-separated}) \Rightarrow \left[ \forall M \left( (X, \kappa \in M \wedge \kappa \subset M) \rightarrow X \cap M \overset{\text{closed}}{\subset} X \right) \right]$$

It is unknown if Theorem 2.10 can be fully generalized to be a biconditional statement in the case of  $\kappa > \omega$ .

We sometimes find it useful to look at how large left-separated subspaces can be in a space that is not necessarily left-separated.

**Definition 2.12.** *Let  $X$  be a topological space and let*

$LS(X) = \sup\{|Y| : Y \subset X \text{ such that } Y \text{ is left-separated}\}$ . *We call  $LS(X)$  the left-separation degree of  $X$ .*

Clearly if  $X$  is left-separated,  $LS(X) = |X|$ . The following theorem is a known result and it provides us with a lower bound on the left-separation degree for any Hausdorff space.

**Theorem 2.13.** *Let  $X$  be a countable Hausdorff topological space. Then  $X$  is left-separated.*

Proof.

Let  $X$  be as above. Since  $X$  is countable, we may order it as an  $\omega$ -sequence. Let  $x \in X$  and  $\mathcal{F} = \{\overline{U} \mid U \in \mathcal{N}_x\}$ . Since  $X$  is Hausdorff, distinct points can be separated by open sets, so let  $y \in X$ ,  $U_y \in \mathcal{N}_x$ , and  $V_y \in \mathcal{N}_y$  such that  $U_y \cap V_y = \emptyset$ . Then  $U_y \subset \overline{U}_y \subset V_y^c$ , so  $y \notin \overline{U}_y$ . Notice  $\bigcap \mathcal{F} = \{x\}$  is closed since it is the intersection of closed sets. Thus singleton sets are closed in a Hausdorff space, and as  $X$  is ordered as an  $\omega$  sequence, every initial segment is the finite union of closed sets, and so every initial segment is closed and  $X$  is left-separated.

□

Then the degree of left-separation is at least  $\omega$  for any infinite Hausdorff space.

We may obtain a better lower bound for the degree of left-separation for some spaces with the following theorem which follows from considering when Theorem 2.8 does not hold for every elementary submodel, but holds for elementary submodels up to a certain size.

The following theorem was proven by Stanley [5] for  $\lambda = \omega_1$ , I prove the generalization below.

**Theorem 2.14.** *Let  $X$  be a topological space and  $\lambda$  be a cardinal such that  $\lambda \leq \kappa = |X|$  and for every elementary submodel  $M$  such that  $X \in M$  and  $|M| < \lambda$ , we have that  $X \cap M$  is a closed subset of  $X$ . Then there exists a set  $X_\lambda \in [X]^\lambda$  such*

that  $X_\lambda$  is left-separated in order type  $\lambda$ . Further, the initial segments  $X_\lambda$  are closed in  $X$ .

Proof.

Let  $X$  and  $\lambda$  be as in the hypothesis. As in Theorem 2.8, let  $(M_\alpha)_{\alpha < \kappa}$  be a family of countable elementary submodels where for each  $\alpha < \kappa$  we have a finite collection  $(N_{\alpha_i})_{i < n_\alpha}$  with  $n_\alpha \in \omega$  such that

1.  $X \in M_\alpha$
2.  $|X \cap M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta| = \omega$
3.  $\bigcup_{\beta < \alpha} M_\beta = \bigcup_{i \in n_\alpha} N_{\alpha_i}$

For each  $\alpha < \lambda$ , let  $X_\alpha = X \cap M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta$  where  $X_0 = X \cap M_0$ . Let  $X_\alpha = \{x(\alpha, n) \mid n \in \omega\}$  be a well ordering of  $X_\alpha$  in order type  $\omega$ . Let  $<_\ell$  be a well ordering of  $X$  such that for any  $x(\alpha, n) \in X_\alpha$  and  $x(\beta, m) \in X_\beta$ , we have  $x(\alpha, n) <_\ell x(\beta, m)$  if and only if  $\alpha < \beta$  or  $\alpha = \beta$  and  $n < m$ . That is,

$$x(\alpha, n) <_\ell x(\beta, m) \iff [\alpha < \beta \vee (\alpha = \beta \wedge n < m)].$$

This is the lexicographical ordering on  $\bigcup_{\alpha < \lambda} X_\alpha$ . Note that this orders  $\bigcup_{\alpha < \lambda} X_\alpha$  in order type  $\lambda$ . We show that all initial segments of  $\bigcup_{\alpha < \lambda} X_\alpha$  in this ordering are closed in  $X$ .

**Claim.**  $\bigcup_{\beta < \alpha} X_\beta$  is closed in  $X$  for each  $\alpha < \lambda$ .

Proof.

We proceed by transfinite induction on  $\alpha < \lambda$ . In the base case,  $X_0 = X \cap M_0$  which is closed by hypothesis as  $|M_0| = \omega < \lambda$ . We handle the limit and successor cases simultaneously.

Fix  $\alpha < \kappa$ . Then

$$\bigcup_{\beta < \alpha} X_\beta = \bigcup_{\beta < \alpha} (X \cap M_\beta \setminus \bigcup_{\delta < \beta} M_\delta) = X \cap \bigcup_{\beta < \alpha} M_\beta.$$

Let  $(N_{\alpha_i})_{i < n_\alpha}$  with  $n_\alpha \in \omega$  such that  $\bigcup_{\beta < \alpha} M_\beta = \bigcup_{i \in n_\alpha} N_{\alpha_i}$ . Then

$$X \cap \bigcup_{\beta < \alpha} M_\beta = X \cap \bigcup_{i \in n_\alpha} N_{\alpha_i} = \bigcup_{i \in n_\alpha} (X \cap N_{\alpha_i}).$$

Since each  $|N_{\alpha_i}| < \alpha < \lambda$ , we have  $X \cap N_{\alpha_i}$  is closed by hypothesis for every  $i \in n_\alpha$ .

Then

$$\bigcup_{\beta < \alpha} X_\beta = \bigcup_{i \in n_\alpha} (X \cap N_{\alpha_i})$$

is the finite union of closed sets and thus closed. This proves our claim. □

Let  $X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha$ . By the previous claim,  $X_\lambda$  is left-separated. For each  $\alpha < \lambda$ , we have  $|X_\alpha| = \omega$  by step two in our construction, and so

$|X_\lambda| = |\bigcup_{\alpha < \lambda} X_\alpha| = \omega \cdot \lambda = \lambda$ ; and since the left-separation of  $X_\lambda$  is witnessed by  $<_\ell$ , the order type of  $X_\lambda$  is  $\lambda$ . Therefore we have a subspace  $X_\lambda \in [X]^\lambda$  that is left-separated in order type  $\lambda$  such that initial segments of  $X_\lambda$  are closed in  $X$ . □

Recall that  $X_\lambda$  being left-separated in order type  $\lambda$  does not mean that  $\lambda$  is the smallest order type witnessing the left-separation of  $X_\lambda$ .

While Theorem 2.14 gives us a left-separated subset of a specific size, we are unsure if a stronger conclusion can be drawn. Can we pick the countable elementary submodels differently to get a different  $X_\lambda$ ? If so, how many other  $\lambda$  sized left-separated subspaces can we find? Is  $\lambda$  the smallest order type witnessing left-separation?

Theorem 2.14 does give us a lower bound for the degree of left-separation, but the theorem may hold for some  $\gamma > \lambda$ , so we do not obtain an upper bound. We note that the result of this theorem is stronger than just obtaining a lower bound; the initial segments of  $X_\lambda$  are actually closed in  $X$ .

It is not known if there are spaces which do not attain their degree of left-separation. For example, is there a space  $X$  with  $|X| = \omega_\omega$  that has no left-separated subspaces of size  $\omega_\omega$ , but has a left-separated subspace of every smaller cardinality? If such an  $X$  exists, then  $LS(X)$  is the supremum of the cardinalities of the left-separated subspaces of  $X$ , which would give us  $LS(X) = \omega_\omega$ .

We conclude this chapter by noting that the degree of left-separation of a space that is not left-separated can be equal to the cardinality of the space itself. As will be shown in Chapter 4, there exist two left-separated subsets of  $A$  and  $B$  of  $2^{\mathfrak{c}}$  such that  $|A| = \omega$  and  $|B| = \mathfrak{c}$ , but  $A \cup B$  is not left-separated. But  $B$  is left-separated as a subspace of  $A \cup B$ , so  $LS(A \cup B) = \mathfrak{c}$ .

CHAPTER 3  
PROPERTIES THAT IMPLY LEFT-SEPARATION

With the tools that we have developed, spaces with countable properties are a natural place to begin looking for left-separation.

Several of the following theorems use countable properties of refinements of an open cover of a topological space to prove left-separation. The aim of the following definition is to clarify the interplay between the elementary submodels and these refinements.

**Definition 3.1.** *Let  $X$  be a topological space with an open cover denoted by  $\mathcal{O}$ . A refinement  $\mathcal{V}$  of  $\mathcal{O}$  is a family of sets such that each  $V \in \mathcal{V}$  is a subset of some  $O \in \mathcal{O}$ . We say that  $\mathcal{V}$  is an open refinement if  $\mathcal{V}$  is a family of open sets, and  $\mathcal{V}$  is a point-countable refinement if for any  $x \in X$ , we have that  $x$  is only contained in countably many sets in  $\mathcal{V}$ .*

**Definition 3.2.** *A space  $X$  is metalindelöf if every open cover of  $X$  has an open point-countable refinement.*

In [5], Stanley proved the following theorem. I provide a new proof below.

**Theorem 3.3.** *Let  $X$  be a metalindelöf topological space such that  $X$  is right-separated in order type  $\omega_1$ . Then  $X$  is left-separated.*

Proof.

Let  $\{x_\alpha \mid \alpha < \omega_1\}$  be a well ordering of  $X$  witnessing right-separation. Let  $M$  be a countable elementary submodel such that  $X \in M$ . As  $X$  is right-separated, let

$\mathcal{U} = \{[x_0, x_\alpha] \mid \alpha \in \omega_1\}$  be an open cover of  $X$ . Notice for each  $\alpha$ ,  $[x_0, x_\alpha]$  is countable. Since  $X$  is metalindelöf, let  $\mathcal{V} \in M$  be a point-countable open refinement of  $\mathcal{U}$ . Let  $x \in \overline{X \cap M}$  and  $\mathcal{V}_x = \{V \in \mathcal{V} \mid x \in V\}$ . Let  $\widehat{V} \in \mathcal{V}_x$ . Since  $x \in \overline{X \cap M}$ , we have  $\widehat{V} \cap (X \cap M) \neq \emptyset$ . Let  $y \in \widehat{V} \cap (X \cap M)$  and  $\mathcal{V}_y = \{V \in \mathcal{V} \mid y \in V\}$ . Notice as  $y, \mathcal{V} \in M$ , we have  $\mathcal{V}_y \in M$ . Since  $\mathcal{V}$  is a point-countable refinement,  $|\mathcal{V}_y| = \omega$  and so  $\mathcal{V}_y \subset M$ . Noting that  $\widehat{V} \in \mathcal{V}_y \subset M$  and  $\mathcal{V}_y \subset \mathcal{V}$  which is a refinement of  $\mathcal{U}$ , we have  $\widehat{V} \subset [x_0, x_\alpha]$  for some  $\alpha \in \omega_1$  and so  $\widehat{V}$  is countable. Since  $\widehat{V}$  is countable,  $\widehat{V} \subset M$  and remembering that  $x \in \widehat{V}$  gives us that  $x \in M$ , which in turn means  $x \in X \cap M$ . Since this holds for each  $x \in \overline{X \cap M}$ , we have  $\overline{X \cap M} = X \cap M$ , so  $X \cap M \stackrel{\text{closed}}{\subset} X$ . Thus by Theorem 2.8,  $X$  is left-separated. □

Generally, a space being metalindelöf is not enough to guarantee that it is left-separated, but it is if in addition, the space has a "small enough" open set around each point. Realizing that the right-separation of  $X$  in Theorem 3.3 was only used to find a countable initial segment to contain our open set  $V$  naturally leads to the following theorem. Before we state the theorem, we define what we mean by "small enough".

**Definition 3.4.** *A space  $X$  is locally countable if every  $x \in X$  has a countable open neighborhood.*

Scheidecker proves the following theorem in [5].

**Theorem 3.5.** *Let  $X$  be a topological space such that  $X$  is metalindelöf and locally countable. Then  $X$  is left-separated.*

Proof.

Let  $X$  be as in the hypothesis and let  $M$  be an elementary submodel of  $X$  such that  $X \in M$ . Let  $\mathcal{U}$  be an open cover of  $X$  witnessing that  $X$  is locally

countable. Let  $\mathcal{V} \in M$  be a point-countable open refinement of  $\mathcal{U}$  and let  $\mathcal{V}_x = \{V \in \mathcal{V} | x \in V\}$  for each  $x \in X$ . Let  $x \in \overline{X \cap M}$  and  $\widehat{V} \in \mathcal{V}_x$ . Notice that  $\widehat{V} \cap (X \cap M) \neq \emptyset$  since  $\widehat{V} \in \mathcal{V}_x$ , so let  $y \in \widehat{V} \cap (X \cap M)$ . Since  $y \in M$  and  $|\mathcal{V}_y| = \omega$ , we have that  $\mathcal{V}_y \subset M$ , and so we have  $\widehat{V} \in \mathcal{V}_y \subset M$ . Since  $\widehat{V} \in \mathcal{V}$  and  $\mathcal{V}$  is a refinement of an open cover of  $X$  witnessing that  $X$  is locally countable,  $|\widehat{V}| = \omega$  and so  $\widehat{V} \subset M$ . Thus since  $x \in \widehat{V}$ , we have that  $x \in M$ , and so  $\overline{X \cap M} = X \cap M$  and  $X \cap M \stackrel{\text{closed}}{\subset} X$ . Therefore  $X$  is left-separated by Theorem 2.8.

□

In contrast to locally countable spaces, we now look at spaces whose open sets must be "large." The next theorem requires three new definitions. Given below is our definition of "large", which we refer to as neat.

**Definition 3.6.** *Let  $X$  be a topological space with topology  $\tau$ . We define  $\Delta(X)$  to be  $\min\{\kappa | \exists U \in \tau (|U| = \kappa)\}$ . We say  $X$  is neat if  $\Delta(X) = |X|$ .*

That is, every non-empty open set in a neat space must have the same cardinality as the space itself.

**Definition 3.7.** *Let  $X$  be a topological space. We denote the density of  $X$  by  $d(X)$  and define  $d(X) = \inf\{|A| : A \subset X \wedge \overline{A} = X\}$ , i.e. the infimum of the cardinalities of the dense subsets of  $X$ .*

**Definition 3.8.** *A subset  $A$  of a topological space  $X$  is said to be nowhere dense if  $\overline{A}$  does not contain a non-empty open set. Equivalently,  $A$  is nowhere dense if the interior of the closure is empty, or  $\overline{A}^\circ = \emptyset$ . A set that is not nowhere dense is somewhere dense.*

The union of finitely many nowhere dense sets is still nowhere dense, but the union of countably many nowhere dense sets might be somewhere dense. Towards a



clearer understanding of nowhere dense sets, consider the following examples in  $\mathbb{R}$  under the Euclidean topology.

**Example 3.9.** The Cantor set (denoted by  $\mathcal{C}$ ) is a nowhere dense subset of  $\mathbb{R}$ . By construction,  $\mathcal{C}$  contains no intervals and is also a closed subset of  $\mathbb{R}$ , and so  $\overline{\mathcal{C}} = \mathcal{C}$  does not contain any non-empty open intervals.

**Example 3.10.**  $A = \mathbb{Q} \cap (0, 1)$  is a somewhere dense subset of  $\mathbb{R}$ . As the rationals are dense in any interval,  $\overline{A} = [0, 1]$ , so  $\overline{A}^\circ = (0, 1)$  which is a non-empty open set.

The following theorem is due to Scheidecker and Stanley [3]. We would also like to thank Mark Ronnenberg for his contributions to this theorem.

**Theorem 3.11.** *Let  $\kappa$  be a cardinal. Let  $\alpha < \kappa^+$ . Let  $X = \bigcup_{\beta < \alpha} X_\beta$  be left-separated in order type  $(\kappa^+ \cdot \alpha)$  where  $\bigcup_{\beta < \alpha} X_\beta$  witnesses the left-separation order type. Suppose  $X$  is neat. Then*

1.  $d(X) = \kappa^+$

*or*

2. for every  $\beta < \alpha$ ,  $X_\beta$  is nowhere dense.

Proof.

By way of contradiction, suppose that  $X = \bigcup_{\beta < \alpha} X_\beta$  is left-separated in order type  $\kappa^+ \cdot \alpha$  such that

1.  $\Delta(X) = \kappa^+$
2.  $d(X) < \kappa^+$
3. there exists some  $\beta < \alpha$  such that  $X_\beta$  is somewhere dense.

Fix  $\beta < \alpha$  such that  $X_\beta$  is somewhere dense and let  $U = \overline{X_\beta}^\circ$ . Since  $X$  is left-separated,  $\overline{X_\beta} \subset \bigcup_{\gamma \leq \beta} X_\gamma$  and so  $U = \overline{X_\beta}^\circ \subset \overline{X_\beta} \subset \bigcup_{\gamma \leq \beta} X_\gamma$ . Noting that  $X_\beta$  is somewhere dense, we have that  $U = \overline{X_\beta}^\circ \neq \emptyset$ , so  $\Delta(X) = \kappa^+$  gives us that  $|U| = \kappa^+$ . Notice that if  $U \cap X_\beta = \emptyset$ , then since  $U$  is open,  $U \cap \overline{X_\beta}$  would be empty. However,  $U \subset \overline{X_\beta}$  and  $U \neq \emptyset$ . Thus  $U \cap X_\beta \neq \emptyset$ . Notice that  $U \cap X_\beta = U \setminus \bigcup_{\gamma < \beta} X_\gamma$  is open by left-separation. Thus  $|U \cap X_\beta| = \kappa^+$ . Further, as  $U \cap X_\beta$  is open and  $X_\beta$  is somewhere dense, we have  $d(U \cap X_\beta) \leq d(X) \leq \kappa$ . Let  $V \subset U \cap X_\beta$  such that  $|V| = \kappa$  and  $U \cap X_\beta \subset \overline{V}$ . Since  $V \subset X_\beta$  and  $|V| = \kappa$  while  $X_\beta = \kappa^+$ ,  $V$  can not be cofinal in  $X_\beta$ . Since  $V$  is not cofinal in  $X_\beta$ , let  $A$  be an initial segment of  $X_\beta$  such that  $V \subset A$ . Then since  $X$  is left-separated,  $\overline{V} \cap X_\beta \subset A$  and  $|A| = \kappa$  gives us that  $|\overline{V} \cap X_\beta| = |\overline{V} \cap A| = \kappa$ . However,  $U \cap X_\beta \subset \overline{V}$ , so  $|\overline{V} \cap X_\beta| = \kappa^+$ . This contradiction proves that  $X$  is not left-separated, and thus proves the theorem. □

The inspiration for Theorem 3.11 came from noticing the general idea while proving that the real numbers under the Euclidean topology are not left-separated. As such, the proof that  $\mathbb{R}$  is not left-separated follows the proof of Theorem 3.11. It is unknown if the result holds in general when  $X$  is not neat.

CHAPTER 4  
UNIONS OF LEFT-SEPARATED SPACES

Proving that the union of left-separated spaces is still left-separated is non-trivial. Even finding an example of two left-separated spaces whose union is not left-separated is not easy. To highlight the difficulty, consider two left-separated spaces  $X$  and  $Y$  and their union  $X \cup Y$ . We know we can find well orderings of  $X$  and  $Y$  witnessing left-separation as subspaces, but  $X$  and  $Y$  might have accumulation points in each other that destroy any naive attempts to find a well ordering of  $X \cup Y$  witnessing left-separation.

We know that the accumulation points that cause us the most trouble are the cluster points of both  $X$  and  $Y$ . Scheidecker and Stanley [3] proved the following lemma.

**Lemma 4.1.** *Let  $X$  and  $Y$  be left-separated spaces. Let  $Z = \overline{X} \cap \overline{Y}$ . If  $Z$  is left-separated, then  $X \cup Y$  is left-separated. Further, the left-separation order type of  $ord_\ell(X \cup Y) \leq ord_\ell(Z) + \max\{ord_\ell(X), ord_\ell(Y)\}$ .*

Proof.

Let  $X, Y$  and  $Z$  be as in the statement of the lemma. Let  $\{z_\alpha | \alpha < \kappa\}$  witness that  $Z$  is left-separated. In addition,  $Z$  is a closed subset of  $X \cup Y$ . The remaining points are  $X \setminus \overline{Y} \cup Y \setminus \overline{X}$ . Notice that no point of  $X \setminus \overline{Y}$  is a cluster point of  $Y \setminus \overline{X}$  and vice versa. As  $X$  and  $Y$  are left-separated, both  $X \setminus \overline{Y}$  and  $Y \setminus \overline{X}$  are left-separated. Therefore, we can simply alternate points of these two sets maintaining their left-separating well orders inherited from  $X$  and  $Y$ . Thus,  $X \cup Y$  is left-separated and  $ord_\ell(X \cup Y) \leq ord_\ell(Z) + \max\{ord_\ell(X), ord_\ell(Y)\}$ .

□

The previous lemma gives us a reasonable place to start when we look at the union of two left-separated spaces. When the union of our spaces is locally countable, we can use elementary submodels to establish our well ordering. The following theorem is a significant result of Scheidecker and Stanley [3].

**Theorem 4.2.** *Let  $X$  and  $Y$  be left-separated spaces where  $\text{ord}_\ell(X) = \kappa \geq \omega_1$  and  $\text{ord}_\ell(Y) = \omega_1$ . If  $X \cup Y$  is locally countable, then  $X \cup Y$  is left-separated and  $\text{ord}_\ell(X \cup Y) \leq \kappa \cdot 2$ .*

Proof.

Let  $X$  and  $Y$  be as stated in the hypothesis of the theorem. Let  $Z = \overline{X} \cap \overline{Y}$ . We will prove that  $Z$  is left-separated in order type at most  $\kappa + \omega_1$ . We will consider  $Y$  as a point set to be  $\omega_1$ .

**Claim.** *For each  $\alpha \in Y$ , there exists  $\beta \in [\alpha + \omega, \omega_1)$  such that  $[\alpha, \beta)$  is an open subset of  $Y$ .*

Proof.

Let  $M$  be a countable elementary submodel such that  $Y \in M$ . Let  $\delta = M \cap \omega_1$  and let  $\alpha < \delta$ . Clearly,  $\alpha \in M$ , and so  $\alpha + \omega \in M$ . We will show that  $[\alpha, \delta)$  is an open subset of  $Y$ . Let  $\gamma \in [\alpha, \delta)$ . Then  $\gamma \in M$ . Let  $\mathcal{N}_\gamma$  be the family of open neighborhoods in  $Y$  containing  $\gamma$ . Since  $Y$  is locally countable and left-separated, there exists  $U \in \mathcal{N}_\gamma \cap M$  such that  $U$  is countable and  $U \subset [\gamma, \omega_1)$ . Since  $U$  is countable and  $U \in M$ , we have  $U \subset M$  and so  $U \subset [\gamma, \delta)$ . Therefore,  $[\alpha, \delta)$  is an open subset of  $Y$ . As  $\delta > \alpha + \omega$ , the claim then follows. □

We can now re-order  $Y$  in order type  $\omega_1$  so that for each  $\alpha \in \text{LIM}$ ,  $[\alpha, \alpha + \omega)$  is an open subset of  $Y$ . For each  $\alpha \in \text{LIM}$ , let  $U_\alpha$  be a countable open subset of  $X \cup Y$  so that  $U_\alpha \cap Y = [\alpha, \alpha + \omega)$ .

For each  $z \in Z \cap Y$ , let  $\alpha(z) \in \text{LIM}$  such that  $z \in U_{\alpha(z)}$ . We now use the family  $\{U_{\alpha(z)} | z \in Z \cap Y\}$  to isolate a portion of  $Z$  which resides in  $X$  and which we will put first in our well-ordering of  $X \cup Y$ . Let

$$Z^- = Z \setminus \bigcup_{z \in Z \cap Y} U_{\alpha(z)}.$$

Since  $Z^- \subset X$ ,  $Z^-$  is left-separated. Well order  $Z^-$  as

$$\{z(-1, \gamma) | \gamma < \lambda\}$$

witnessing left-separation in order type  $\lambda \leq \kappa$ .

We now look at the remaining portion of  $Z$ . For each  $\alpha \in \text{LIM}$ , let

$$Z_\alpha = (Z \cap U_\alpha) \setminus Z^-.$$

Note that each  $Z_\alpha$  is countable and thus we can order each one as an  $\omega$ -sequence.

**Claim.** For every  $\alpha, \beta \in \text{LIM}$ , if  $\alpha \neq \beta$ , then  $\overline{Z_\alpha} \cap \overline{Z_\beta} = \emptyset$ , and  $U_\alpha \cap U_\beta \cap Y = \emptyset$ .

*Proof.*

Fix  $\alpha, \beta \in \text{LIM}$  so that  $\alpha \neq \beta$ . Notice that  $U_\alpha \cap U_\beta$  is open in  $X \cup Y$  and further that  $U_\alpha \cap U_\beta \subset X$ . Therefore,  $(U_\alpha \cap U_\beta) \cap \overline{Y} = \emptyset$  and thus  $(U_\alpha \cap U_\beta) \cap Z = \emptyset$ . Thus, we now have that  $Z_\alpha \subset U_\alpha$  and similarly  $Z_\beta \subset U_\beta$ , thus  $\overline{Z_\alpha} \cap \overline{Z_\beta} = \emptyset$ .

□

We can now list each  $Z_\alpha$  as an  $\omega$ -sequence  $\{z(\alpha, n) | n \in \omega\}$ . This well orders all of  $Z$  as

$$\{z(-1, \gamma) | \gamma < \lambda\} \cup \{z(\alpha, n) | \alpha \in \text{LIM} \wedge n \in \omega\}.$$

We now need to show that this well ordering witnesses that  $Z$  is left-separated. Let  $z \in Z$ . There are two cases to consider:  $z \in Z^-$  or  $z \in Z_\alpha$  for some  $\alpha \in \text{LIM}$ . As  $Z^-$  is left-separated and comes first in our well ordering, there is nothing to prove in this first case. Let  $\alpha \in \text{LIM}$  so that  $z \in Z_\alpha$ . Then  $z \in U_\alpha$  and  $U_\alpha \cap Z_\beta = \emptyset$  for all  $\beta \neq \alpha$ . Further, there exists a  $z' \in Z \cap Y$  so that  $z \in U_{\alpha(z')}$  and  $U_{\alpha(z')} \cap Z^- = \emptyset$ . Thus, initial segments of  $Z$  are closed. This shows that  $Z$  is left-separated in order type at most  $\kappa + \omega_1$ .

Thus, now by Lemma 4.1 we have that  $X \cup Y$  is left-separated and

$$\text{ord}_\ell(X \cup Y) \leq (\kappa + \omega_1) + \kappa = \kappa \cdot 2.$$

□

If  $\text{ord}_\ell(Y) = \omega_1 \cdot 2$ , we can consider it as  $Y = Y_1 \cup Y_2$  where  $Y_1$  and  $Y_2$  are left-separated subspaces of  $Y$  such that  $\text{ord}_\ell(Y_1) = \text{ord}_\ell(Y_2) = \omega_1$ . Then  $X \cup Y_1$  is left-separated in order type  $\kappa \cdot 2$  by the previous theorem, and similarly,  $(X \cup Y_1) \cup Y_2$  is left-separated in order type  $(\kappa \cdot 2) \cdot 2 = \kappa \cdot 2^2$ . The generalization of the theorem follows from repeating this process finitely many times and is given by Scheidecker and Stanley [3].

**Theorem 4.3.** *Let  $X$  and  $Y$  be left-separated spaces so that  $X \cup Y$  is locally countable. Suppose  $\text{ord}_\ell(X) = \kappa$  and  $\text{ord}_\ell(Y) = \omega_1 \cdot n$  where  $n \in \omega$ . Then  $X \cup Y$  is left-separated, and further  $\text{ord}_\ell(X \cup Y) \leq \kappa \cdot 2^n$ .*

It is unknown if the theorem holds for  $\text{ord}_\ell(Y) \geq \omega_1 \cdot \omega$ .

While it is difficult to prove that the union of left-separated spaces is left-separated in general, it is easy to find examples that are not too complicated; e.g. any countable union of countable Hausdorff spaces is countable and Hausdorff, thus left-separated. With this in mind, it is interesting that we have not been able

to find any simple examples of a union of left-separated spaces which is not left-separated. As referenced in Chapter 2, the best we are able to do requires two subsets of  $2^{\mathfrak{c}}$ . We quote Scheidecker and Stanley [3] here: "Juhász, Soukup and Szentmiklóssy constructed a space  $X$  which is a countable dense subset of  $2^{\mathfrak{c}}$  and a space  $Y$  which is a  $G_\delta$ -dense, left-separated subset of  $2^{\mathfrak{c}}$  with  $|Y| = \mathfrak{c}$ . As  $X$  is countable it is also left-separated. In [6], Soukup and Stanley proved that  $X \cup Y$  is not left-separated."

Informally defined, we say  $Y$  is  $G_\delta$ -dense if  $Y$  has non-empty intersection with every  $G_\delta$  set in  $2^{\mathfrak{c}}$ .

CHAPTER 5  
FUTURE WORK AND OPEN QUESTIONS

We conclude with several of our open questions which outline the general direction of our future work. Many of these questions are in regards to generalizing the theorems in this paper; of particular interest to us is whether or not we can develop similar tools to work with right-separated spaces.

**Question 1.** *Is there an equivalent to Theorem 2.8 for right-separated spaces? i.e. can we put a condition on the intersections of every elementary submodel with a space that guarantees right-separation?*

In Chapter 2, we were left with an unsatisfying partial generalization of Theorem 2.10.

**Question 2.** *Let  $X$  be a topological space such that  $|X| > \kappa$  and  $X$  has a point- $\kappa$  base. Suppose that for every elementary submodel  $M$  such that  $X, \kappa \in M$  and  $\kappa \subset M$ , we have that  $X \cap M$  is a closed subset of  $X$ . Is  $X$  left-separated? What if this condition holds for a space without a point- $\kappa$  base?*

We ended Chapter 2 with Theorem 2.14 which proved the existence of a left-separated subset if certain conditions are met. Can we conclude more?

**Question 3.** *Is there a stronger conclusion to Theorem 2.14? In particular, if Theorem 2.14 holds for  $X$  with elementary submodels of size less than  $\lambda$ , is every  $\lambda$  sized subset of  $X$  left-separated?*

**Question 4.** *Does the conclusion of Theorem 3.11 hold when we remove the condition that  $X$  is neat?*



**Question 5.** *Let  $X$  and  $Y$  be left-separated spaces so that  $X \cup Y$  is locally countable. Suppose  $\text{ord}_\ell(X) = \kappa$  and  $\text{ord}_\ell(Y) = \omega_1 \cdot \omega$ . Is  $X \cup Y$  left-separated? If so, is  $\text{ord}_\ell(X \cup Y) \leq \kappa \cdot \omega$ ?*

**Question 6.** *Let  $X$  and  $Y$  be left-separated such that  $X \cup Y$  is locally countable. Is  $X \cup Y$  left-separated?*

At the end of Chapter 4 we gave the example of two left-separated spaces whose union was not left-separated. Is there a simpler example?

**Question 7.** *Is there a simple example of two left-separated spaces whose union is not left-separated?*

The next two questions came from attempting to find a simpler example. We wondered if we could make the union left-separated by tweaking the properties of the two spaces or adding a condition on their union.

**Question 8.** *Let  $X$  and  $Y$  be topological spaces such that  $X$  is countable and dense in  $Y$  and  $Y$  is left-separated. What conditions can we add to guarantee that  $X \cup Y$  is left-separated?*

**Question 9.** *If  $X$  is countable and  $Y$  is left-separated such that  $X \cup Y$  is the countable union of nowhere dense sets, is  $X \cup Y$  left-separated?*

Another possible route would be to suppose that  $Y$  is not  $G_\delta$ -dense.

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