Maggie and her students: Talking and reasoning in a reform-based mathematics classroom

Debra Louise Semm Rich
University of Northern Iowa

Copyright ©2011 Debra Louise Semm Rich
Follow this and additional works at: https://scholarworks.uni.edu/etd

Part of the Educational Methods Commons, and the Science and Mathematics Education Commons

Recommended Citation
Semm Rich, Debra Louise, "Maggie and her students: Talking and reasoning in a reform-based mathematics classroom" (2011). Dissertations and Theses @ UNI. 325.
https://scholarworks.uni.edu/etd/325

This Open Access Dissertation is brought to you for free and open access by the Student Work at UNI ScholarWorks. It has been accepted for inclusion in Dissertations and Theses @ UNI by an authorized administrator of UNI ScholarWorks. For more information, please contact scholarworks@uni.edu.
Copyright by

DEBRA LOUISE SEMM RICH

2011

All Rights Reserved
MAGGIE AND HER STUDENTS: TALKING AND REASONING
IN A REFORM-BASED MATHEMATICS CLASSROOM

An Abstract of a Dissertation
Submitted
in Partial Fulfillment
of the Requirements for the Degree
Doctor of Education

Approved:

Dr. Lynn Nielsen, Committee Chair

Dr. Michael Licari
Dean of the Graduate College

Debra Louise Semm Rich
University of Northern Iowa
December 2011
ABSTRACT

In this qualitative case study, I explored how Maggie, the teacher, and her students used language to advance students' conceptual understanding of mathematics in a reform-based mathematics classroom. Specifically I sought to describe how Maggie’s reform-based mathematics classroom environment impacted the quantity and quality of mathematical language spoken by her students. Maggie’s eighth-grade mathematics classroom was observed during a course of study on data and statistics. Three semi-structured interviews with Maggie, artifacts provided by her, and fourteen 52-minute classroom lessons, supported with field notes, were analyzed.

Environmental themes were revealed as they related to Maggie, her beliefs, goals, knowledge, and what she said. Maggie’s talk was framed by five productive teacher-talk moves. The quantity of student talk was analyzed according to three variations of the three-phase reform-based mathematics lesson (Direct Instruction, Guided Discovery, Open-Ended Exploration) and three talk formats (Whole-Class Discussion, Partner Talk, Small-Group Discussion). Due to the Maggie’s diminished role during Partner Talk and Small-Group Discussion, an in-depth analysis was conducted to determine what students were talking about when they worked independently. This revealed a qualitative change over the course of the unit. Further analysis revealed that students engaged in two mathematical Discourses: Procedural Discourse and Conceptual Discourse. Since the focus of this study was on how language was used to advance students’ conceptual understanding of mathematics, this type of talk became a focus of further analysis. Ten categories of students’ mathematical reasoning emerged from the data.
This study provides a rich description of a reform-based mathematics classroom environment, and the quantity and quality of student talk impacted by it. Implications for educational leadership, literacy education, mathematics education, and teacher education, as well as recommendations for future studies, are shared.
MAGGIE AND HER STUDENTS: TALKING AND REASONING
IN A REFORM-BASED MATHEMATICS CLASSROOM

A Dissertation
Submitted
in Partial Fulfillment
of the Requirements for the Degree
Doctor of Education
Approved:

Dr. Lynn Nielsen, Committee Chair
Dr. Timothy Gilson, Committee Member
Dr. Elizabeth Hughes, Committee Member
Dr. Rick Traw, Committee Member
Dr. Barry Wilson, Committee Member

Debra Louise Semm Rich
University of Northern Iowa
December 2011
DEDICATION

I dedicate this dissertation to Maggie and her colleagues everywhere who believe that all students can learn and work tirelessly to help them do so.
ACKNOWLEDGEMENTS

There are many people who have supported me throughout my doctoral studies and research endeavors. I would specifically like to thank my family, friends, doctoral faculty, advisor, committee chair and committee, and the participants in the study.

First, I thank my husband, Gil, my rock, and unwavering champion. He encouraged me to persevere when I wasn’t sure if I could, and supported me in ways too numerous to mention. I am forever grateful for his unselfish patience. I thank my daughter, Karen, whose daily phone calls during our mutual commute to different universities gave us time to share our research as she also pursued her doctoral studies. She inspires me to see the good in everyone and to be a lifelong learner of interesting phenomena. Thanks, also, to her husband, Nathan, for his kind words of encouragement.

I thank my daughter, Laura who shares stories of her students with me. She reminds me of the challenges facing inner-city children and the deeper reasons that called us both into the education profession. She inspires me to savor the moment and hear the beauty in the rhythm of life. I thank her husband, Joel, who engages me in conversations about mathematics education and pedagogy as he pursued his own graduate degree, and who helped troubleshoot my computer and technology issues long distance. Also thanks to both families for blessing me with beautiful grandchildren during the time period that I pursued my research. Abigail and Noah inspire me to see the wonder and joy in the world. I want to acknowledge my stepson, Jeremy. No matter the challenges I might face, they pale in comparison to those he encounters daily. He inspires me to be resilient.

Finally, I would like to posthumously thank my father who nearly lived long enough to
share in the joy of my last graduation. He instilled a deep value of education in me and always encouraged me to follow my dreams. I am forever grateful for his endearing love.

Second, I thank my friends who have inquired as to my progress, shared words of encouragement, and celebrated important milestones along the way. I appreciate their understanding when I had studying to do and papers to write. I am grateful for their support and deeply value our friendships.

Third, I thank the graduate faculty who mentored me throughout my program. I thank my doctoral advisor, Dr. Victoria Robinson, who guided me from day one of my graduate program. I value her positive outlook and sage advice. I thank Dr. John Smith, who met weekly with me for over a year as I formulated my research topic. He saw promise in my ideas long before I could articulate them well. I also thank him for challenging me to become a critical thinker. I no longer view the world in the same way. I especially thank Dr. Lynn Nielsen, my dissertation chair. He always made time to meet with me, no matter how busy his schedule, whether in person or through e-mail during the wee hours of the evening. His feedback on my work was immediate, thorough, insightful, and constructive. I am thankful for his guidance and his genuine interest in my work, in my family, and in me. I also thank the other four members of my committee—Dr. Timothy Gilson, Dr. Elizabeth Hughes, Dr. Rick Traw, and Dr. Barry Wilson—who generously gave their time and support to read, meet, and provide feedback about my research.

Finally, I thank Maggie for generously sharing her thinking about reform-based mathematics education and graciously welcoming me into her classroom to observe her
teaching. I also thank Landon, Diana, Andrew, Evie, Dylan, Connor, Chloe, Tony, Ava, Matt, Tyler, Gavin, and Dan, for agreeing to let me listen in on their conversations during 14 mathematics lessons. I am indebted to Maggie and each her students for their hospitality, candidness, and willingness to allow us all to learn from them.
# TABLE OF CONTENTS

LIST OF TABLES ........................................................................................................... xiv

LIST OF FIGURES ....................................................................................................... xvi

CHAPTER I: INTRODUCTION ....................................................................................... 1

Just Another Reform? ................................................................................................. 1

A Focus on Learning .................................................................................................. 3

Why an Interest in Mathematical Talk? .................................................................... 6

An Outside View on Becoming an Insider ............................................................... 9

CHAPTER II: WHAT IT COULD BE, WHERE WE ARE, AND HOW WE GOT HERE ........................................................................................................... 13

A Vision for School Mathematics ........................................................................... 14

Student Achievement in Mathematics: A Broken System ...................................... 15

Closing the Achievement Gap or Igniting a War? .................................................... 17

Ideological Tension: A Deep Divide ........................................................................ 19

Humanist Voice ......................................................................................................... 21

Progressive Voices Challenge the Discourse ......................................................... 22

The Developmentalist Voice ................................................................................... 25

The Social Efficiency Voice ..................................................................................... 28

The Humanist Voice Talks Back ............................................................................. 34

Developmentalist Voices Translated ..................................................................... 37

Social Efficiency Voices In Humanists’ Clothing .................................................. 40

The Social Meliorist Voice ...................................................................................... 45
### Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speaking To Everyone: A Slogan System?</td>
<td>48</td>
</tr>
<tr>
<td>The Debate Heats Up</td>
<td>50</td>
</tr>
<tr>
<td>Rigor Versus Relevance</td>
<td>51</td>
</tr>
<tr>
<td>Content Versus Pedagogy</td>
<td>52</td>
</tr>
<tr>
<td>Democracy Versus Elitism</td>
<td>53</td>
</tr>
<tr>
<td>A National Agenda: Humanist Voices in Social Meliorists' Clothing</td>
<td>54</td>
</tr>
<tr>
<td>A Parallel War</td>
<td>56</td>
</tr>
<tr>
<td>Common Core State Standards</td>
<td>58</td>
</tr>
<tr>
<td>It Matters to This One</td>
<td>60</td>
</tr>
<tr>
<td>Statement of the Problem</td>
<td>60</td>
</tr>
<tr>
<td>Purpose of the Study</td>
<td>61</td>
</tr>
<tr>
<td>Assumptions</td>
<td>61</td>
</tr>
<tr>
<td>Definition of Terms</td>
<td>62</td>
</tr>
<tr>
<td>CHAPTER III: UNPACKING THE RESEARCH QUESTION</td>
<td>68</td>
</tr>
<tr>
<td>Language: A Promise of Hope?</td>
<td>68</td>
</tr>
<tr>
<td>The Language of Mathematics</td>
<td>71</td>
</tr>
<tr>
<td>The Symbolic System</td>
<td>72</td>
</tr>
<tr>
<td>The Syntactic System</td>
<td>74</td>
</tr>
<tr>
<td>The Semantic System</td>
<td>74</td>
</tr>
<tr>
<td>Difficult English Constructions</td>
<td>77</td>
</tr>
<tr>
<td>Genre of “Story” Problems</td>
<td>78</td>
</tr>
<tr>
<td>Misinterpretation</td>
<td>79</td>
</tr>
</tbody>
</table>
Maggie's use of instructional time ................................................................. 155
Maggie's availability, high expectations, and encouragement ................. 158
Maggie's Beliefs About Mathematics Education: “If I Could Rule the World” ... 161
Maggie’s Long-Term Mathematics Goals: Learning How To Learn .......... 167
Maggie’s Short Term Goals for the Unit: Becoming Wise Creators and Wise Consumers of Data .............................................................. 172
Maggie’s Expertise: Content Knowledge For Teaching ......................... 186
How Maggie Organizes Her Lessons ......................................................... 190
Whole-Class Discussion ........................................................................ 190
The direct instruction lesson ................................................................. 192
The guided discovery lesson ................................................................. 193
The open-ended exploration lesson ..................................................... 194
Purposeful decisions ........................................................................ 195
Partner Talk .......................................................................................... 198
Small-Group Discussion ........................................................................ 201
How Maggie Implements Her Lessons: Weaving It All Together .............. 204
Three-Phase Reform-Based Mathematics Lesson .................................. 205
Meaningful Distributed Instruction ..................................................... 218
Independent Work ................................................................................ 223
What Maggie Says ............................................................................. 224
This Is For Everybody ........................................................................ 224
What Do You Think? Why? .............................................................. 228
How Do You Know? How Could You Check? ...................................... 229
Spontaneously Checking and Self-Correcting .................................................. 276

Partner Talk and Small-Group Discussion ........................................................ 279

Lexical Density .................................................................................................... 280

Procedural and Conceptual Discourses ............................................................... 285

Mathematical Reasoning ....................................................................................... 289

Category 1: I don’t know ..................................................................................... 292

Category 2: Agrees or disagrees with partner without evidence of independent reasoning ............................................................................................................. 295

Category 3: Personalizes the data ......................................................................... 297

Category 4: Responds without evidence of reasoning ............................................ 301

Category 5: Indicates thinking but provides no explanation .................................. 302

Category 6: Asks a question to seek understanding ............................................... 305

Category 7: Responds with partial or incomplete reasoning .................................. 307

Category 8: Responds with simple reasoning ....................................................... 309

Category 9: Responds with complex reasoning ..................................................... 312

Category 10: Self-checking response .................................................................... 315

Off-topic talk ....................................................................................................... 320

Reflection ........................................................................................................... 325

Summary ............................................................................................................. 325

CHAPTER VI: WHAT DID I LEARN? WHY DOES IT MATTER? AND WHERE DO WE GO FROM HERE? ................................................................. 328

What Did I Learn From Maggie and Her Students? ............................................ 328

Why Does It Matter? ............................................................................................ 334
Educational Leadership ................................................................. 336

Literacy Education ........................................................................ 337

Mathematics Education .............................................................. 339

Teacher Education ....................................................................... 341

Where Do We Go From Here? ....................................................... 342

REFERENCES ............................................................................... 344
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2010-2011 Eighth Grade Iowa Tests of Basic Skills Class Scores for Mathematics Sub-Tests: Problem Solving and Concepts and Estimation</td>
</tr>
<tr>
<td>2</td>
<td>Eighth Grade Mathematics Class: Use of Instructional Time</td>
</tr>
<tr>
<td>3</td>
<td>Eighth Grade Mathematics Instruction: Amount of Time Engaged In Three Talk Formats</td>
</tr>
<tr>
<td>4</td>
<td>Eighth Grade Mathematics Instruction: Use of Whole-Class Discussion</td>
</tr>
<tr>
<td>5</td>
<td>Eighth Grade Mathematics Instruction: Use of Partner Talk</td>
</tr>
<tr>
<td>6</td>
<td>Eighth Grade Mathematics Instruction: Use of Small-Group Discussion</td>
</tr>
<tr>
<td>7</td>
<td>Eighth Grade Mathematics Instruction: Type of Lesson</td>
</tr>
<tr>
<td>8</td>
<td>Eighth Grade Mathematics Instruction: Cognitive Complexity of Questions and Mathematical Tasks That Engaged Students During Partner Talk and Small-Group Discussion</td>
</tr>
<tr>
<td>9</td>
<td>Eighth Grade Mathematics Instruction: Mathematical Vocabulary First Introduced by Teacher During Data and Statistics Unit</td>
</tr>
<tr>
<td>10</td>
<td>Eighth-Grade Mathematics Instruction: Quantity of Student Talk Compared to Teacher Talk</td>
</tr>
<tr>
<td>11</td>
<td>Eighth-Grade Mathematics Instruction: Quantity of Student Talk Compared to Teacher Talk While Engaged in Three Talk Formats</td>
</tr>
</tbody>
</table>
12 Eighth-Grade Mathematics Instruction:
Quantity of Student Talk Compared to Teacher Talk
While Engaged in Three Whole-Class Discussion Lesson Models

13 Eighth-Grade Mathematics Instruction:
Density of Student Talk While Engaged in
Three Whole-Class Discussion Lesson Models, Partner Talk, and
Small-Group Discussion

14 Eighth-Grade Mathematics Instruction:
Quantity of Student Talk While Engaged in
Three Whole-Class Discussion Lesson Models, Partner Talk, and
Small-Group Discussion

15 Eighth Grade Mathematics Instruction:
Percent of Students Familiar With 5 Common
Graphical Representations of Data

16 Eighth Grade Mathematics Instruction:
Increase in the Lexical Density of Student Talk
From Lesson 2 to Lesson 12

17 Eighth Grade Mathematics Instruction:
Comparison of Student Responses Focused on
Mathematical Procedures and Mathematical Reasoning
While Engaged in Partner Talk

18 Eighth Grade Mathematics Instruction:
Comparison of Student Responses Focused on
Mathematical Procedures and Mathematical Reasoning
While Engaged in Small-Group Discussion
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A Knowledge Package for Understanding the Meaning of Division By Fractions ........................................... 85</td>
</tr>
<tr>
<td>2</td>
<td>Lexical Density of Assessment Items, <em>National Assessment of Educational Progress: Mathematics Assessment, Grades 4 and 8.</em> ............. 122</td>
</tr>
<tr>
<td>3</td>
<td>Kids Who Say Reading Books Not Required for School Is Important............ 178</td>
</tr>
<tr>
<td>4</td>
<td>Average Hours of Sleep on Workdays/Weeknights ............................................... 207</td>
</tr>
<tr>
<td>5</td>
<td><em>Top 100 Most-Viewed You Tube Videos</em> .......................................................... 216</td>
</tr>
<tr>
<td>6</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 1: I Don’t Know ................................................................. 294</td>
</tr>
<tr>
<td>7</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 2: Agrees or Disagrees Without Evidence of Independent Reasoning .......................................................... 298</td>
</tr>
<tr>
<td>8</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 3: Personalizes the Data ................................................................. 300</td>
</tr>
<tr>
<td>9</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 4: Responds Without Evidence of Reasoning ................................................. 303</td>
</tr>
<tr>
<td>10</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 5: Indicates Thinking But Provides No Explanation ........................................... 306</td>
</tr>
<tr>
<td>11</td>
<td>Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 6: Asks a Question To Seek Understanding ........................................... 308</td>
</tr>
</tbody>
</table>
12 Students' Use of Mathematical Reasoning
During Partner Talk and Small-Group Discussion
Category 7: Responds With Partial or Incomplete Reasoning..........................310

13 Students' Use of Mathematical Reasoning
During Partner Talk and Small-Group Discussion
Category 8: Responds With Simple Reasoning.............................................313

14 Students' Use of Mathematical Reasoning
During Partner Talk and Small-Group Discussion
Category 9: Responds With Complex Reasoning.........................................316

15 Students' Use of Mathematical Reasoning
During Partner Talk and Small-Group Discussion
Category 10: Self-Checking Response.........................................................319
CHAPTER I

INTRODUCTION

North winds knew that spring break was over. They blew away any memories of swimsuits and four-wheelers that enticed young adolescents outside for a week to bask in the fleeting Midwest March sunshine. There was a chill in the air on Monday morning when they returned to school. It was first of 14 days that I would spend in Maggie's eighth-grade mathematics classroom. She, her students, and her students' parents had graciously agreed to allow me to audio- and video-record every word they said during the entire upcoming unit on data and statistics. I wanted to explore how they used language to advance students' conceptual understanding of mathematics in a reform-based mathematics classroom. Specifically, I was interested in how Maggie's reform-based classroom environment impacted the quantity and quality of mathematical language spoken by her students. Deep understanding of a concept is difficult to measure on a standardized test. With careful observation, was it possible to document shifts in student thinking and reasoning by listening to what they said? In order to find out, I focused my study at the root level of meaning, the very discourse in which Maggie and her students engaged to communicate and reveal learning in real time.

Just Another Reform?

Spending most of my 30 plus year career, prior to administrative roles, in the literacy field, I am familiar with shifts in educational practice. From basal readers to whole language to Reading First (2002), I have experienced them all. Was reform-based
mathematics education just another “latest-greatest phase” for the mathematics field? Or did this environment truly make a difference for students?

Teachers, principals, and other educational leaders are faced with ubiquitous calls for school reform. Yet much like chicken soup for the flu, the term has become the answer for what ails us. A search of “school reform” on the U.S. Department of Education’s website provided a list of 2300 articles, programs, and initiatives. Unfortunately, many principals and other school leaders have become jaded by the proverbial pendulum swings of reforms that have come and gone; that were mismanaged, ill-conceived, or short-lived initiatives for change. This has resulted in school leaders who manage the status quo rather than seek ways to engage their students in learning (Huberman, 1988).

Before prematurely dismissing the latest call for reform, it is worthwhile to understand what is being advocated at the state (Iowa Department of Education, 2010; Iowa Education Summit, 2011) and national levels (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000; National Mathematics Advisory Panel [NMAP], 2008), with which Maggie was aligned. Given the importance of mathematics education as a core academic subject in the school curriculum and the national interest in Science, Technology, Engineering, and Mathematics (STEM) fields (NMAP, 2008), it is crucial that principals and other school leaders make sound educational decisions that provide the greatest benefit to students. It is worthwhile to understand reform-based mathematics education, because this time, the reform truly is different. Student learning is at the core.
A Focus on Learning

Nearly a decade prior, DuFour (2002) proposed a radical idea: *learning* should become the preoccupation of the school. For too long, principals have asked, “What are the teachers teaching?” and “How can I help them to teach it more effectively?” Instead, the key question principals and others should be asking is, “To what extent are the students learning the intended outcomes of each course?” When educators shift from a focus on teaching to a focus on learning, it results in a substantive structural and cultural transformation of the school. In the process, “principals function as *learning leaders* rather than *instructional leaders*” (p. 13, italics in original).

How principals and other educational leaders (e.g., assistant principals, curriculum coordinators) function as learning leaders in this era of reform movements depends greatly on their views of subject matter. According to Burch and Spillane (2003), leadership activity in mathematics was quite different from leadership activity in literacy in the eight schools they studied, all part of the Chicago Public School District.

Overwhelmingly, literacy was seen by 83% of the leaders as a subject that is involved in all disciplines. Well over half of the leaders depicted literacy as a broad measure of student progress toward essential thinking and organizational skills. Leaders encouraged all teachers, not just reading specialists, to engage in opportunities to develop and discuss curriculum, and often cited school-developed literacy activities such as parent programs, reading competitions, and curriculum development groups, as critical to improving instruction. Leaders frequently observed classrooms *during literacy instruction* and engaged teachers in professional discussions about students’ learning.
They analyzed students’ work and met with small groups of teachers to talk about their practice. Through these discussions, the leaders learned how teachers utilized research to inform their work so they began to connect faculty with university partners. The leadership developed a participatory approach to improving literacy instruction because they viewed literacy as a subject that involved everyone. Therefore, everyone was engaged in the literacy Discourse.

On the other hand, leaders placed much less emphasis on teacher participation in decision-making and curriculum development in the area of mathematics. Over half emphasized having teachers adhere closely to the sequence of curriculum. Leaders frequently attributed improvements in mathematics instruction to the use of the new highly sequenced mathematics textbook series they had just adopted. Leaders also cited the need to provide teachers with additional professional training in mathematics. Support for teachers came from external sources in the way of staff developers who said, “This is what you’re going to do” (Burch & Spillane, 2003, p. 528), materials, and supplies. Only 37% of principals and 14% of assistant principals were personally involved in mathematics reforms. The few who did meet with teachers and analyzed student achievement test scores acknowledged the presence and support for more internal forms of expertise. For most, however, mathematics reform was relegated to the external “experts” because they viewed mathematics as a highly defined body of knowledge. Leaders remained on the outside of the mathematics Discourse and as a result, most everyone else was, too.
The practice of these school leaders reflected specific ideas about the nature of teaching and learning. The subject matter differences they identified, although important, do not represent universal truths. They exist as perceptions, which were influenced by leaders' own sense making and enacted through their reform strategies and the choices they made about engaging and validating teacher expertise. These administrators thought the school community had the expertise to reform literacy instruction, but the expertise for reforming mathematics instruction lie outside the school.

I'm not sure how widespread these perceptions are regarding mathematics reform; nevertheless, they are cause for concern. Three issues are problematic. First, the traditional notion of mathematics and the reform practices implemented by the leadership in Burch and Spillane's (2003) study were incongruent with the mathematics reform proposed by the National Council of Teachers of Mathematics (NCTM, 1989, 2000) concerning best practices in mathematics instruction. Second, the enactment of mathematics reform in this manner devalued the role of the teacher as a professional. Third, there was minimal focus on whether the students were learning, as advocated by DuFour (2002). If principals and other school leaders are to lead reform movements (Fullan, 2009), then our decisions must be informed ones.

I decided to take up DuFour's challenge to research student learning, specifically in the mathematical domain. I framed the study in the field of the "new Literacy Studies," which is a plural set of social practices that replaces the traditional notion of literacy and mathematics as "in-the-head" cognitive processes with that of a socio-cultural approach, acknowledging the "full range of cognitive, social, interactional,
cultural, political, institutional, economic, moral, and historical contexts” (Gee, 2008, p. 2), after the collaborative work of Bernstein (1971; 2000), Halliday and Hasan (1989). More encompassing and deeper than previously conceptualized, Gee defined this new Discourse (with a capital “D”) as a way of “behaving, interacting, valuing, thinking, believing, speaking, and often reading and writing, that are accepted as instantiations of particular identities... by specific groups” (Gee, 2008, p. 3). In other words, I paid attention to more than just the literal meaning of the words Maggie and her students uttered. I also listened for their beliefs, goals, values, attitudes, and contextual innuendos that were all wrapped up in the deeper meaning of each spoken text.

Why an Interest in Mathematical Talk?

Life is a synthesis of experiences. We are a summary of our yesterdays, an interpretation of our today, in pursuit of our tomorrows. We quest to soothe our restless wonderment, asking, “Why?” and “What if?” driving onward seeking connections, insight, and purpose. This study draws from my multiple identities lived over three decades as a professional educator, first as an elementary classroom teacher, then as a reading specialist, university reading methods instructor, consultant for a national text publisher, Reading Recovery® teacher, district language arts coordinator, professional development provider, department of education curriculum writer, private literacy consultant, regional reading specialist, district curriculum coordinator, elementary principal, regional agency board director, and member of a university mathematics department.
Intrigued by observing students as language learners, I began to study the topic in earnest during my graduate master’s work in the area of reading education. This culminated with my thesis research in the area of emergent literacy focused on teacher talk during kindergarten sharing time. Following graduation, I was hired to teach university courses in the area of language development and continued to be a student of the language-learning process as I delved more deeply into the research. When I returned to teaching at the elementary level, I honed my observational kid-watching skills in my literacy classroom and as a Reading Recovery® teacher. Later, as a district language arts coordinator, Area Education Agency language arts consultant, and principal, I’ve had the unique opportunity to conduct hundreds of classroom observations. I’m always fascinated whether I participate in or observe the teaching-learning process. Something magical happens when it’s going well.

The seed for this particular study was planted those many decades ago, although I only became cognizant of its growth during the past several years when I was the lead literacy specialist in a university mathematics department. Realizing that some considered this paradoxical position, nevertheless, I believed that the divide between literacy and mathematics did not need to be the chasm described in Snow’s (1959) essay on Two Cultures. As I listened to my colleagues discuss ways to help students make sense of mathematics, I heard reasons for the selection of problem contexts to provide purpose for solving mathematics problems. I heard debate over visual representations of problems that would make plausible thinking visible. I heard value placed upon divergent paths to solutions. These were all features of reform-based mathematics
instruction. I listened to internalize the Discourse of mathematics at a deeper conceptual level, to try making sense that is so desired by this group of competent mathematics educators, but I often found myself struggling to do so.

My challenge in making sense of the mathematics my colleagues presented was not due to my own lack of mathematical background or disdain for the topic, nor from their attempt to exclude, but from what Gee (2008) described as the problem of "recognition and being recognized" (p. 159) which is the key [emphasis added] to Discourses (Gee, 2005). Discourses are identity tool kits complete with socially shared ways of acting, talking, and believing.

A paraphrased example that Gee (2008) has used to make this point is simple but powerful. If I (female, middle-aged, educator) walk into a biker bar, I may speak the language of the setting, but I don’t speak the Discourse. The biker community would not recognize me as a “real biker babe,” but rather as an outsider by my appearance, actions, and use of language. There are innumerable Discourses in any society. Being – doing a Harley motorcycle rider, a university professor, a gamer, a dancer, a mathematics student, a neuroscientist, and a deer hunter are all Discourses (Gee, 2005; 2008). “Discourses are all about how people ‘get their acts together’ to get recognized as a given kind of person at a specific time and place” (Gee, 2008, p. 155).

In the mathematics department, I was recognized as a “real” literacy specialist, a “real” school administrator, but not a “real” mathematics educator even though I had taught mathematics early in my career. What gave me away was my need for explicitness in language to provide me with clarity of the posed problem, the possibilities
of thinking, and ultimately, problem solving. When people share a common Discourse, the need for explicitness is diminished. Implicitness and vague reference are used to signal that the "utterance is in an informal social language used to achieve solidarity" (Gee, 2005, p. 42). I did not share the Discourse. I was not recognized as an insider. Upon reflection of this experience, I wondered how students become insiders, or recognized members, of academic Discourses in school. I also wondered what happened if they didn’t.

This is an important question to ask in light of research regarding the role of language, the social conditions of learning, and its impact on the identity of the learner as a member of the larger academic community (Cobb, 2004; Gee, 1999; Moje & Lewis, 2007). For many years, mathematics was seen as a discourse in which only privileged white males were invited to participate. If, as an educational leader, I am to ensure all students are valued members of the learning community, I must make certain every student’s voice is included in the discussion.

**An Outside View on Becoming an Insider**

To study this question seriously, I realized being an outsider was a good place to be if I were to “learn” about mathematics education, as defined by Gee’s (2008) Learning Principle. He wrote,

One cannot critique one Discourse with another one (which is the only way to seriously criticize and thus change a Discourse) unless one has meta-knowledge about both Discourses.... Teaching that leads to learning uses explanations and analyses that break down material in to its analytic “bits” and juxtaposes diverse Discourses and their practices to each other (pp. 177-178).
While I didn’t plan to “change” a Discourse, I did plan to make connections across Discourses as I noticed patterns and constructed new ideas. This outside perspective afforded me the opportunity to take notice of things I might miss if I were an insider due to the hegemony of the mathematics Discourse (Apple, 1990). I attempted to synthesize the theories and research drawn from the fields of mathematics and literacy in order to provide a more comprehensive discussion of the context, concepts, and challenges. Interestingly, many connections can be made across the disciplines, but, for obvious reasons, very different perspectives are voiced. As Veel (1999) explained,

Like the language itself, research on the language of mathematics is itself very different from research on other areas of language education. Most descriptions of mathematical language are to be found in journals of mathematics education, not in journals of language education. To the mathematician, the research on mathematical language in language education journals frequently appears to be woefully inadequate in its understanding of mathematical knowledge. To the linguist, the research in mathematics journals seems horribly simplistic in the role it assigns to language in learning. The result of all this is that language educators and mathematicians rarely talk to one another. (p. 185, italics in original)

Fortunately, this is starting to change as “mathematics educators and language and literacy educators... begin to forge ongoing collaborative relationships” (Cobb, 2004, p. 337).

As a “Learner” of a Discourse, I have different goals and engage in different practices than when I “Acquire” a Discourse. We are all born into our primary Discourse. For most of us, we acquire secondary Discourses by engaging with social institutions beyond the family such as schools, workplaces, stores, government offices, businesses, churches, and so forth (Gee, 2008). When we value certain aspects of these secondary Discourses, we incorporate them into the primary Discourse (through early
socialization) of our children. This is what mathematics students do when they become insiders. Gee (2008) distinguished between learning and acquisition when he wrote,

Any Discourse is... for most people most of the time mastered only through acquisition, not learning. Thus, literacy (fluent control or mastery of a... Discourse) is a product of acquisition, not learning, that is, it requires exposure to models in natural, meaningful, and functional settings, and (overt) teaching is not liable to be very successful – it may even initially get in the way. Time spent on learning and not acquisition is time not well spent if the goal is mastery in performance. (p. 177)

If we apply Gee’s (2008) definition to academic Discourse, then learning to “talk science” or “talk math” involves more than just thinking, speaking, reading, and writing as a set of linguistic forms. It also involves actions, behaving and interacting as students “do science” or “do mathematics,” and it involves beliefs and values (Lampert, 1990) that “are more important than mere skills for successful later entry into specific ... Discourses ‘for real’ ” (Gee, 2008, p. 158, italics original). Learning, then, results in a shift in identity. Deep, participatory learning involves not only the “stuff” of the discipline, for example mathematics content, but also how to think and act like a mathematician, even if one does not eventually enter the mathematics profession. Learning both involves and requires participation in a Discourse community. Discourse communities include face-to-face groups as well as ideational groupings that share ways of knowing, thinking, believing, acting, and communicating (Moje & Lewis, 2007).

Throughout this study, I attempted to construct accurate Discourses (with a capital D) and synthesize important concepts in the fields of leadership, literacy, and mathematics from the research. I also attempted to take steps toward synthesizing three areas of concern: that the nature of mathematical knowledge be well understood, that the
role of language in constructing and exchanging this knowledge be fully appreciated, and that educational leaders perpetuate a laser-like focus on the learning processes in their schools.
CHAPTER II

WHAT COULD BE, WHERE WE ARE, AND HOW WE GOT HERE

According to popular belief, "the facts of mathematics are universally true, its procedures universally correct, and both completely independent of culture" (Schoenfeld, 2004, p. 253). Schoenfeld reinforced this point with the following example. Suppose you draw a triangle on a flat surface. No matter what triangle you draw, the sum of its interior angles will be a straight angle. Math is math. It is a discipline dominated by computation and rules without reasons.

In contrast, another definition to consider is that "mathematics is a science of pattern and order" (National Research Council [NRC], 1989, p. 31). When mathematicians find and explore the regularity or order and then make sense of it, mathematics becomes a social construction of knowledge. Mathematical activity can be seen as a form of linguistic performance (Brown, 1999) when verbs such as "explore," "investigate," "conjecture," "solve," "justify," "represent," "formulate," "discover," "construct," "verify," "explain," "develop," "describe," and "use" become part of the language of doing mathematics (Van de Walle, Karp, & Bay-Williams, 2010).

Embodying the very essence of Discourse, the particular definition to which you ascribe influences the way you behave, interact, value, think, believe, speak, and often read and write about math. It is this second definition that underlies the type of experience the National Council of Teachers of Mathematics envisioned for students when they first proposed reform of mathematics education through a trio of documents: Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989),
Professional Standards for Teaching Mathematics (NCTM, 1991), and Assessment Standards for School Mathematics (NCTM, 1995), which were further refined and updated in Principles and Standards for School Mathematics (NCTM, 2000). NCTM (2006) later published Curriculum Focal Points to provide additional guidance to practitioners.

A Vision for School Mathematics

The Standards (NCTM, 2000) were based on the principles of Equity, Curriculum, Teaching, Learning, Assessment, and Technology and addressed the content areas of Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability. In addition to guiding content, the Standards also described the process standards of Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. The Council proposed the following vision for school mathematics:

Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all, with accommodations for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (p. 3)
The Council admitted this vision is not realized in the vast majority of classrooms, schools, and districts across the nation, which are often places of elitism, exclusion, confusion, and disengagement.

Student Achievement in Mathematics: A Broken System

For the past 30 years, the research evidence is consistent and compelling concerning the weaknesses in mathematical performance of U.S. students. They may be able to perform computational exercises, but have a limited understanding of basic mathematical concepts (NRC, 2001). According to the most recent results of the National Assessment of Educational Progress (NAEP; NCES, 2009a) less than one-third of fourth- and eighth-grade students demonstrated proficiency in mathematics.

American students achieved in mathematics at a "mediocre level by comparison to peers worldwide (NMAP, 2008, p. xii). Trends in Mathematics and Science Study (TIMSS) reported mathematics and science achievement of U.S. fourth- and eighth-grade students compared to that of students in other countries (Gonzales et al., 2008). The latest international mathematics data for TIMSS, administered in 2007, was released in September 2009. Only 10% of U.S. fourth grade students scored at the Advanced level falling behind fourth-grade students from seven countries ($p < .05$) including Singapore, Hong Kong Special Administrative Region (SAR) of the People’s Republic of China, Chinese Taipei, Japan, Kazakhstan, England, and the Russian Federation. Performance by U.S. eighth-grade students was less promising, with six percent achieving at an Advanced level falling behind eighth-grade students from a slightly different list of seven
countries \((p < .05)\). These included Chinese Taipei, Republic of Korea, Singapore, Hong Kong SAR, Japan, Hungary, and the Russian Federation.

The Program for International Student Assessment (PISA) is a system of international assessments that focus on 15-year-olds' capabilities in reading literacy, mathematics literacy, and science literacy and is sponsored by the Organization for Economic Cooperation and Development (OECD; Baldi, Jin, Skemer, Green, & Herget, 2007). The most recent PISA assessment of mathematics literacy was administered in 2006, with results reported in December 2007. Fifteen-year-olds from 31 jurisdictions (23 OECD and eight non-OECD jurisdictions) had a higher average score than their U.S. peers.

Furthermore, significant disparities in mathematics achievement related to race and socioeconomic status persist (Baldi et al., 2007; Gonzales et al., 2008; National Center for Education Statistics [NCES], 2009a). The U.S. Department of Education report, Status and Trends in the Education of Hispanics (NCES, 2003) revealed that in 1998, about one-quarter of Hispanic, African-American, and American Indian/Alaska Native students completed advanced mathematics courses, whereas about one-half of White and Asian/Pacific Islander students did so. In order to succeed in high-level mathematics such as Algebra II and Calculus, all students must have access to and succeed in such gateway courses as Algebra I (Holloway, 2004). However, this report (NCES, 2003) showed that 59% of Hispanics completed only middle-level mathematics courses, eight percent took low-level courses, and seven percent completed nonacademic courses or no mathematics courses at all. When data was further disaggregated, it
revealed that Hispanic and African American students were over-represented in the demographic category of poverty (Fass & Cauthen, 2008).

Such dire results caused leaders from the professional mathematics community to declare that the “the system that translates mathematical knowledge into value and ability for the next generation – is broken and must be fixed” (NMAP, 2008, p. xiii). The National Math Advisory Panel (NMAP, 2008) warned,

During most of the 20th century, the United States possessed peerless mathematical prowess – not just as measured by the depth and number of the mathematical specialists who practiced here but also by the scale and quality of its engineering, science, and financial leadership, and even by the extent of mathematical education to its broad population. But without substantial and sustained changes to its educational system, the United States will relinquish its leadership in the 21st century. … it is yet more fundamental to recognize that the safety of the nation and the quality of life – not just the prosperity of the nation – are at issue. (p. xi)

Members of the business community called for “a renewed commitment to improving science, technology, engineering, and mathematics (STEM) education in the United States” (Tapping America’s Potential, 2008) and the mathematics community, itself, appealed for national policy (NMAP, 2008).

Closing the Achievement Gap or Igniting a War?

Implementing NCTM’s (2000) proposed vision would not happen without an ambitious agenda. The Council identified multiple prerequisites if the goal were to be reached including a solid mathematics curricula, competent and knowledgeable teachers who could integrate instruction with assessment, education policies that enhanced and supported learning, classrooms with ready access to technology, and a commitment to both equity and excellence.
One might expect some resistance to change efforts of this magnitude, (Hall & Hord, 2011; Joyce & Showers, 2002; Marzano, Waters, & McNulty, 2005), but the notion of “resistance” was an understatement. Teachers were being asked to teach in new ways and not given much support (Schoenfeld, 2004). When teachers feel uncomfortable with a curriculum they are not prepared to implement, they will either be reticent to do so, if at all, or they will find a way to “domesticate” it (Joyce & Showers, 2002). Parents expressed frustration, feeling ill prepared to help their children with homework as these new practices and new materials were different from the traditional texts they had experienced in school. Instead of sequential presentations of formulas and pages of practice problems, new texts featured “colorful illustrations, assignments with fun, lively names, and sidebars discussing topics from the environment to Yoruba mathematics” (Rosen, 2000, p. 61, as cited in Schoenfeld, 2004, p. 272). To top it off, the suggestion of reform divided the professional mathematics community. It should be noted, however, “the resistance to change [was] not based on the purported success of traditional curricula…. but on the fear that the replacement [would] be even worse” (Schoenfeld, 1994, p. 73), though no data confirmed these fears. Rigor was pitted against relevance. Content weighed against pedagogy (Klein, 2003). Democracy countered elitism. Battle lines were drawn. “War” was declared (Schoenfeld, 2004)!

Before long, the general public entered the fray. American education became every bit as polarized as its political system (Wallis, 2006), aligning ideologically along party and religious lines (Schoenfeld, 2004). On the crimson right, conservatives called for “back to the basics.” They believed hard work built character; therefore, school was
not meant to be fun. They expected students to sit still, listen to their teacher, and drill their times tables. On the opposite, indigo extreme, progressives believed that school should take its cues from a child's interests. Learning should be like breathing, natural and relaxed (Wallis, 2006).

Unfortunately, in math wars, as in all wars, there are casualties to innocent parties (Schoenfeld, 2004). When extremists battle, America's children who should be well served by mathematics education, end up whiplashed as they are jerked sharply from right to left and back again with each pendulum swing (Wallis, 2006). The Discourse was turning into a shouting match.

**Ideological Tension: A Deep Divide**

It gives one pause to understand how significant the historical, social, political, and economic context shapes the seemingly benign classroom mathematical discussions between teachers and students as they explore such topics as irrational numbers and parabolas. Four voices, or perspectives of mathematics battled for dominance of the Discourse since the early 1900s (Schoenfeld, 2004; Stanic, 1986). These voices from the past play a significant role in the mathematics Discourse of the present. This on-going meta-narrative determined what knowledge was of most worth, the nature of mathematics that was learned, and who got to learn it.

As cited above, the immediate origins of the mathematics debate can be traced to NCTM's proposed reform movement (1989, 1991, 1995, 2000). A historical perspective suggests, however, that underlying issues dividing the mathematics community and the general public go back much farther and may not be much different from past
disagreements over the best ways to educate our nation’s children (Klein, 2003). As noted over nine decades ago, “Few subjects taught in... school elicit more contradictory statements of view than does mathematics. What should be taught, how much of it, to whom, how, and why are matters of disagreement. There is every variety of position” (National Education Association [NEA], 1920, p. 9).

In the following section, each perspective will be placed in historical context followed by a brief description of how that voice dominated the thinking of that time. As each perspective took its turn at shaping mathematics education, it also took its turn swinging the proverbial educational pendulum.

When pendulums swing, they seem to return to the original place, but can never return to the original time. Perspectives are revisited, but as Donmoyer (1979) explained, “The conceptual clarity of the pendulum image ... tends to obscure differences in social, political, and economic forces at work during each period” (p. 555). Furthermore, Brown (1999) noted, “ideas are not inherited prepackaged and intact, but rather, each new generation will engage in tasks that give rise to new understandings of what might be seen as old ideas” (~ 4).

Remnants of many of these historical perspectives found their way into NCTM’s (2000) *Principals and Standards for School Mathematics*, and will be noted within each section. NCTM was a powerful voice in its own right. When it called for reform, members of each perspective hoped NCTM sided with them and felt threatened if they saw something that appeared to challenge their view. This was something that added ammunition to war chests and haunted the reform movement for years to follow.
Humanist Voice

The humanists were the singular voice of mathematics Discourse in the 19th century. Led by Charles William Eliot, president of Harvard University, and William Torrey Harris, U.S. Commissioner of Education, humanists believed in “mental discipline” and the ability to reason (Stanic, 1986). Learning mathematics enabled one to learn to think logically in general. They also believed in the cultural value of mathematics and saw merit in its study on that basis (Stanic, 1987, as cited in Schoenfeld, 2004).

Even though the humanists represented the traditional perspective of mathematics education, the era represented a significant time in its history. Issues raised by groups, such as the Conference on Mathematics of the Committee of Ten on Secondary Schools (NEA, 1893) that proposed a prescribed sequence of mathematics courses for high schools, and the committee appointed by the Chicago Section of the American Mathematical Society (NEA, 1899) that established a standardized mathematics curricula for high schools and academies, were not without controversy. According to Stanic (1986), the humanists represented the viewpoint that mathematics Discourse should be an important part of the school curriculum. Unfortunately, few students had access to it. Less than seven percent of 14-year-olds attended high school in 1890 (Stanic, 1987, p. 150, as cited in Schoenfeld, 2004, p. 256).

Traces of the humanist perspective appeared in the NCTM (2000) Principals and Standards document, which recognized one need for mathematics study had to do with mathematics as a part of our cultural heritage. “Mathematics is one of the greatest
cultural and intellectual achievements of humankind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects" (p. 4).

**Progressive Voices Challenge the Discourse**

As throughout the history of education, the trifocal tensions of society, subject matter, and the individual interacted to shape curricula in schools (Marsh & Willis, 2007). At the turn of the 20th century, Progressives confronted the conventional humanist perspective and became participants in the mathematical Discourse. Two psychologists, G. Stanley Hall and Edward L. Thorndike, a pediatrician, Joseph Mayer Rice, and a sociologist, Lester Frank Ward, laid the intellectual foundations for this challenge (Larabee, 2010; Stanic, 1986).

G. Stanley Hall initiated the child study movement in the 1880s. He believed that the pre-adolescent child developed to his or her best when allowed to go through the stages of evolution freely, and not forced to follow constraints (Grezlik, 1999). He urged that formal teaching of arithmetic be delayed until later in the school program believing the earliest school years should be focused on concrete experiences, which established readiness for later learning (Saracho & Spodek, 2008). This was based on his understanding that mathematics involved inductive rather than deductive reasoning, which was the common approach at the time.

*It is just because induction, in these sciences, is so rapid and complete that we have a body of conclusions now so extensive that they can be applied deductively. Their place in education rests upon understanding this. The child repeats the race — therefore, in his studies of numbers and quantities, his mind must be kept at first intuitive rather than in logical attitudes. Proof, which actually came late in the
development of the science of mathematics, must come late in the child.”
(Partridge, 1912, p. 260)

The new mathematics curriculum needed to be appropriate for the students' given stage of
development (Larabee, 2010), for boys, anyway. Hall asserted, “Girls preponderate in high-school Latin and algebra because of custom and tradition, whereas they preponderate in English and history classes... from inner inclination” (Hall 1903, p. 446, as cited in Kliebard & Franklin, 2003, p. 403). He made it clear that certain subjects were simply too strenuous for girls to bear causing “over-brained work,” questioning their ability to study mathematics and potential use of the subject.

Edward L. Thorndike, beginning in 1901, conducted a series of experiments that cast doubt on the value of mental discipline and the possibility of transfer of training from one activity to another (Klein, 2003). Instead of viewing curriculum as a medium for developing mental faculties, he argued that curriculum constituted the substance of learning, since he determined the transferability of knowledge was a myth. This meant that it no longer made sense to pursue a liberal education through Latin, poetry, or mathematics. Instead, educators needed to design curricula to match the abilities and future occupational roles of particular students (Larabee, 2010). Around that time, Joseph Mayer Rice, physician turned publicist, traveled the country studying schools. He harshly criticized schools for their methodology of teaching and advocated that they adopt a more “scientific” pedagogical method (Graham, 1966; Stanic, 1986). A culmination of his work, Scientific Management of Education was published in 1913 (Graham, 1966).
It was probably Lester Frank Ward’s experience with poverty and hard work that prompted him to devote his academic life to advocating for social justice. As a sociologist, Ward did not seem to think too highly of mathematics or mathematicians. Rather, the goals of equal opportunity and fair distribution of existing knowledge were much clearer than the content of the school curriculum that would help achieve those goals (Stanic, 1986). Ward advocated a planned, or “telic,” society (“sociocracy”) in which nationally organized education would be the dynamic factor (Lester Frank Ward, 2011). He believed that for economic development of society, it should institute a universal and comprehensive system of education, regulate competition, connect people together on the basis of equal opportunities and cooperation, and promote the happiness and freedom of everyone. His idea of promoting equality of women, social classes, and of races was seen as revolutionary for the time (Lester Frank Ward, 2008).

While the Progressives had a strong aversion for the traditional humanist curriculum, differences within the movement were fundamental. Three voices emerged. One voice was called the pedagogical progressives and aligned with Hall. They were led by John Dewey, William Heard Kilpatrick, and others, and fostered the Developmentalist perspective. For developmentalists, all subject matter became of secondary importance as the focus of the curriculum moved from bodies of knowledge to the individual student (Stanic, 1986). The second voice was dubbed the administrative progressives. Followers of E. L. Thorndike and Rice, led by David Snedden and others, developed the social efficiency perspective (Larrabee, 2010), which focused on the societal importance of mathematics (Stanic, 1986). The third voice was called the social meliorists. They were
led by Lester Frank Ward and others, but were the least developed interest group at the turn of the century. They did not pose the same threat to the humanists as the developmentalists or social efficiency educators did. Though these diverse perspectives distinctly impacted the field, many of the groups’ leaders were colleagues and served the profession concurrently.

The Developmentalist Voice

The pedagogical progressives, led by John Dewey, brought a new voice to the mathematics Discourse. They were focused primarily on developing a new process of teaching and learning in the classroom. These developmentalists grounded learning in the interests, needs, and developmental capacities of the individual student and organized child-centered instruction around the principle of stimulating the student’s natural desire to learn about the world through active engagement in discovery. Children were involved in self-directed projects and activities, with the focus on learning to learn rather than learning specific bodies of knowledge. They developed classroom processes that modeled and promoted values of community, cooperation, justice, and democracy (Larabee, 2010). According to the developmentalists’ perspective, mathematics should be taught “only insofar as developmental experience showed it to be necessary, and the extent of a person’s interests and abilities in mathematics were the fundamental criteria in determining the extent to which that person should study mathematics” (Stanic, 1986, p. 192).

In 1915, William Heard Kilpatrick, professor of education at Teachers College Columbia University and a protégé of John Dewey, was asked by the National Education
Association's Commission on the Reorganization of Secondary Education (CRSE) to chair a committee to study the problem of teaching mathematics in the high schools. The mathematics committee included no mathematicians and was composed entirely of educators (NEA, 1920).

Kilpatrick was most noted for developing the Project Method (Kilpatrick, 1918) for early childhood education, which organized curriculum and classroom activities around a subject's central theme. Consistent with Dewey, Kilpatrick believed children should direct their own learning according to their interests, allowed to explore their environment, and experience learning through the natural senses (Gutek, 2009).

This progressive philosophy was apparent in the NEA's (1920) preliminary report, *The Problem of Mathematics in Secondary Education*, which challenged the use of mathematics to promote mental discipline noting that nothing in mathematics should be taught unless its probable value could be shown, and recommended differentiation of the mathematics curricula according to interest and ability once students completed a common introductory course. Four groups of users of mathematics were distinguished:

(a) The "general readers" who will find their use of mathematics beyond arithmetic confined largely to the interpretative function.
(b) Those whose work in certain trades will make limited, but still specific, demand for the "practical" use of mathematics.
(c) Those whose practical work as engineers or as students of certain sciences requires considerable knowledge of mathematics.
(d) Those who specialize in the study of mathematics with a view either to research or to teach or to the mere satisfaction of extended study in the subject. (p. 15)

On first read, one could interpret the formation of these mathematics user groups as influenced by social efficiency thinking, described more fully below. One important
caveat distinguished this report, however. It noted, "there must be no caste-like perpetuation of economic and cultural differences... keep[ing] wide open the door of further study for those who may later change their minds" (NEA, 1920, p. 14). The Developmentalists strongly believed in student access to a broad range of educational and social opportunities in order to challenge the existing social structure (Larrabee, 2010).

Recognizing the developmental progression of mathematics education was at the heart of the NCTM Standards (2000). The introductory comments for each grade band, described children's maturation levels as they explored, talked about, and experienced mathematics. Children at the prekindergarten level through grade 2

...learn through exploring their world; thus, interests and everyday activities are natural vehicles for developing mathematical thinking.... Young students are active, resourceful individuals who construct, modify, and integrate ideas by interacting with the physical world and with peers and adults. They make connections that clarify and extend their knowledge, thus adding new meaning to past experiences. They learn by talking about what they are thinking and doing and by collaborating and sharing their ideas. (pp. 74 -76)

In grades 3 through 5,

[m]ost students enter .... with enthusiasm for, and interest in, learning mathematics. In fact, nearly three-quarters of U.S. fourth graders report liking mathematics (Silver, Strutchens, and Zawojewski, 1997). They find it practical and believe that what they are learning is important. (p. 143, citation in original)

In grades 6 through 8,

[a]s they enter adolescence, students experience physical, emotional, and intellectual changes that mark the middle grades as a significant transition point in their lives. During this time, many students will solidify conceptions about themselves as learners of mathematics – about their competence, their attitude, and the interest and motivation. These conceptions will influence how they approach the study of mathematics in later years, which will in turn influence their life opportunities. (p. 211)
In grades 9 through 12, students develop in multiple ways – becoming more autonomous and yet more able to work with others, becoming more reflective, and developing the kinds of personal and intellectual competencies that they will take into the workplace or into postsecondary education. (p. 287)

The Social Efficiency Voice

During this same time period, another voice joined the Discourse to oppose the humanists. The Social Efficiency perspective focused on mathematics’ role in society in contrast to the Developmentalists’ focus on the individual.

The findings of E. L. Thorndike were used to challenge the justification for mathematics as a form of mental discipline and contributed to the view that mathematics education should be for purely utilitarian purposes. One of the prominent leaders of this movement was David Snedden, professor of educational sociology at Teacher’s College Columbia University, and later Massachusetts’ Commissioner of Education. His 689-page tome, *Educational Sociology* (Snedden, 1922) became a standard in the field. In it, Snedden promoted the idea that each subject (e.g., history, English, Latin, mathematics, science) had to meet the test of social usefulness and that the efficient society resembled a winning "team group" with above-average people as leaders and the rest as followers; each group was trained for its specific role and fulfilled its proper function (Knoll, n.d.).

The social efficiency perspective saw education as a means to solve major social problems, particularly to maintain social order and promote economic growth. They thought of schools as the place to prepare students for their predetermined social roles.

Early 20th-century America was impacted by three social factors. First, there was a continuing influx of immigrants. Second was America’s expanding role in world
affairs reflected in its involvement in World War I. Third was the devastation of the Great Depression (Stanic, 1986). Emerging social conditions included a highly differentiated industrial economy and a growing urban population, stratified by class and ethnicity.

The social efficiency voice argued that these conditions required a newly stratified structure of secondary schooling and called for the tools of John Franklin Bobbitt and W. W. Charters “scientific curriculum-making” (Stanic, 1986, p. 196) to create distinct forms of curriculum for students with different levels of intelligence and different social trajectories in order for them to become productive workers in the wide variety of occupations that characterized the new economy. Snedden (1922), concerned about the burgeoning numbers of school enrollees lamented,

Again, all children between twelve and fourteen are now required, or soon will be required, to attend school. In the “good old days” large proportions of the children of these ages who had lost interest in school, or who were mentally slow or morally difficult, were either allowed silently to fold their tents and steal away, or else they were no less quietly “elbowed out” of school life. Now we have them all with us; and since the mountain of their abilities (or deficiencies) will not come to the Mohammds of uniform school standards, these standards must go to them. That change of front spells more kinds of curriculum flexibility than we have yet dreamed of. (p. 502)

Snedden (1922) raised the following questions: “How much arithmetic is needed by all? For what purposes? How many and what kinds of people should be required or advised to study the several secondary school mathematical subjects as now standardized?” (p. 493). “Would realistic ‘job analysis’ studies show that in these callings any considerable need will be encountered for the use of applications of algebra or geometry?” (p. 509).
He acknowledged, "Elementary arithmetic clearly ranks with reading and writing as among the most valuable and necessary tools of civilized society (p. 495), but also stated, "Algebra...is a nonfunctional and nearly valueless subject for 90% of all boys and 99% of all girls – and no changes in method or content will change that" (Osborne & Crosswhite, 1970, p. 211, as cited in Klein, 2003, Historical Outline: 1920 to 1980 section, ¶ 7). Snedden made it a point that he saw no need for required mathematics beyond sixth-grade arithmetic (Stanic, 1986). Fortunately, his proposal was not fully implemented, but his ideas did influence the type of mathematics to which students would be exposed.

As Commissioner of Education in Massachusetts, Snedden made two important appointments that built support for his cause. First, he hired Charles A. Prosser as deputy commissioner for industrial education. In 1912, Prosser resigned this post to become full-time executive director of the National Society for the Promotion of Industrial Education and the primary author of the Smith-Hughes Act (1917) that established the aims and funding for a national system of vocational education. Also in 1912, Snedden appointed Clarence Darwin Kinsley as the Federal Board of Vocational Education’s agent for high schools. Kinsley was later named the chair of the National Education Association (NEA) Committee of Nine on the Articulation of High School and College and general chair of the NEA Commission on the Reorganization of Secondary Education (CRSE), the same commission for which Kilpatrick chaired the mathematics committee. Many of the social efficiency ideas were embodied in the CRSE’s (1918) influential report, The Cardinal Principles of Secondary Education (Larabee, 2010).
At the beginning of World War II, almost three-fourths of children aged 14 to 17 attended high school and nearly half of them graduated. This expanding population, once more, put pressure on the system. While the curriculum, up to this point had remained mostly unchanged, the student body was much more diverse and ill prepared than before (Stanic, 1987, as cited in Schoenfeld, 2004).

By the mid-1940s, the Life Adjustment Movement emerged. Charles Prosser, friend of Snedden who helped pass the 1917 Smith-Hughes National Vocational Education Act, led the cause. The Life Adjusters believed that secondary schools were "too devoted" to an academic curriculum and should prepare students for the world of work (Loss & Loss, n.d.). It was estimated that 60% or more of all public school students lacked the intellectual capability for college work or even for skilled occupations and would need a school curricula that prepared them for every day living (Ravitch, 2000, as cited in Klein, 2003).

By 1949, the Life Adjustment Movement had substantial support among educators, and was touted by numerous federal and state education agencies. In 1951, and again in 1954, the United States Office of Education’s Commission on Life Adjustment Education for Youth published reports used as blueprints for action. Some educators suggested that in order to avoid stigmatizing the students in these programs, non-academic studies should be available to all students (Klein, 2003). The Life Adjustment movement succeeded in instituting its "therapeutic curricula – geared toward the development of personal hygiene, sociability and personality, and habits of mind" (Loss & Loss, n.d., Life Adjustment Progressivism section ¶ 2).
Appropriate high school courses included mathematics programs focused purely on practical problems such as consumer buying, insurance, taxation, and home budgeting, but not on algebra, geometry, or trigonometry. The students in these courses would become unskilled or semiskilled laborers, or their wives, and they would not need an academic education. Instead they would be instructed in home, shop, store, citizenship, and health. Life Adjustment advocates promised they could meet the needs of all American students (Ravitch, 2000, as cited in Klein, 2003). Once again, a significant proportion of the population was denied participation in the mathematics Discourse and consequently, denied access to a comprehensive mathematics education.

The legacy of this movement is still seen in schools: curriculum tracking in secondary schools, differentiation of instruction to the academic skills and social trajectories of individual students, the use of standardized testing for student placement, and the shift from purely academic studies to those of a more practical (e.g., vocational) nature (Larabee, 2010).

The Principals and Standards for School Mathematics (NCTM, 2000) supported the practical application of mathematics to real world matters. “The need to understand and be able to use mathematics in everyday life and in the workplace has never been greater and will continue to increase” (p. 4). NCTM (2000) acknowledged mathematics was important for life. “Knowing mathematics can be personally satisfying and empowering. The underpinnings of everyday life are increasingly mathematical and technological. For instance, making purchasing decisions, choosing insurance or health plans, and voting knowledgeably all call for quantitative sophistication” (p. 4). The
organization confirmed the need for mathematics in the workplace. "Just as the level of mathematics needed for intelligent citizenship has increased dramatically, so too has the level of mathematical thinking and problem solving needed in the workplace, in professional areas ranging from health care to graphic design" (NCTM, 2000, p. 4).

Traces of the social efficiency perspective's agenda were seen in NCTM's (2000) *Principles and Standards for School Mathematics*. The introductory section for the *Standards* for grades 9 through 12 promoted differentiation of instruction within the core curriculum and advanced courses that could be considered tracking.

High school students with particular interests could study mathematics that extends beyond what is recommended here in various ways. One approach is to include in the program material that extends these ideas in depth or sophistication. Students who encounter these kinds of enriched curricula in heterogeneous classes will tend to seek different levels of understanding. They will, over time, learn new ways of thinking from their peers. Other approaches make use of supplementary courses. For instance, students could enroll in additional courses concurrent with the program. Or the material proposed in these Standards could be included in a three-year program that allows students to take supplementary courses in the fourth year. In any of these approaches, the curriculum can be designed so that students can complete the foundation proposed here and choose from additional courses such as computer science, technical mathematics, statistics, and calculus. Whatever the approach taken, all students learn the same core material while some, if they wish, can study additional mathematics consistent with their interests and career directions. (p. 289)

The Assessment Principle placed a high priority on assessment:

When teachers have useful information about what students are learning, they can support their students' progress toward significant mathematical goals. The instructional decisions made by teachers - such as how and when to review prerequisite material, how to revisit a difficult concept, or how to adapt tasks for students who are struggling or for those who need enrichment - are based on inferences about what students know and what they need to learn. Assessment is a primary source of the evidence on which these inferences are based, and the decisions that teachers make will be only as good as that evidence. (p. 23)
And the introductory section of the *Standards* for grades 9 through 12 addressed the functional nature of mathematics in the workplace.

Most advanced high school mathematics has rigorous, interesting applications in the work world. For example, graphic designers routinely use geometry. Carpenters apply the principles of trigonometry in their work, as do surveyors, navigators, and architects.... Algebra pervades computing and business modeling, from everyday spreadsheets to sophisticated scheduling systems and financial planning strategies. Statistics is a mainstay for economists, marketing experts, pharmaceutical companies, and political advisers. (Hoachlander, 1997, p. 135, as cited in NCTM, 2000, p. 288)

**The Humanist Voice Talks Back**

Mathematics was seen as a foundation for the nation's military and economic preeminence. In times of perceived national crisis, mathematics curricula received significant attention. This was most memorable during World War II, when it became a public scandal that so many military recruits lacked the basic arithmetic skills needed for bookkeeping and gunnery (Klein, 2003), nor had the technical expertise to support military supremacy in the cold war years (Lappan, 1997). Changes in society at large worked against the Life Adjustment agenda. As a result of World War II, the appearance of radar, cryptography, navigation, atomic energy, and other technological advances changed the economy and underscored the importance of mathematics in the modern world (Klein, 2003). It was also the era of Joseph McCarthy's communist witch-hunt. The perceived fondness for feel-good classroom instruction, along with international understanding through education, infuriated conservative 1950s America (Loss & Loss, n.d.). Parents wanted their own children educated and not merely adjusted (Klein, 2003).

Historian Arthur Bestor led the charge against the Life Adjustment's perspective of anti-intellectualism, arguing in his *Educational Wastelands* (1985/1953) and *The*
Restoration of Learning (1955), that the prior decade's emphasis on vocational instruction and life management skills marginalized the traditional core subjects. He felt it was impossible to be a fully educated person without at least some exposure to traditional liberal studies.

Joined by Robert Maynard Hutchins, president of the University of Chicago, advocate of the Great Books curriculum, and James Bryant Conant, president of Harvard University, the tone of the national conversation on education changed dramatically as more educators and public officials began to rethink the direction of American education. The dominant educational perspective once again revisited the 19th-century humanist perspective of education as mental discipline as the traditional academic studies in the liberal arts, mathematics, and the hard sciences were embraced. (Loss & Loss, n.d.). In the Discourse of mathematics, only the humanist voice could be heard.

In response to this concern, the National Science Foundation (NSF) was established in 1950, to support mathematics, science, and mathematics and science education. With NSF support, a range of curricula with "modern" content was produced. It was dubbed "New Math" because for the first time, mathematicians were actively contributing to K-12 school mathematics curricula and some of the content was actually new. Aspects of higher-level mathematics including set theory, modular (clock) arithmetic, and the symbolic logic of computer languages were embedded in the new curriculum (Schoenfeld, 2004). These efforts received little attention until 1957, when the Soviet Union caught the United States off guard with its successful launch of the Sputnik satellite (Klein, 2003). The media declared this a national humiliation. One
billion dollars were allocated for education through the *National Defense Education Act of 1958*, with provisions, among others, to improve K-12 mathematics education. That same year, the American Mathematical Society established the School Mathematics Study Group (SMSG) to develop a new curriculum for high schools. The NCTM also created its own curriculum committee, which published *The Secondary Mathematics Curriculum in 1959* (NCTM, 1959), along with many other groups that emerged during this period.

By the early 1970s, New Math was dead. Programs that dealt with number bases other than base ten, as well as a relatively heavy emphases on set theory or more unusual topics, tended to confuse and alienate even the most supportive parents. There were instances in which abstractness for its own sake was overemphasized leaving many teachers ill equipped to deal with the demanding content of the New Math curricula (Klein, 2003).

The NCTM's (2000) *Principles and Standards for School Mathematics* supported the scientific and technical community. "Although all careers require a foundation of mathematical knowledge, some are mathematics intensive. More students must pursue an educational path that will prepare them for lifelong work as mathematicians, statisticians, engineers, and scientists" (p. 4).

NCTM (2000) also reminded educators to avoid the mistakes of the New Math era by focusing on mathematics content and processes that are worth the time and attention of students. A long list of possible ideas, skills, concepts, and processes are delineated, but nowhere was set theory, modular arithmetic, or symbolic logic mentioned.
The rigors of discrete mathematics, found in computer sciences, were addressed, however. A Discrete Mathematics Standard for students in grades 9 through 12 was included in the 1989 *Curriculum and Evaluation Standards for School Mathematics*, but by the 2000 edition, only three areas, combinatorics, *iteration* and recursion, and vertex-edge graphs were distributed across the *Standards*. “Combinatorics is the mathematics of systematic counting. Iteration and recursion are used to model sequential, step-by-step change. Vertex-edge graphs are used to model and solve problems involving paths, networks, and relationships among a finite number of objects” (NCTM, 2000, p. 31).

**Developmentalist Voices Translated**

All the while the social efficiency supporters and humanists were taking turns reforming mathematics, two prominent voices of the Developmentalist perspective were speaking from across the globe. Jean Piaget and Lev Vygotsky would significantly influence mathematics education, but first their ideas had to be translated into English. Swiss psychologist, Jean Piaget, developed two important theories in the late 1920s (see Piaget, 1928) that were central to the *school* of cognitive theory known as "cognitive constructivism," though these ideas were not “discovered” until the late 1950s after translation into English (see Inhelder & Piaget, 1958). Piaget developed the notion that children are not “blank slates” but rather creators of their own knowledge. Integrated networks, or “cognitive schemas” are both the product of constructing knowledge and the tools with which additional knowledge can be constructed. Piaget suggested that schemas *might change* in two ways. Assimilation is the process of fitting in new concepts with prior knowledge, thus expanding the exiting network. Accommodation
takes place when a new concept does not fit in with the existing network, causing the brain to revamp or replace existing schemas (Van de Walle et al., 2010). This is an underlying theory for the notion of conceptual understanding, which will be addressed in the next chapter.

Piaget's Stage Theory proposed that children's thinking did not develop entirely smoothly, but rather in growth spurts that occurred at about 18 months, seven years, and again, at around 12 years. This was taken to mean that before these ages children were not capable, no matter how bright, of understanding things in certain ways (Atherton, 2010). Developmentalists used Piaget's stages as the basis for scheduling the school curriculum. Topics such as algebra were not taught until students became "formal thinkers" beginning around age 12 (Schoenfeld, 2004). Later scholars determined Piaget's schedule was too rigid. Many children managed concrete operations earlier than he thought, and some people never attained formal operations, or at least were not called upon to use them (Atherton, 2010).

Piaget's peer, Russian psychologist Lev Semenovich Vygotsky (1962; 1978) agreed with much of Piaget's constructivist thought. Vygotsky, however, was a "social constructivist" who criticized Piaget's idea that the construction of knowledge was an independent act. He recognized the role of language and the social role of other people (i.e., adults, peers) in facilitating children's learning. Vygotsky is best known for his Zone of Proximal Development (ZPD), defined as "the distance between the actual developmental level as determined by independent problem solving and the level of
potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86).

Another major concept in sociocultural theory is "semiotic mediation," as revealed through oral or written discourse. It is the "mechanism by which individual beliefs, attitudes, and goals are simultaneously affected by and affect sociocultural practices and institutions" (Forman & McPhail, 1993, p. 215). Social interaction is essential for mediation. The community of learners is affected by the culture the teacher creates and also the broader social and historical culture of their peers. Thus, "learning is dependent on the learners (working in their ZPD), the social interactions in the classroom, and the culture within and beyond the classroom" (Van de Walle et al., 2010, p. 21).

Cobb (1994) maintained that both the constructivist and sociocultural perspectives were complementary and useful to mathematics education. He argued that theories developed from the constructivist perspective focus on what students learn and the processes by which they do so, while the sociocultural perspective informs theories of the conditions for the possibility of learning.

Piaget's schema theory played a major role in mathematics education (Van de Walle et al., 2010), forming the basis for The Connection Standard in Principles and Standards for School Mathematics (NCTM, 2000).

Instructional programs from prekindergarten through grade 12 should enable all students to recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics in context outside of mathematics. (p. 64)
Vygotsky's (1962; 1978) work influenced NCTM's thinking regarding language, curriculum, and the role of the teacher. According to NCTM (2000), the role of language and social interaction played an important part in mathematics education and included a process standard addressing the role of communication. The Communication Standard in Principles and Standards for School Mathematics (NCTM, 2000) stated that,

Instructional programs from prekindergarten through grade 12 should enable all students to organize and consolidate their mathematical thinking through communication; communicate their mathematical thinking coherently and clearly to peers, teachers, and others; analyze and evaluate the mathematical thinking and strategies of others; and use the language of mathematics to express mathematical ideas precisely. (p. 60)

NCTM (2000) hinted at Vygotsky's (1978) ZPD when teachers were prompted “to support students without taking over the process of thinking for them and thus eliminating the challenge” (p. 19). The NCTM (2000) did not advocate, as the followers of Piaget advised, that children wait until age 12 to learn algebraic principles. “The mathematics curriculum must be coherent.... and well articulated across the grades” (p. 11). “Each of these ten Standards applies across all grades, prekindergarten through grade twelve” (p. 30). Algebra at the prekindergarten level may involve “sort[ing] stickers into groups having similar traits such as color, size, or design and order[ing] them from smallest to largest” (p. 91).

Social Efficiency Voices In Humanists' Clothing

Despite the call for traditional curriculum in the 1950s, mathematicians' version of “new math” was not what was expected. Once again, the theme of the 1970s was “back to the basics.” While this sounded like the humanist voice was speaking up again, it was the social efficiency voice that sounded the alarm. In the 1950s, demands for
excellence were optimistic. As Bruner (1960) clarified, excellence "refers not only to schooling the better student but also helping each student achieve his optimum intellectual development" (p. 9). This time, the hope was for mere adequacy as different objectives, justification, and accountability made it clear it was a new era (Donmoyer, 1979).

As during earlier social efficiency periods, school enrollments surged. This time, baby boomers entered high school. These demographic pressures exacerbated earlier problems. "High school mathematics was (still) for the elite" (Schoenfeld, 2004, p. 264). Focused primarily on skills and procedures, the mathematics curricula resembled the traditional curriculum suggested in the Report of the Committee of Ten on Secondary Schools in 1893 (NEA, 1893): arithmetic in first through eighth grades, algebra in ninth grade, geometry in 10th grade, a second year of algebra with some trigonometry in 11th grade, and pre-calculus in 12th grade. The only sustaining change from the new math era was high school calculus for advanced students. Students who took the standard sequence were prepared for postsecondary education. Others were placed in "business math" or "shop math" in order to satisfy graduation requirements (Schoenfeld, 2004).

Where the case for educational reform in the 1950s and 1960s rested on national welfare, and after Sputnik, national survival, the reasons were much more individualistic in the 1970s. Donmoyer (1979) cited a California court case of an illiterate San Francisco high school graduate that provided the impetus for legislating mandatory graduation competencies. Where graduation competency requirements did exist, they were minimal. Most states required only one or two years of high school mathematics
(Schoenfeld, 2004), and in states such as Oregon, these were “not limited to general reading, writing, and computational skills, but include[d] items such as balancing a checkbook and filling out an income tax form” (Donmoyer, 1979, p. 556). Mathematics education hearkened back to the Life Adjustment perspective that reformers of the 1950s and 1960s feverishly argued against.

Starting in the mid-1970s, the majority of states created minimum competency tests in basic skills, and almost half of them required students to pass these tests as a condition for graduation from high school (Schoenfeld, 2004). While critics in the 1950s and early 1960s emphasized testing and even called for the establishment of a national examination system, the sluggish economy and centralization of policy making resulted in a demand for accountability that did not exist during the earlier era. Test scores were published in local newspapers and were tied to federal and state money (Donmoyer, 1979).

By the end of the decade, the results of “basics” instruction were in. Students demonstrated little ability to problem solve, which was not surprising, given that the curricula emphasized only mastery of core mathematical procedures, but the performance on the procedures had not improved either (Schoenfeld, 2004). Test scores bottomed out by the early 1980s and in response, the NCTM published An Agenda for Action (S. Hill, 1980). The NCTM proposed that the exclusive goal on basics was wrong and that primary goal of mathematics instruction was to have students develop problem-solving skills. “Back to the basics” was replaced by “problem solving.”
In the meantime, research in problem solving flourished. Academics from anthropology, artificial intelligence, cognitive psychology, computer science, education, linguistics, and philosophy joined to form a new interdisciplinary field of cognitive science (see Gardner, 1985). Studies of expert mathematicians showed that there were critical aspects of mathematical competence that were not addressed in the past. Previously it was thought that mathematical competence was directly related to what one "knows," the facts, procedures, and conceptual understandings of the field. Researchers learned that competent mathematicians used a wide range of problem-solving strategies to make sense of new problem contexts or access unknown solution methods, and persevered while grappling with tough challenges. They were also able to communicate the results of their mathematical work effectively, both orally and in writing (Schoenfeld, 2004).

Despite this extensive body of research, changes to textbooks were trivial (Schoenfeld, 2004). Problems such as "7 - 4 = ?" might be replaced by a sheet of exercises that looked like John had 7 apples. He gave 4 apples to Mary. How many apples does John have left?" (p. 258.) Often, a page or two of word problems were added to the end of chapters that went essentially unchanged.

Despite all the efforts by mathematics educators, two invisible forces were seen as roadblocks to high student achievement: textbooks and tests (NRC, 1989). While independence is the prima facie hallmark of U.S. education, it was largely a myth. It is true that educational policy is not set by the U.S. Department of Education, but by each of the 50 state departments of education and local independent school districts, a legacy
of constitutional authority that reserves to the states all matters not expressly granted to
the federal government. In reality, however, the United States had a de facto national
curriculum in mathematics chosen by members of the invisible committees in the
textbook adoption states of California, Texas, and New York and an assessment system
chosen by anonymous officials who selected standardized tests. Teachers taught only
what was in the textbook and students learned only what was on the test (NRC, 1989;
Schoenfeld, 2004). Textbooks were slow to adopt any changes as noted above regarding
the problem-solving debacle and tests were criticized for not measuring what mattered
most (Calkins, Montgomery, & Santman, 1998). Both were focused primarily on skills
and procedures. Millions of students were marginalized from serious mathematical
thought and they continued to be excluded from the Discourse of mathematics.

The NCTM (2000) Principals and Standards for School Mathematics took up the
cause and endorsed problem solving, with an entire process standard devoted to the topic.
Problem solving, according to the NCTM (2000) had a much more expanded meaning
than the word problems of the 1980s. In this context, problem solving meant engaging
students in a task for which the solution method was unknown in advance. By drawing
on their prior knowledge, students would develop new mathematical understandings.
Problem solving was to be the major means of learning and doing mathematics. Students
were expected to have frequent opportunities to formulate, grapple with, and solve
complex problems that required a significant amount of effort. They were also
encouraged to reflect on their thinking.

Instructional programs from prekindergarten through grade twelve should enable
all students to build new mathematical knowledge through problem solving; solve
problems that arise in mathematics and in other contexts; apply and adapt a
variety of appropriate strategies to solve problems; monitor and reflect on the
process of mathematical problem solving. (p. 52)

The benefits of problem solving included “ways of thinking, habits of persistence and
curiosity, and confidence in unfamiliar situations that will serve them well outside the
mathematics classroom” (p. 52).

The Social Meliorist Voice

While the social meliorists’ voice was a part of the Discourse since the turn of the
century, it spent most of that time listening. Finally, after years of watching the majority
of students excluded from participation, it was time to speak up. Social meliorists
focused on schools as potential sources of social justice, calling for “equality of
opportunity through the fair distribution of extant knowledge” (Stanic, 1987, p. 152, as
cited in Schoenfeld, 2004, p. 256.) As noted above, one of the early leaders of the
movement was Lester Frank Ward, professor at Brown University and the first president
of the American Sociological Association. Ward argued a benevolent government that
provided education to all and protected the weak from the strong best served the
economic development of society (Lester Frank Ward, 2008). The idea of a universal
and comprehensive system of education, in the face of an economic crisis, began to take
shape in the 1980s.

During this time, the U.S. economy plummeted while the deficit soared. To make
matters worse, Asian markets, especially Japanese, were thriving. In the midst of a major
economic recession, politicians diverted attention from failing policies and
mismanagement and began linking the economic problems with ineffective public schools.

The National Commission on Excellence in Education (NCEE) produced *A Nation at Risk: The Imperative for Educational Reform* (NCEE, 1983). It called for strengthened high school graduation requirements, and recommended standardized achievement tests, rigorous textbooks, increased homework, a longer school calendar, and highly qualified teachers. *It also acknowledged that state and local officials had the primary responsibility for financing and governing schools, but that the Federal government should have the primary responsibility to identify the national interest in education. The report concluded with a call to “all segments of our population” (NCEE, 1983, A Final Word section, ¶ 3) to rally behind educational reform in order to restore “America’s place in the world” (¶ 4). The committee articulated this as “preeminence in commerce, industry, science, and technological innovations” (NCEE, 1983, A Nation At Risk section, ¶ 1).

In 1985, the NRC established the Mathematical Sciences Education Board as a means to sustain attention on the issues of mathematics instruction. The Council produced a series of reports. *Everybody Counts* (NRC, 1989) recognized the central role of mathematics in the economic growth of the country. It called for all students to become mathematically literate in the information age, including women, minorities, and the disabled, citing “the need for equity in opportunity and for excellence in results” (p. 28-29). The following year, *A Challenge of Numbers* (Madison & Hart, 1990), was published, which detailed the troubling demographics surrounding the traditional
curriculum. The attrition rate from mathematics, from ninth grade on, was roughly 50% per year, and worse for Latinos and African American students.

Knowledge of any type (Hurn, 1993; Turner, 1960), but specifically mathematical knowledge was seen as a powerful medium for social access and upward social mobility (Schoenfeld, 2004). From a functionalist viewpoint, education in a democratic society "reduces intolerance and prejudice, and increases support for civil liberties" (Hurn, 1993, p. 46). Consequently, lack of access to mathematics was a barrier that left people socially and economically disenfranchised. Civil-rights worker, Robert Moses (Moses & Cobb, 2001), argued,

...the most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy. I believe that the absence of math literacy in urban and rural communities throughout this country is an issue as urgent as the lack of registered Black voters in Mississippi was in 1961. (p. 5)

NCTM responded through a quick succession of publications: *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995). The democratic spirit was found in the NCTM's (1989) *Standards*. "New social goals for education included (1) mathematically literate workers, (2) lifelong learning, (3) opportunity for all, and (4) an informed electorate" (p. 3). The *Standards* identified five general goals for all students: "(1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) they learn to
reason mathematically” (NCTM, 1989, p. 5). These goals were grounded in assumptions that learning was an active process rather than one of memorization and practice.

NCTM’s (2000) *Principles and Standards for School Mathematics* also addressed the theme of equity through its Equity Principle. “Excellence in mathematics education requires equity – high expectations and strong support for all students” (p. 12). It was important to note that equity was not defined as equality. “Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students” (p. 12), however, “Mathematics can and must be learned by all students” (p. 13 italics in original). Appropriate accommodations included further assistance, assessment accommodations, increased time to complete assignments, additional resources, enrichment programs, or technological tools and environments. This required access to high-quality instructional programs and highly qualified teachers.

Similar to the reports of from other various professional groups throughout the years, NCTM had little reason to expect these would be unlike the others, which had little lasting impact on the curriculum and were mostly of historical interest. Most such reports tended to come and go. None of the authors or others involved had any idea of the ultimate magnitude of the response to their document would be. This time, the Discourse community listened.

**Speaking To Everyone: A Slogan System?**

What made NCTM’s voice so powerful? The *Standards* documents appealed to both radical and conservative movements (Schoenfeld, 2004). On the radical side, the
Standards were seen as challenging many of the assumptions underlying the traditional curriculum. On the conservative side, the document was written for the broad NCTM membership: mathematics teachers. Authored by two-dozen writers, draft copies of the Standards were widely distributed for feedback, and revisions were made. The cost of this consensus building was precision.

According to Apple (1992), the Standards volumes were close to being a kind of “slogan system” due to three characteristics. “First, they must have a penumbra of vagueness so that powerful groups or individuals who would otherwise disagree can fit under the umbrella” (p. 413) so that coalitions to support a movement for curricular change can be built. Second, “They need to be specific enough to offer something to practitioners here and now” (p. 414). Finally, “a slogan system seems to need to have the ability to charm.... It offers us a sense of imaginative possibility and in doing so generates a call to and a claim for, action” (p. 414).

The Standards successfully addressed the needs of stakeholders who held various educational perspectives described above. Among them were the economic modernizers and conservatives (social efficiency voices) who wanted to transform mathematics into economically useful knowledge and wanted a technically prepared and flexible workforce; the process-oriented and constructivist educators and researchers (developmentalist voices) who wanted more dynamic and interactive styles of curriculum and teaching; democratically inclined educators (social meliorist voices) who sought to make mathematics somewhat more based on community needs and who questioned the patterns of differential achievement; mathematicians who were concerned about the state
of mathematics as a field; advocates of a national common culture and common
curriculum; cultural conservatives (all humanist voices) worried about the decline of
standards and the loss of America’s supremacy; and teachers who work under difficult
and uncertain conditions in schools (Apple, 1992).

As noted throughout the previous sections, traces of each perspective were found
within the original Standards (NCTM, 1989) and the revised version (NCTM, 2000).
Each philosophical group read something into the document that resonated with them.
Sometimes they read more into the document than was actually there (Schoenfeld, 2004).
Interestingly, one state assessment assessed students’ mathematical competency through
portfolios containing students’ work on extended projects. At the same time, another
state administered basic skills multiple-choice tests. Both did so in the name of the
Standards. Some materials produced during this time were considered “flaky,” and some
classroom practices seemed “dubious.” The Standards were blamed for them all.
(Schoenfeld, 2004).

The Debate Heats Up

With each round of reform, new ideas were espoused that were rooted in deeply
held beliefs. While the Standards (NCTM, 1989; 2000) attempted to acknowledge these
beliefs, the document proved to be its own worse enemy.

In the following section, I will summarize the philosophical beliefs of the four
dominant voices and contrast them with each other. I will also highlight the ways in
which the Standards exacerbated the argument. After a century of debate, consequential
questions remain. These questions cause all of us to wrestle with our own beliefs regarding significant issues relevant to the mathematical Discourse and the role of school.

Rigor Versus Relevance

At the heart of the humanist/social efficiency debate was the issue of rigor versus relevance. Rigor is the quality of being extremely thorough, exhaustive, or accurate (Oxford Dictionaries, 2010). The humanists valued the traditional high school curriculum, which was designed to be rigorous and intended for those who pursued higher education. In contrast, relevance pertained to appropriateness to the matter at hand (Oxford Dictionaries, 2010), in other words, the direct usefulness or utility of mathematics. The social efficiency proponents argued that mathematics should be applicable to the workplace.

Traditionalists acknowledged that half of the students dropped out of the mathematics pipeline each year after grade nine, but saw this 50% attrition rate as a confirmation that mathematics is hard. They suspected that lowering standards was the only way to achieve greater success rates in mathematics (Schoenfeld, 2004). The “increased attention/decreased attention” charts found in the Standards intensified these fears. Traditionalists viewed NCTM’s Standards (1989) as challenging many of the tenets of a traditional curriculum when the following practices were suggested for decreased attention:

- complex paper-and-pencil computations, long division, rote practice, rote memorization of rules, teaching by telling, relying on outside authority (teacher or an answer key), memorizing rules and algorithms, manipulating symbols, memorizing facts and relationships, the use of factoring to solve equations, geometry from a synthetic viewpoint, two column proofs, the verification of
complex trigonometric identities, and the graphing of functions by hand using tables of values. (Shoenfeld, 2004, pp. 267-268)

The consequential question of concern is, "What knowledge is of most worth?" (Apple, 1992, p. 422).

Content Versus Pedagogy

The source of the humanist/developmentalist debate is best understood as a prolonged struggle between content and pedagogy. On one hand, the humanists argued that prioritizing a concentrated content prohibited them from implementing too much student-centered discovery learning, because that particular pedagogy required more time than rigorous content requirements allow. On the other hand, the developmentalists acknowledged the choice of a constructivist pedagogy obviously limited the amount of content that could be presented to students (Klein, 2003), but maintained that it was more important for students to understand what they were learning than to simply "cover" material (Van de Walle et al., 2010).

The Standards (NCTM, 1989) were seen as a challenge to the "content-oriented" voice of mathematics that dominated for more than a century. Each of the three grade bands began with the following four standards: mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections. Only after these four process standards were described, was the traditional mathematical content addressed. Interestingly, this order was reversed in the 2000 version. The consequential question of concern is, "What is the nature of math that is learned?" (Schoenfeld, 2004, p. 255).
Democracy Versus Elitism

At the foundation of the humanist/social meliorist debate was the issue of democracy versus elitism. These two competing ideologies, democracy and elitism (capitalism), define the very structure of American society. In a democracy, common individuals are involved in participative decision making either through direct vote or through elected representatives. Social equality and respect for diversity are valued. In contrast, the American economy is based on capitalism characterized by open competition in a free market. Private or corporate owners produce and distribute goods and services with economic growth proportionate to increasing accumulation and reinvestment of profits.

The Standards, reinforced by the NCTM's call for "mathematics for all" and the equity agenda in Everybody Counts (NRC, 1989) clearly aligned with the social meliorists' perspective. There was a long history of data indicating that race and socioeconomic status correlated with mathematics performance, with dropout rates, and with economic opportunity (Dalton, Glennie, & Ingels, 2009; Kozol, 1991; NSF, 2010; Secada, 1992). The Standards and the reform movement it represented were seen as a threat to the current social order (Schoenfeld, 2004).

Nonetheless, in reality it was the traditional curriculum, touted by the humanists, which was a means to the perpetuation of privilege. The consequential question of concern is, "Who gets to learn mathematics?" (Schoenfeld, 2004, p. 255).
A National Agenda: Humanist Voices in Social Meliorists' Clothing

Shortly after the Standards (NCTM, 1989) were published, Apple (1992) offered a critique of the document. His primary thesis expressed concern for the role the Standards would play within the growing conservative movement in education of the time. His insightful foresight was a premonition of things to come when he asked the most contentious and consequential question, "Whose knowledge is of most worth?" (p. 422; Apple, 2000). As Apple (1992) explained, "there is a complex relationship between what comes to be called official knowledge in schools and the unequal relations of power in the larger society" (p. 422).

A little more than a decade after the NRC (1989) called for national standards and the NCTM (1989) Standards were published, a national agenda for education was set forth with passage of the most significant piece of legislation to impact public schools since Brown v. Board of Education (1954) and P.L. 94-142 Amendment to the Education for All Handicapped Children Act (1975) later reauthorized as the Individuals with Disabilities Education Act (IDEA, 1990, 2004). While Brown and IDEA guaranteed equal access to education, No Child Left Behind Act (NCLB, 2002) promised equal outcomes as a result of education, measured by state tests in the areas of reading, mathematics, and science. While flaws of the legislation were well documented (American Federation of Teachers [AFT], 2006; Dobbs, 2005; Jennings, 2010; NEA, 2006; Paley, 2007), the premise was laudable.

With its strong message of equity, many thought that NCLB (2002) landed squarely in the social meliorist camp. To understand how that was not the case, we must
think critically about who has authority of the mathematics Discourse. Did NCLB (2002) really open the discussion to everyone? Or was it simply a guise for the humanist voices to dominate once more?

This is the fundamental question critical activists, most notably Paulo Freire (2009; Shor, 1987) among others, confronted. Marilyn Frankenstein, mathematics professor at Brooklyn College, City University of New York, provided some insight. In the following example, Frankenstein (1987) considered how the elite convinced the mathematically illiterate that social welfare programs were likely places for budget cuts because they knew the illiterate did not research the numbers to reveal that “welfare” to the rich dwarfed any meager subsidies given to the poor. Though her example is now dated, the point is still relevant: “...in 1975 the maximum payment to an Aid for Dependent Children family of four was $5000 and the average tax loophole for each of the richest 160,000 taxpayers was $45,000” (Babson & Brigham, 1978, p. 37 as cited in Frankenstein, 1987, p. 193). Or how, “in 1980, $510 million in tax money paid for new airports so that private pilots would not land their planes at large commercial airports” (Judis, & Moberg, 1981, p. 22, as cited in Frankenstein, 1987, p. 193). Despite the NCTM’s (2000) call for “problems that come from [children’s] worlds” (p. 52), Frankenstein’s examples would not provide context to the problem-solving situations presented America’s textbooks.

To reinforce how powerful the humanist voice is, I pause the discussion of the Math Wars briefly to discuss what happened in the Reading Wars, a parallel struggle
within the literacy community, and the effect that NCLB's (2002) Reading First component had on the outcome.

A Parallel War

The Reading First component of NCLB (2002) pumped billions of dollars into America's poorest and lowest performing schools "to apply scientifically based reading research—and the proven instructional and assessment tools consistent with this research—to ensure that all children learn to read well by the end of third grade" (Reading First, 2002, Program Description section, ¶ 1).

While Reading First was not without critics (Office of Inspector General, 2007), it accomplished two things. First, it "assumed the role of conventional wisdom in reading instruction, albeit by mandate rather than groundswell" (Pearson, 2004, p. 220) effectively ending the "Reading Wars" that had engaged the professional reading community during previous decades. While the debate was portrayed in the media as a war between whole language and phonics, it was much more complex (see Pearson, 2004, for an analysis).

The Reading Wars had much to do about the view of authority, which represented a microcosm of the larger historical traditional versus progressive debate about American schooling. From the traditionalists' viewpoint, the role of schooling was to provide authoritative knowledge of right and wrong. It was the teacher's responsibility to determine what was right and to make sure the students learned it. It was irrelevant and inappropriate for students to discuss their feelings and individualistic thoughts in school. What was important was that students were taught what they should know.
Phonics instruction was associated with discipline, structure, and authority. The whole language movement was associated with reader response journals and invented spelling, seen by traditionalists to be as undisciplined and as individualistic as it could get (Schoenfeld, 2004). This traditionalist viewpoint fit the agenda of the New Right, a powerful and well-connected political group whose membership included Diane Ravitch, E.D. Hirsch, and Lynn Cheney (Berliner & Biddle, 1995), wife of Dick Cheney, Vice President during the Bush administration. Through the mandate of "scientifically" based research, the traditionalist viewpoint was carefully woven into NCLB, the first piece of legislation passed during that administration. It is important to note that Diane Ravitch (2010) has since retracted her position.

Second, due to the federal attention given to reading instruction, the domain monopolized the attention of principals and teachers. Legislation determined the approved content and methodology (National Reading Panel, 2000; Reading First, 2002), time devoted to instruction (Reading First, 2002), required reading products and assessments to be used in many states (Office of Inspector General, 2007), and technical assistance or professional development provided to practicing educators (Reading First, 2002).

An unintended consequence of prioritizing reading instruction in NCLB (2002) resulted in the marginalization of mathematics education with less attention and direction given to it at the federal level (Cervetti, Pearson, Bravo, & Barber, 2006). No comparable component to Reading First was included in NCLB for mathematics. As a
result, the math wars continued to rage. It is yet to be seen if they will meet the same outcome.

Common Core State Standards

As recently as June 2, 2010, the Common Core State Standards (CCSSI, 2010), in the areas of mathematics and English language arts, were published. The Council of Chief State School Officers and the National Governors Association Center For Best Practices led the effort, with input from the NEA, AFT, NCTM, the National Council of Teachers of English (NCTE), and the general public. NCTM released a public joint statement with the National Council of Supervisors of Mathematics (NCSM), the Association of State Supervisors of Mathematics (ASSM), and the Association of Mathematics Teacher Educators (AMTE) in support the goal of the Common Core in their effort “to describe a coherent, focused curriculum that has realistically high expectations and supports an equitable mathematics education for all students” (NCTM, 2010).

According to the website (CCSSI, 2010), the standards are not a curriculum, nor do they prescribe pedagogy. The focus continues on content (humanist voice). Upon review, there are familiar aspects to the document. The Standards for Mathematical Practice describe a variety of expertise that mathematics educators at all levels should seek to develop in their students. These are based on two influential documents. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency outlined in the NRC’s (2001) report, Adding It Up: adaptive
reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition. They are delineated as follows:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning. (CCSSI, 2010, pp. 6-8)

The *Common Core Content Standards* are similar to the previous NCTM (2000) content standards, but are not stranded through each grade as before. For example at the kindergarten level, the standards include Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Measurement and Data, and Geometry. By grade 12, the content standards are organized by conceptual categories: Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability. Readers were assured that this effort was state led, without the involvement of the federal government, although state education agencies were encouraged to leverage federal money (i.e., Title 1 or other federal programs) to assist with implementation costs (Achieve, 2010).

It is too early to tell if the Common Core will effectively end the math wars as Reading First essentially ended the Reading Wars (Pearson, 2004), mend the broken system (NMAP, 2008), or have any impact on student achievement. If history has any role to play, another reform is all ready brewing. The humanist voice is gaining strength.
It Matters to This One

To some, this latest round of reform might be seen as another swing of the pendulum, and it most likely is; but with each oscillation, new ways of doing, of being, refine our mathematical thinking. The mathematics Discourse has a long history, confounded by many voices throughout time. Each leaves a residue, its mark (Moje & Lewis, 2007) on the professional community as a whole and on the individual as a learner. Once again, this newest reform effort raises questions about the values of a mathematics education, by redefining what constitutes mathematics and by advocating new pedagogical practices (Klein, 2003).

In the meantime, it becomes another child’s opportunity to access admission into the mathematics Discourse. It matters, for too long the majority of population was denied and for those who were admitted, too many were marginalized to the fringes of participation.

Statement of the Problem

The impetus for reform, in both the content and pedagogy of mathematics education can be traced to knowledge gained from research, the influence of the professional mathematics community represented by the National Council of Teachers of Mathematics, and the public and political pressure for change in mathematics education due largely to the poor performance of U.S. students in national (NCES, 2009a) and international standardized tests of mathematics (Baldi et al., 2007; Gonzales et al., 2008).

In response to these issues, state standards, No Child Left Behind (NCLB, 2002), and the new Common Core State Standard Initiative (CCSSI, 2010) pressed for higher
levels of achievement, more testing, and increased teacher accountability. The reform agendas of NCTM and other perspectives, in addition to the public and political sector pulled teachers and pushed students in different directions.

When considering the implications and ramifications of such reform, it became necessary to examine closely what happened in a reform-based classroom by listening carefully to the classroom discourse. It was important for educational leaders to understand the reform they were asked to support and to study the effect it had on students’ learning.

**Purpose of the Study**

The purpose of this study was to explore how teachers and students used language to advance students' conceptual understanding of mathematics in a reform-based mathematics classroom. Specifically this study sought to describe how the reform-based mathematics classroom environment impacted the quantity and quality of mathematical language exhibited by students.

**Assumptions**

It was assumed that the Mathematics lessons that were observed were typical for this teacher. It was also assumed that the teachers’ and students’ use of language provided a window into their thinking.
Definition of Terms

The following terms used in this research are defined according to their use:

**Discourse with a capital “D”**

A Discourse with a capital “D” is composed of distinctive ways of speaking/listening and often, to writing/reading *coupled* with distinctive ways of acting, interacting, valuing, feeling, dressing, thinking, believing, with other people and with various objects, tools, and technologies, so as to enact specific socially recognizable identities engaged in specific socially recognizable activities. (italics in original, Gee, 2008, p. 155)

**discourse with a lower-case “d”**

A discourse with a lower-case “d” is a connected series of utterances located in a text or conversation. (*Oxford Dictionaries*, 2010)

**Reform-based Mathematics Curriculum**

The phrase “reform-based mathematics curriculum” also known as "standards-based mathematics curriculum" refers to curriculum, including guiding frameworks and student and teacher materials, reflecting and enacting recommendations for mathematics curriculum and instructional practice outlined in the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM 2000). Standards-based mathematics programs have the following characteristics:

Comprehensive. They are based on the broad content of the national standards at each grade level: Number and Operations, Algebra, Geometry, Measurement, and
Data Analysis and Probability. They also incorporate the mathematical processes of Problem Solving, Reasoning and Proof, Communication, Connections, and Representation.

Coherent. They are woven together as a whole, with ideas connecting to each other. They are not repetitive, and the sequence from one grade to the next gives students the preparation they need for the next learning step.

Depth. Important and pivotal "big ideas" are developed in increasing depth as student mature.

Engaging. They challenge all students intellectually and encourage active learning. This enables all students to both participate and grow in learning.

Motivating. They teach mathematics through realistic situations and applications, giving both an understandable approach and a reason to learn the mathematics.

(Adapted from Trafton, Reys, & Wasman, 2001, pp. 259-263)

The Guided Discovery Lesson

Guided-discovery lessons allow students to explore and develop ideas through a careful sequence of tasks and questions. The teacher knows what results are desired and guides students toward these results (Rubenstein, Beckmann, & D. R. Thompson, 2004).

The Open-Ended Exploration Lesson

Open-ended exploration lessons provide students opportunities to explore mathematics without preconceived notions of the paths or results that might be obtained. The teacher anticipates students' responses, including understandings
and misunderstandings, but realizes that the investigation might lead down unanticipated routes. Teachers must have a strong understanding of the mathematics and the confidence to invite students to proceed along undiscovered paths (Rubenstein et al., 2004).

The Integrating Direct Instruction with Guided Discovery Lesson

Integrating direct instruction with guided discovery lessons typically begin with the teacher providing direct instruction when the students need to learn to use some tool, need to learn vocabulary, or need to become proficient in some procedure that they are not likely to discover on their own. Effective direct instruction is often integrated with guided discovery, to enable students to explore relationships using the information provided in the direct instruction portion of the lesson (Rubenstein et al., 2004).

Revoicing

In a revoicing move, the teacher essentially tries to repeat some or all of what the student has said, and then asks the student to respond and verify whether or not the teacher’s revoicing is correct (Chapin, O’Connor, & N. C. Anderson, 2009, pp. 13-14).

Repeating: Asking Students To Restate Someone Else’s Reasoning.

The teacher asks one student to repeat or rephrase what another student has said, and then immediately follows up with the first student (Chapin et al., 2009, p. 15).
Reasoning: Asking Students To Apply Their Own Reasoning To Someone Else's Reasoning.

After a student has made a claim, and the teacher has made sure that students have heard it and have had time to process it, the teacher can move on to elicit student reasoning about the claim (Chapin et al., 2009, p. 15).

Adding On: Prompting Students For Further Participation

The teacher uses the revoicing move to clarify the position that has emerged, and to model how to talk respectfully to the originator of the position. Then the teacher asks others to contribute, prompting them to either state agreement or disagreement, or to add other comments. (Chapin et al., 2009, p. 16)


Wait time comes into play after a student has been called on. After the teacher has called on a particular student, that student should be given at least a minimum five seconds to organize his or her thoughts (Chapin et al., 2009, p. 17).

Whole-Class Discussion

In this class format, the teacher is in charge of the class, however, the teacher is not primarily engaged in delivering information or quizzing. Rather, he or she is attempting to get students to share their thinking, explain the steps in their reasoning, and build on one another's contributions. Whole-class discussions give students the chance to engage in sustained reasoning. The teacher facilitates and guides quite actively, but does not focus on providing answers directly. Instead, the focus is on the students' thinking. (Chapin et al., 2009, p. 19)
Partner Talk

In this talk format, the teacher asks a question and then gives students a short time, perhaps a minute or two at the most, to put their thoughts into words with their nearest neighbor. (Chapin et al., 2009, p. 21)

Small-Group Discussion

In small-group discussion format, the teacher typically gives students a question to discuss among themselves, in groups of three to six. The teacher circulates as groups discuss and doesn’t control the discussions, but observes and sometimes interjects when appropriate (Chapin et al., 2009, p. 21).

Precision of Vocabulary

Precision of vocabulary recounts the accuracy or correctness and sophistication in vocabulary. (Chall, Jacobs, & Baldwin, 1990, p. 145)

Vagueness

Vagueness has to do with thinking or communicating in an unfocused or imprecise way indicating uncertainty by the speaker. (Rowland, 2000)

Lexical Density

Lexical Density is measured by determining the average number of content words, or lexical items, per clause. (Halliday & Hasan, 1985, pp. 61-72 as cited in Veel, 1999, p. 203)

Hedge

A hedge is a type of vagueness within the field of linguistics. It includes the words “sort of,” “about,” “approximately” – words that have the effect of blurring
precise measures, as well as words and phrases such as “I think,” “maybe,” “perhaps,” which hedge the commitment of the speaker to that which he or she asserts. Students often use hedges to shield themselves from being accused of error (Rowland, 2000, p. 57-58; 209). A shield is one major type of hedge. The use of a shield indicates some uncertainty in the mind of the speaker in relation to some judgment or opinion. Examples of shields include “Well, I think...,” “There is evidence that’s been presented...,” “probably,” or “Maybe...” (Rowland, 2000, p. 59). An approximator is another major type of hedge. The use of an approximator is to modify the judgment or opinion, making it more vague. Examples of approximators include “about,” “a little bit,” “around,” “approximately,” “sort of,” or “somewhat.” (Rowland, 2000, p. 60) Note: Two categories of hedges, shields and approximators, will not be distinguished in this study.
CHAPTER III
UNPACKING THE RESEARCH QUESTION

"What we cannot think, that we cannot think: we cannot therefore say what we cannot think.... The limits of my language mean the limits of my world.... Whereof one cannot speak, thereof one must be silent." Ludwig Wittgenstein (1918, p. 74; 90)

The purpose of this study was to explore how teachers and students used language to advance students' conceptual understanding of mathematics in a reform-based mathematics classroom. Specifically this study sought to describe how the reform-based mathematics classroom environment impacted the quantity and quality of mathematical language exhibited by students.

Language: A Promise of Hope?

The reality of the matter is that most students in American schools are not proficient in mathematics (Baldi et al., 2007; Gonzales et al., 2008; NCES, 2009a), yet for the first time in history, all students are expected to demonstrate deep conceptual understanding and communicate logical thinking, while solving novel problems (CCSSI, 2010; NCLB, 2002; NCTM, 2000). In the decade since NCLB (2002), little progress has been made in students' mathematics achievement, with significant gaps remaining between the poor performance of middle class white students and the poorer performance of minority and low socio-economic students (Baldi et al., 2007; Gonzales et al., 2008; NCES, 2009a).

Researchers are beginning to ask an important question by looking at data through a different lens. According to Secada (1992), research conducted prior to the 1960s compared recent bilingual immigrants and/or their children, who tended to be from
lower-SES backgrounds, with monolingual groups from middle- and upper-class backgrounds. This was a problem because it failed to consider social class as a confounding variable when comparing bilingual populations to monolingual English speakers (Hakuta, 1986, as cited in Secada, 1992). More recent research on mathematics achievement addressed this issue by creating matched groups along SES indices and/or through statistical adjustments of the data.

The new question being asked is, "Which is the primary causal agent [of mathematics achievement]: SES or language proficiency?" The answer is becoming clear, according to Secada (1992). "Regardless of how the issue is posed, the consistent finding has been that language proficiency, however it is measured, .... is related to mathematics achievement and to learning" (pp. 638-639). Developing a strong command of the language and being able to communicate mathematically may hold promise for increasing the mathematics achievement for students of low socio-economic status (SES), according to Secada's (1992) interpretation of the research, and may improve mathematics achievement for all.

One goal of reform-based mathematics instruction is to foster more dialogic interactions during classroom discourse (Ball, 1993; Chazan & Ball, 1995; Cobb, Wood, & Yackel, 1993; Knuth & Peressini, 2001; Lampert & Blunk, 1998; NCTM, 2000) in order to share the responsibility and authority for explaining mathematics with students (Forman, McCormick & Donato, 1998; Hamm & Perry, 2002) and assessing students' understanding (Wallach & Even, 2005). The Communication Standard in Principles and Standards for School Mathematics (NCTM, 2000) stated,
Instructional programs from prekindergarten through grade 12 should enable all students to organize and consolidate their mathematical thinking through communication; communicate their mathematical thinking coherently and clearly to peers, teachers, and others; analyze and evaluate the mathematical thinking and strategies of others; and use the language of mathematics to express mathematical ideas precisely. (p. 60)

Communicating mathematically continued to be an integral part of the new

*Common Core State Standards for Mathematics* (2010). Two of the eight *Standards for Mathematical Practice* (CCSSI, 2010) specifically addressed this idea. The first had to do with students communicating their thinking as they construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. *Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.* (CCSSI, 2010, Standards for Mathematics Practice section ¶ 4)

The second addressed students’ attention to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers
with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions. (CCSSI, 2010, Standards for Mathematics Practice section ¶ 7)

In this chapter, I attempt to unpack the concepts underlying the overall purpose of my study. I will address five major areas: the language of mathematics and why it proves to be so difficult for so many, what it means to understand conceptually, the nature of classroom discourse including the quantity and quality of teacher talk and the quantity and quality of student talk, embedded in environment of the reform-based mathematics classroom.

The Language of Mathematics

Mathematics may indeed be "the new literacy" (Schoenfeld, 1995, as cited in NMAP, 2008); "at the least, it is essential for any citizen who is to be prepared for the future" (NMAP, 2008, p. 5). Developing communicative competence means knowing how to use language to communicate in various social situations and how to use the language appropriate to the context (Pimm, 1987).

This is a formidable task due to the cognitive complexity of the mathematical language coupled with limited contextual support. It is considered by some (Bernstein, 2000) to be the most challenging of the academic discourses.

Described as a language of its own (Kane, Byrne, & Hater, 1974; Pimm, 1987; Wakefield, 2000), mathematics has its own grammar, symbols, and semantics. Seemingly common English words take on multiple meanings within the mathematics classroom. Specialized vocabulary abounds. Meaning-bearing symbols and notations are
ubiquitous. Concepts and connections within and across the content domain add to the possibility for confusion.

Pimm (1987) argued that the *construction of meaning* rather, than the question of rigor, is the central problem facing mathematics education. Posing the idea that mathematics should be learned as one would learn a second *language*, through communicative *language* teaching, rather than formal, deliberate, and largely rule-based instruction, symbols would sink into the background of attention bearing the same relation to the mathematical meanings they convey as words do in ordinary language. The ... expressed hope may, however, be a forlorn one. This is because the symbol- *transforming* aspect of mathematics, which has proven to be such a powerful analytical tool, relies for fluency on the ability of the transformer to disconnect the symbol from its referent, and work with the symbol alone. (p. 204)

Language is composed of three basic systems (Clay, 1991): symbolic, syntactic (grammatical structure), and semantic (meaning), integrated within a context to convey thought, and in written form, a punctuation system to provide clarity. This same systemic organization can be applied to mathematics.

The Symbolic System

While conventional English uses 26 letters of the alphabet to symbolically represent approximately 44 phonemes, there are four main conventional types of symbols used in mathematics. Due to the sheer number of symbols, *it can* be challenging to remember them all.

Logograms are specially invented shapes that stand for whole words. Everyday language examples include "$" for "dollars" or "&" for "and." Mathematical logograms are not used outside a *mathematical* context. Examples include the Arabic numerals 0, 1,
... 9. Also $+, -, \times, \div, \%, \sqrt{\cdot}, \Rightarrow, \Leftrightarrow, \therefore, \in, \cup, \cap, \subset, \infty, \equiv, \circ, \sum, \prod$, and $\equiv$. These symbols can also cause confusion because of the multiple meanings assigned to some of them. For example, the symbol "−" can indicate subtraction, division, negative number, and is used to separate the numerator from the denominator in a fraction.

Pictograms are a type of stylized symbol, but are clearly an interpretable image of the object itself. Examples include $\angle$ for angle, $\square$ for square, $\bigcirc$ for circle, and $\Delta$ for triangle. Geometric pictograms include $Z$ to indicate alternate angles and $F$ to indicate corresponding angles.

Many punctuation symbols ordinarily found in standard English orthography are also widely used in mathematics, though not for punctuation. The colon "::" signifies a ratio as in "a:b," punctuates a function as in "f:A$\Rightarrow$B," or is used as a separator in the description of a set "x:x\geq2." The comma "," and period "." are both used as decimal points (France and United States respectively). Other punctuation symbols include the exclamation mark "!," parentheses "( )," brackets "{ }," "[ ]," the asterisk "\times," and forward slash "/."

Alphabetic symbols are used extensively in mathematics. These include both the lower and upper case Roman alphabet, a, b, ..., z, A, B, ..., Z and the Greek alphabet, $\alpha, \beta, ..., \omega$. A further convention is the idea of using consecutive letters for like objects, when no other factors are operating. Fractions are typically represented as $\frac{a}{b}$ or $\frac{p}{q}$ instead of $\frac{n}{d}$, which would be easier to remember as the numerator and denominator. Some specific conventions are used for particular letters. Capital C is used for circumference;
m represents mean and s, standard deviation. To compound the issue, different ways of representing symbols may or may not alter their meaning. For example, the position of the numeral 2 denotes different values in the following series: 2, 20, 200, \( \frac{1}{2} \), 2\(^3\). The size of the symbol matters when comparing the 3 in 23 with 2\(^3\). Finally, orientation makes a difference when comparing the symbols < and >, but does not when rotating geometric figures, such as a square [square].

O’Halloran (2005) contended that students’ understanding of how mathematical language works and how that language translates into mathematical symbols may be the key factor in their success. She feared that if “the grammar of mathematical symbolism is not taught from a linguistic perspective, the cumulative effect is that many students fail [emphasis added] mathematics because they simply do not understand how (or why) mathematical symbolism functions as a resource for meaning” (p. 202).

The Syntactic System

Mathematics has its own grammatical structure. The symbol cluster ‘8+ +4 -’ is as ungrammatical as a mathematical expression as ‘put cat the mat on the’ is as an English one. The Associative Property \([a + (b + c) = (a + b) + c]\), the Commutative Property \([(a + b = b + a) or (ab = ba)]\), and Distributive property \([a(b + c) = ab + ac]\) are provided as examples.

The Semantic System

Mathematics has its own semantic system, which has to do with meaning. It is every bit as complex as the symbolic and syntactic structures, because of the multiple terminologies used.
Sometimes mathematics uses everyday English words (e.g., cost per person, miles per hour) to mean the same thing that the words mean in everyday English. At other times, everyday English is used to mean something different in the mathematical context (e.g., face, degree, power, product, mean, real imaginary, rational, natural). Differences in words such as "some," which means "not all," or "any," which means "every" confuse novice mathematicians. Seemingly simple words, such as "by," can be used in multiple ways, meaning to multiply (e.g., ten "by" two), divide (e.g., ten "by" two), or refer to length and width of a rectangle (e.g., ten "by" two). Finally, mathematics, like many other content areas, has its own specialized vocabulary (e.g., quadrilateral, parallelogram, multiplicand, numerator, hypotenuse; Pimm, 1987).

Bernstein (2000) considered mathematics, along with the sciences (e.g., physics, biology), especially rigorous due to its vertical discourse. Vertical discourses have a "hierarchical knowledge structure” and a “horizontal knowledge structure.”

A “hierarchical knowledge structure” describes the way that mathematics builds up a hierarchy of technicality very quickly, with each level of understanding dependent on understanding of the previous level and one more step away from any everyday meaning of the word. For example, consider the following hierarchy of terms that represent one-, two-, and three-dimensional concepts. At the first level, the technical mathematical words generally mean the same in everyday language. The word “length” means “how long something is,” the word “width” means “how far across something is,” and the word “height” means “how far off the ground something is.” At the second level, the technical terms learned at the first level (length, width) are combined to form a new
technical term "area." The everyday meanings (how long something is, how far across something is) of the first level are no longer referred to. Area may be understood as length "times" width, but to truly understand area, one must understand the notion of "covering" a surface. At the third level, technical terms generated at the first and second levels (height, area) are combined to make "volume." Volume may be thought of as height "times" area, but is more fully understood as the "filling" of a space. The technical term "volume" is at least two steps away from an everyday definition, which makes it difficult to explain in everyday terms. This "length-area-volume" example is a relatively simple one (Veel, 1999).

Consider the many levels of technicality in following example taken from more advanced mathematics: "If \( p \) is positive at point \( P \) on a curve, then the tangent is positive at that point and its function is said to be an increasing function at \( P \)" (Jones & Couchman, 1981, p. 232, as cited in Veel, 1999, p. 199, italics in original). Any notion of an "everyday" meaning is literally unthinkable.

In addition to the technicality of the vocabulary used in the example above, it is also difficult to comprehend because of the high number of content words, coupled with low contextual support, within a single sentence. Lexical density is the average number of content words, or lexical items, per clause (Halliday & Hasan, 1985, 61-72, as cited in Veel, 1999, p. 203). The greater the lexical density of a sentence, the more "packing-in" of meanings there are, which qualitatively differentiates it from everyday talk. Expressions become even denser when mathematical symbols are used to represent actual words. Veel and Coffin (1996, as cited in Unsworth, 2000) found, when studying
academic subjects, there was a progressive increase in lexical density as the subject increased in cognitive complexity.

*Students of mathematics are challenged by the magnitude of words to learn, but this learning is compounded by the fact that each strand of mathematics has a unique discourse.* Bernstein, (2000) described this second part of vertical discourse as a “horizontal knowledge structure.” The vocabulary of one strand of mathematics is not translatable to another. For example, the symbols, grammar, technical words, phrases, and modes of argument for Number and Operations are unique to those for each of the other strands: Geometry, Measurement, Algebra, and Data Analysis and Probability. When a word does appear in more than one strand, it can have different meanings (e.g., “square” and “base” found in number and geometry contexts). In order to be a proficient mathematical language user, one must be fluent in multiple discourses.

**Difficult English Constructions**

Even if the mathematical language in a problem seems relatively straightforward, the grammatical features of academic English can cause problems for students. For example, the following sixth-grade problem appeared in the *Massachusetts Comprehensive Assessment System Mathematics Test* (2003, as cited in Bielenberg & Fillmore, 2004/2005):

> Students in Mr. Jacob’s English class were giving speeches. Each student’s speech was 7 to 10 minutes long. Which of the following is the *best* estimate for the total number of student speeches that could be given in a 2-hour class?” (p. 46, italics in original)

The only technical mathematical term used in this item is the expression “best estimate.” But consider the complexity of the English grammatical features. This question contains
a complex noun phrase, which contains a complex prepositional phrase, which contains a relative clause construction, which contains a passive construction (Bielenberg & Fillmore, 2004/2005). This structure can be challenging for any student, but especially English Language Learners (Martiniello, 2008).

Genre of “Story” Problems

Most students approach a story problem in the same way they approach a story. When approaching a word they don’t know in a story, students often try to figure it out from context of other words in the sentence, by looking at the pictures to gain possible meaning, or thinking about what they already know about the story that could help them. These strategies are often not helpful when trying to understand unknown words in a story problem.

Dubbed “word-problemese” (Lave, 1993, as cited in Wiest, 2003), word problems have their own genre. They are short, which causes several problems. Without sufficient contextual support and the natural redundancy of language, they can be difficult to understand. In addition, the problem contexts are often invented so students have opportunities to solve “real-world” problems. Since students do not have adequate background knowledge about the topic from which to draw, and the pictures rarely have anything to do with the actual mathematics of the problem, they are left with few clues to meaning. Add to this is the fact that the language used in the problem is often more complex than it needs to be. The problem that appeared very simple to those of us who are familiar with the genre of word problems is incomprehensible for novice learners.
When students apply reading strategies that are successful for other genres to mathematical word problems, they experience failure. This leads them to develop other strategies. They often look for clue words, or they choose an operation based on the absolute or relative size of the numbers (Wiest, 2003).

Misinterpretation

Sometimes, misinterpretation or mishearing of a similar sounding word causes confusion. Consider the following problem posed by Celedón-Pattichis (2003, as cited in Irujo, 2007): “On Saturday, 203 children came to the swimming pool. On Sunday, 128 children came. How many more children came to the pool on Saturday than on Sunday?” Most teachers would assume that the “how many more” construction caused a particular student to conclude an incorrect answer. Upon asking a student who had solved the problem incorrectly to reveal his thinking, it was learned that the student had misinterpreted than as then. The student thought the question was asking how many of the students who came to the pool on Saturday then also came on Sunday. The facts given in the problem were insufficient for the student to answer the perceived question.

At other times, a mathematical word can sound very similar to an everyday English word. When a problem is presented only orally to a child, confusion may occur. The following sound-alike word pairs have found to be especially problematic: court/quart, attitude/altitude, spear/sphere, tents/tenths, have/half, and sense/cents (Adams, 2003). No wonder a child is puzzled by the problem, “Which represents the larger quantity, one have or four tents?”
Summary

This section demonstrated that cognitive complexity coupled with limited contextual support causes mathematical language to be considered the most challenging of the academic discourses. The plethora of symbols, its specific grammatical structure, technical vocabulary, and multiple meanings of words can be overwhelming. In addition, the confounded language and minimal contexts used in word problems add to the difficulty.

Accessing mathematical language and accessing mathematical knowledge are inextricably linked. Failure to provide students access to mathematical language, in effect, acts as a gatekeeper to their access of mathematical knowledge. However, access is only the first step. To fully appreciate mathematics, it requires that students develop conceptual understanding of the language and the knowledge it represents.

Use of Language to Advance Conceptual Understanding

A primary goal of reform-based mathematics is for students to develop conceptual understanding. The NRC's (2001) publication, *Adding It Up: Helping Children Learn Mathematics* was a primary resource for the authors of the *Common Core State Standards* (CCSSI, 2010), and provided the following description of conceptual understanding, one of the strands of mathematical proficiency:

Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kind of contexts in which it is useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (NRC, 2001, p. 118)
Students who developed conceptual understanding were better able to construct viable arguments and critique the reasoning of others.

**Three Types of Knowledge**

One of the primary distinctions between experts and novices was the type and extent of knowledge they acquired as they became more proficient (Paris, Lipson, & Wixson, 1983). Cognitive and developmental psychologists emphasized two types of knowledge: declarative and procedural. Paris et al. (1983) added a third type: conditional.

Declarative knowledge is sometimes referred to as “knowledge that” (Almasi, 2003). It is the “facts” the authors of *Adding It Up* (NRC, 2001) referred to in their definition of conceptual understanding above. Declarative knowledge also refers to one’s beliefs about the task and one’s abilities. For students of mathematics, these beliefs reflect the specific mathematics task and their perceived ability to perform it. For example: “Fractions are hard for me to understand.” Declarative knowledge resides in long-term memory. Derry (1990, as cited in Almasi, 2003) described it “as a large, tangled network” (p. 351, as cited in Almasi, 2003, p. 7) that is only useful if it can be recalled when needed. It is important that learners are able to organize their information and make connections in order to easily access facts. For learners who do not have an extensive amount of declarative knowledge about mathematics, then it is important to help them make connections between new information and their prior knowledge (Almasi, 2003).
Procedural knowledge is sometimes referred to as knowing how (Paris et al., 1983) to do something. It has to do with putting information into action instead of simple recall. In the description of conceptual understanding above, the authors of Adding It Up (NRC, 2001) referred to procedural knowledge as knowing methods. Without procedural knowledge, students of mathematics, for example, would be unable to perform a conventional algorithm.

Conditional knowledge is the ability to know when and why (Paris et al., 1983) a given strategy is important and the kind of contexts in which it is useful. When learning addition facts, for example, students often find it is helpful to apply the doubles strategy when finding sums of doubles plus or minus one (e.g., $7 + 8 = ?$ I know $7 + 7 = 14$ plus one more equals 15).

Making Connections

According to the NRC (2001), it is important that students develop connections among and between all three types of knowledge, declarative, procedural, and conditional, and that they organize their thinking to develop conceptual understanding. In the 1980s, cognitive psychologists expanded on the ideas of Piaget’s schema theory to explain how previous experiences, knowledge, emotions, and understandings have a major effect on what and how we learn (see R. C. Anderson & Pearson, 1984). Bits of facts and other known information, such as sights, sounds, odors, emotions, procedures, and settings stored in memory, are called “schemata” (Rumelhart, 1980). This can be thought of as a series of “mental file folders” (Tompkins & McGee, 1993). As new bits of information are learned, the brain scans for the right folder to file the new information.
These connections between the new and the old continually update and change the schemata, and learning occurs.

Literacy educators defined three types of connections that readers make: text-to-self, text-to-text, and text-to-world (Harvey & Goudvis, 2007). This way of organizing connections may be useful in mathematics, as well, and are alluded to in NCTM’s (2000) Connections standard.

If “text” is thought of as mathematical content and/or procedures, then text-to-self connections are made when students relate the mathematical content with their own life. For example, knowing that the length from their nose to an outstretched arm is about a yard is a useful connection for students to make when estimating length.

Text-to-text connections are made when students relate mathematical content or procedures to each other. Linking the concept of a partitive division problem (e.g., you have 30 pieces of candy and want to divide them equally among your five best friends) with the traditional procedural algorithm is an example of a text-to-text connection.

Text-to-world connections are made when students relate mathematical content or procedures to the outside world. Seeing relationships between mathematics and the natural sciences is an example of a text-to-world connection. These connections can range in complexity from recognizing the hexagonal shape of a honeycomb cell to Newton’s development of the law of gravity.

**Degrees of Conceptual Understanding**

Most people believe that words have complete and established meanings, the sorts of things that lexicographers record as definitions in dictionaries, which reside in
people's heads as concepts (Gee, 2008). When the dictionary definition matches our
definition, or concept, in our heads then all is well. We make the assumption that, since
we all understand words, others somehow have the same definition or concept in their
heads. This theory, that words have fixed meaning, has influenced traditional educational
practice.

But most words do not have fixed meanings. Gee (2008) provided a simple
example to illustrate this point. Take the word "coffee."

If I say, “The coffee spilled, go get a mop,” the word indicates a liquid. If I say,
“The coffee spilled, go get a broom,” the word indicates beans or grains. If I say,
“The coffee spilled, stack it again,” the word indicates tins or cans. If I say,
“Coffee growers exploit their workers,” the word indicates coffee berries and the
trees they grow on. (p. 8)

The word “coffee” is not a definite concept, but a little “story” that included all the
connections about how coffee products are produced and used. (Berries grow on trees,
get picked, get their husks removed, the remaining beans are ground up, used as a
flavoring or made into a liquid which is drunk or used for other purposes.) It does not
matter that one know all aspects of the word “coffee” and still be able to use the word.

It is not that simple in mathematics. If a student fails to develop the complete
“story” for a mathematical idea (i.e., conceptual understanding regarding a particular fact
or method), everything upon which that concept is built is at risk to be misunderstood.
For example, Skemp (1987) noted that trying to understand algebra without ever having
really understood arithmetic is impossible. Due to the hierarchy of concepts and the
connections between them, he cautioned that mathematics might remain a closed subject
to those who fail to develop conceptual understanding along the way. The only hope he
suggested was the availability of carefully crafted *explanations* provided by more capable others.

Knowledge *packages*, Ma's (1999) term for conceptual connections, can be quite extensive for a single topic. For example, the knowledge package for understanding the meaning of division of fractions included the meaning of whole number multiplication, the concept of division as the inverse of multiplication, *models* of whole number division, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, and so forth. Figure 1 represents these connections:

![Diagram of a knowledge package for understanding the meaning of division by fractions](image)

*Figure 1.* A Knowledge Package for Understanding the Meaning of Division by Fractions (Ma, 1999, p. 77)

The degree of students' conceptual understanding depends on the quantity and quality of connections that an idea has with existing ideas. The more connections that
exist among facts, ideas, and procedures, the better understanding one has (Hiebert & Carpenter, 1992). It is not an all-or-nothing proposition, but rather can be thought of as a continuum of understanding. During the process of developing conceptual understanding, students have a range of ideas that they connect to their individualized understanding that may be more or less thorough or more or less well developed. Over time, connections become richer and more extensive as students add new knowledge (NRC, 2001; Van de Walle et al., 2010).

Ma (1999) defined the richness of connections as the depth of understanding. The closer an idea is to the structure of the discipline, the more powerful it will be, and consequently the more topics it will be able to support. In contrast, understanding a topic with breadth is to connect it with those of similar or less conceptual power. For example, to connect subtraction with regrouping with concepts such as the rate of composing or decomposing a higher value unit or the concept that addition and subtraction are inverse operations is a matter of depth. To connect it with the topics of addition with carrying, subtraction without regrouping, and addition without carrying is a matter of breadth. Both depend on thoroughness or the ability to connect all parts of the field. It is this thoroughness that “glues” or connects the knowledge of mathematics into a coherent whole.

**Evidence of Conceptual Understanding**

Since incomplete or partial understanding of concepts may lead to gaps in learning, misconceptions, and frustration (Skemp, 1987), it is important to access students’ thinking to ensure they are making productive connections and not engaged in
tangential thinking. Sometimes students fixate on trivial information that does not enhance understanding, which may lead to distractions and the need to fix up meaning (Harvey & Goudvis, 2007).

One way to access students thinking is to engage them in mathematical conversations. Students demonstrate they are developing conceptual understanding by verbalizing to others the connections they make among and between concepts and representations (NRC, 2001). Wade (1990) found that the process of thinking aloud helps students clarify the relationships between these connections and reveals misunderstandings. This process is useful, however, as an indicator of the student’s conscious processing and not thinking that might be implicit (NRC, 2001; Wade, 1990).

Williams (2002) found that having students construct concept maps, similar to Ma’s (1999) above, was a useful way for students to share the connections and reveal the sophistication of their understanding. Novices were more likely to include trivial or irrelevant concepts, while experts included concepts and the relations that connected these concepts at a higher level of complexity signaling a highly integrated knowledge structure.

Another indicator of conceptual understanding is students' ability to represent mathematical situations in different ways and know how different representations can be useful for different purposes. In order to be fluent in the mathematics Discourse, students need to be able to see how various representations connect with each other, compare the similarities, and contrast the differences between them (NRC, 2001).
Benefits of Developing Conceptual Understanding

Students experience many benefits when they develop conceptual understanding (NRC, 2001). Students who make connections between ideas remember what they have learned better than those who do not. When students use a mnemonic technique to learn a procedure by rote (e.g., SOHCAHTOA to remember how to calculate sine, cosine, and tangent of an angle), it may make the procedure seem easier in the short term, but might not lead to understanding. Students who understand a method or procedure are able to continually monitor its use to determine if they are accurate. They ask themselves if their work makes sense and self-correct if necessary.

Another benefit is students' ability to extend learning to new areas. For example, students who have a well-developed conceptual understanding of place value were more likely than students without such understanding to make use of it when connecting the concepts of multi-digit addition and multi-digit subtraction. Learning to add and subtract multi-digit numbers does not involve entirely new ideas for them.

Students who have developed conceptual understanding have less to learn. They are able to see the deeper similarities between seemingly unrelated situations. As noted earlier, understanding that has been encapsulated into compact clusters of interrelated facts and principles can be summarized in a sentence or a phrase (e.g., rate of change) but can be unpacked if the student wants to explain something, reflect on a concept, or make a connection to a new idea.

In addition, students are more likely to avoid critical errors in solving problems, especially errors of magnitude. For example, if they were multiplying 6.34 x 8.42 and
got 53382.8 for an answer, they could immediately see this was incorrect. They know that $6 \times 8$ is only 48, so multiplying two numbers slightly larger than six and eight must have a product slightly larger than 48. It is obvious to them that the decimal point is incorrectly placed and they will take steps to move it. They realize 53.3828 is a more reasonable answer than their first attempt.

Summary

This section demonstrated that for students to fully understand mathematical language and the knowledge it represents, they must develop conceptual understanding. The more connections students make between declarative, procedural, and conditional knowledge in rich and extensive ways, the more complete their understanding is of a concept. This better enables them to construct and articulate their reasoning, construct worthwhile arguments, and analyze the reasoning of others. They are better able to remember, extend their learning in new areas, have less overall to learn, and avoid critical errors of magnitude.

Unlike other topics, however, partial or incomplete understanding in mathematics leaves students with critical holes in their learning. Because mathematics concepts build upon one another in a hierarchical fashion, these gaps may preclude a student from further learning without thoroughly crafted explanations and supportive reasoning of others. This requires us to explore the talk that occurs in classrooms as teachers and students engage in mathematical discourse. We will turn our focus to teachers' use of language during mathematics lessons.
One cannot underestimate a teacher's influence on student learning (Darling-Hammond, 1999; Fennema & Franke, 1992; Sanders & Rivers, 1996). In order to achieve the vision set forth by reformers (NCTM, 1989; 2000), most teachers are required to reshape their teaching practices, construct different roles for themselves and different expectations for their students, and teach in new ways they have never experienced before (Darling-Hammond & McLaughlin, 1995). These new ways of teaching call for new ways of talking, for the long-standing cultural view is that “teaching is talking” (Stubbs, 1983, p. 17).

NCTM recognized the importance of classroom talk in reformed classrooms by focusing three of the six Professional Standards (1991) on discourse. “The discourse of a classroom—the ways of representing, thinking, talking, agreeing and disagreeing—is central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing” (p. 34).

Teachers’ use of language and the manner in which they guide classroom discussions are guided by their goals, beliefs, and knowledge (Ball, 1996; Brendefur & Frykholm, 2000; Schoenfeld, 1998). This talk reflects both the teachers’ planned and moment-to-moment decisions and actions in the classroom (Schoenfeld, 1998). In turn, it determines, to a large extent, what students learn in their classrooms (Fennema & Franke, 1992).

Briefly, teachers’ goals are the object of their ambition. They range from very long-term (e.g., “I want students to make deep and broad connections between
mathematical ideas at the conceptual level”), to medium-term (e.g., “the students need to understand that a fraction is part of a whole unit”), to short-term (“the solutions to this problem should be on the board clearly for their notes”; Schoenfeld, 1998).

Teachers’ beliefs, influenced by their own personal and professional histories (Ball, 1996), determine which goals and actions have high priority (e.g., What counts as “understanding”? How important are formulas? Is a good classroom a quiet classroom?). How the teacher feels about these and other issues determine which goals have the highest priority and in turn shape what the teacher chooses to do (Schoenfeld, 1998).

Teachers’ knowledge has to do with the intellectual resources the teacher brings to the situation. It includes knowledge of the content, the context, and the students. It is both what one knows (the inventory of knowledge) and how that knowledge is organized and accessed (Fennema & Franke, 1992; Schoenfeld, 1998).

Within mathematics, content knowledge can be further broken down into subject matter knowledge and pedagogical content knowledge (Ball, Thames, & Phelps, 2008; H. Hill & Ball, 2009). Subject matter knowledge has to do with the teacher’s own level of conceptual understanding of mathematics (facts, terms, concepts, organization of ideas and connections among them, ways of thinking, arguing, and their growth within the discipline). It is an important factor in how they teach the subject (Schoenfeld, 1998) in both in the selection of worthwhile tasks (Ball, 1996) that elicit thought and consequently rich discussion (Henningsen & Stein, 1997; NCTM, 1991; Stein, Grover, & Henningsen, 1996), and in how they facilitate that discussion (Ball, 1996). Pedagogical content knowledge includes the teachers’ purpose for teaching the subject, knowledge of
curriculum and materials, knowledge of strategies and representations for teaching particular topics, and knowledge of students' understandings and potential misunderstandings, (Ball et al., 2008; H. Hill & Ball, 2009; Schoenfeld, 1998).

Context knowledge has to do with the situations and places in which teachers work. Knowledge of district- and state-level objectives, curricular guidelines, tests, parents, and administrators are all parts of the context of teaching (Ball, 1996).

Knowledge of students is essential to teaching for understanding. Learning more about students and about listening to them can be crucial. Hearing what students say requires one to experience the world through another's perspective, which is difficult given the diversity of perspectives (Ball, 1996).

How the teacher integrates these three areas, beliefs, goals, and knowledge, impacts the quantity and quality of teacher talk in the mathematics classroom (Schoenfeld, 1998). First I will describe the quantity of teacher talk in the classroom and factors that contribute to it. Then I will explore ways that teachers improve the quality of what they say.

Quantity of Teacher Talk in the Mathematics Classroom

In the traditional classroom, nearly 70% of classroom time is dominated by teacher talk (Flanders, 1970). This figure is raised to well over 80% if only lesson time is observed (McHoul, 1978). The most common discourse pattern is characterized by an interchange of teacher questions (initiation), student answers (reply), and teacher comments (evaluation; McHoul, 1978; Mehan, 1979) or teacher feedback (subsequently changed to "follow-up"; Nassaji & Wells, 2000; Sinclair & Coulthard, 1975). This
initiation-reply-evaluation (IRE) or initiation-reply-follow-up (IRF) exchange pattern has consistently been documented to account for 70% to 80% of teacher-student interactions (McHoul, 1978; Mehan, 1979; Nassaji & Wells, 2000; Sinclair & Coulthard, 1975; Wells, 1999), with the teacher responsible for two-thirds of the exchange. Both patterns are highly teacher-directed in that the teacher controls the turn-taking and has ultimate authority over what will be accepted as an appropriate response (Culican, 2007). This pattern of communication is labeled “uni-directional” (Brendefur & Frykholm, 2000, p. 126), or “univocal” (Truxaw & DeFranco, 2008, p. 489), where the “listener receives the ‘exact’ message that the speaker intends for the listener to receive” (Knuth & Peressini, 2001, p. 321). Therefore, the teacher’s goal in a uni-directional/univocal discourse is to transfer new meaning to the listener (Knuth & Peressini, 2001).

Efforts to reform the mathematics classroom have been motivated in part to address the kind of uni-directional/univocal communication pattern described above (Brendefur & Frykholm, 2000). In order to fully implement the Communication Standard (NCTM, 1989, 2000) the Professional Standards (NCTM, 1991) provided the following guidance to teachers:

The teacher of mathematics should orchestrate discourse by–posing questions and tasks that elicit, engage, and challenge each student’s thinking; listening carefully to students’ ideas; asking students to clarify and justify their ideas orally and in writing; deciding what to pursue in depth from among the ideas that students bring up during a discussion; deciding when and how to attach mathematical notation and language to students’ ideas; deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty; monitoring students’ participation in discussions and decide when and how to encourage each student to participate. (p. 35)
Despite teachers' best intentions to create this type discourse in their classrooms, their goals, long-held beliefs about teaching and mathematics, and lack of content knowledge override their best intentions. I will illustrate with an example from research.

Schoenfeld (1998) described an Algebra I lesson segment taught by Mark Nelson. The topic of the entire lesson was the arithmetic of exponents. Mark's goal was to enable the students to deal with the reduction of rational algebraic expressions, with a secondary goal to deal with the special case \[ \frac{xa}{xa} = 1. \] Having worked through examples with the whole class, Mark had the students work in small groups for a while assigning them three problems of which \[ \frac{x_1}{x_2} \] was the third.

Mark's beliefs aligned with the idea that students' comments were provided as springboards for his explanations. Because of this, there was little perceived need to delve deeply into the nature of the students' understandings or misunderstandings. If the student provided an incorrect answer, an option was to simply ignore the response and let another student answer. Mark expected students to have little difficulty with the first two problems and he was correct. He expected some difficulty with the third and planned to deal with the confusion by working through the example at the board.

Mark's knowledge structures (i.e., subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge) varied. Content-wise, the mathematics was simple for him. Pedagogically, Mark expected to use the IRE sequence, which would provide him with the structure that he perceived he needed to keep the students engaged and provide him substance for his summary comments. Since this was his first time
teaching this particular material, he did not have a reservoir of pedagogical content knowledge for this topic.

The lesson segment took about four minutes. Students' provided a variety of responses to the problem, which provided Mark grounds for his working through the expansion for $\frac{x^5}{x^5}$. He wrote $\frac{x^{}x^{}x^{}x^{}x^{}x}{x^{}x^{}x^{}x^{}x^{}x}$ on the board and subsequently canceled x’s by making slash marks through the numerator and denominator. His expectation was that his next question, “So what am I left with?” (Schoenfeld, 1998, Appendix A, line 38) would yield the correct answer, “1.” His plan was to summarize and wrap things up. But, instead, the students called out answers that included “x,” “zero,” “zero over zero,” and “nada” (Schoenfeld, 1998, Appendix A, lines 39; 41).

No longer on familiar ground, Mark attempted to provide an alternative example and chose $\frac{5}{5}$ that involved canceling numbers instead of symbols. The students realized this equaled “1,” but they failed to make the connection he wanted them to make, that the canceling applied to the x’s as well.

Over the next few minutes, Mark tried to explain again without success. Finally, Mark resorted to “telling” the answer when he asserted, “Five over five equals one, so this $\frac{x^{}x^{}x^{}x^{}x^{}x}{x^{}x^{}x^{}x^{}x^{}x}$ (points) is going to equal one as well” (Schoenfeld, 1998, Appendix A, line 76). He reminded the students that when you subtract exponents (as in the original expression $\frac{x^5}{x^5}$) you get $x^0$. He concluded the lesson by writing “$x^0 = 1$” on the board and told the students, “Get this in your notes... any number to the zero power equals 1” (Schoenfeld, 1998, Appendix A, line 87).
Mark did not delve very deeply into student understanding because student responses merely served as contributions for his explanations. A common role many teachers see of themselves is to explain why something works, rather than to shape or respond to the kinds of understanding student comments reveal (Ma, 1999; Schoenfeld, 1998). Had Mark been more prepared, both theoretically and in awareness of students' typical answers, he might not have been caught off guard as he was.

Mark's situation is typical. Orchestrating instructional dialogue is complex (Ball, 1993; Leinhardt & Steele, 2005), but having a comprehensive knowledge package (Ma, 1999) from which to draw is helpful. Teachers run into difficulty when they try to make sense of students' thinking when it is presented as fragmented ideas (Bernstein, 1971; Heath, 1983) or interpret students' unexpected statements and solutions (Ball, 1996), which then must be aligned to the principles of mathematics (Ball, 2000, Ball et al., 2008; Leinhardt & Steele, 2005; Schoenfeld, 1998; Sherin, 2002). They find guiding a class discussion of a mathematical conjecture to be treacherous (Ball, 1996) when they have a partial knowledge structure themselves (Fennema & Franke, 1992; Ma, 1999; Schoenfeld, 1998). When a teachers' subject matter knowledge or content pedagogical knowledge is weak, the last-ditch strategy for dealing with situations where the lesson falls apart is to “tell,” preferably in elaboration or response to the correct answer provided by a student (Schoenfeld, 1998).

Brendefur and Frykholm (2000) cite “Brad” a student teacher who found himself in a similar situation as “Mark,” described above (Schoenfeld, 1998). In fact, Frykholm
(1999, as cited in Brendefur & Frykholm, 2000) found the uni-directional communication pattern to be the preferred teaching style for 85% of beginning teachers.

According to J. P. Smith (1996), "teaching by telling" was also a common teaching practice found among veteran teachers, but for a different reason. Veteran teachers felt a high sense of efficacy when they taught by telling. Given their traditional view of teaching, they felt their central task was to provide clear, step-by-step demonstrations of each procedure, restate steps in response to student questions, provide adequate opportunities for students to practice the procedures, and offer specific corrective support when necessary. If students did not master a procedure, they should repeat their demonstration. They should also provide recurrent opportunities for students to refresh and strengthen their mastery of previously taught content. Teaching by telling provided a clear-cut basis on which teachers built a sense of efficacy. They felt they must project their authority in the classroom. They conceptualized a link between their actions and their students' learning.

In both these settings, novice and veteran teachers tended to promote mathematics as a static body of knowledge, first interpreted and conveyed by the teacher, then passively received by the learners (Brendefur & Frykholm, 2000). The uni-directional/univocal communication pattern is associated with the deductive model of teaching (Truxaw & DeFranco, 2008). It does not provide opportunity for teachers to encourage students' construction of a broad range of mathematical concepts called for by NCTM (1989; 2000) reforms. It does provide, however, for a large quantity of teacher talk.
Quality of Teacher Talk in the Mathematics Classroom

Although meaningful discourse can enhance learning, the mere presence of talk does not ensure that understanding follows. The quality and type of discourse are crucial to helping students think conceptually about mathematics (Brendefur & Frykholm, 2000; Nathan & Knuth, 2003; Truxaw & DeFranco, 2008). In contrast to the unidirectional/univocal communication pattern that yields a high quantity of teacher talk, discourse that is characterized by a give-and-take communication that uses dialogue as a process for thinking is labeled “dialogic” (Truxaw & DeFranco, p. 489). In dialogic communication, the listener initially receives the “exact” message sent by the speaker, and then generates meaning by using dialogue as a “thinking device” (Lotman, 1988, p. 36, as cited in Knuth & Peressini, 2001, p. 321). Therefore, the teachers’ goal for dialogic discourse is to generate meaning (Knuth & Peressini, 2001). It is associated with an inductive model of teaching (Truxaw & DeFranco, 2008).

Reform-based mathematics classrooms embody dialogic discourse in which both teachers and students are responsible for contributing to discussions about mathematics (Knuth & Peressini, 2001) and share the authority for presenting valid and logical argument supported with mathematical evidence (Forman et al., 1998; Hamm & Perry, 2002). When implemented successfully, students in reform-based mathematics classrooms spend more time with each problem, are asked more questions requesting them to describe and explain alternative strategies, talk more using longer responses, and show higher levels of performance on most types of assessment items (Hiebert & Wearne, 1993).
Dialogic discourse requires a level of sophistication and expertise compelling the teacher to know when to "step in" to become part of the mathematics conversation and "step out" to teach the rules and norms of discourse (Rittenhouse, 1998). I will illustrate with an example from research.

Schoenfeld (1998) described a physics measurement lesson segment taught by Jim Minstrell. Jim's lesson on the surface paralleled Mark's lesson described above. As with Mark's lesson, Jim's lesson proceeded according to plan until a classroom event caused him to make an impromptu decision about how to guide the classroom discussion.

Jim engaged his students in a Blood Alcohol Content problem. In this scenario, five individuals measured the blood alcohol level of a person arrested for drunk driving, each of whom had obtained five different numbers. The question Jim posed to his class was, "How could one deal with those data in a sensible way and arrive at a "best value" for the driver's blood alcohol level?" (Schoenfeld, 1998, 6.1 Context section ¶ 3). The class explored reasons for considering some data and not others and the idea that choosing the numbers is a matter of discretion and judgment, not just following the rules. The class explored the three measures of central tendency, mean, median, and mode and discussed which may be more or less meaningful given the situation.

Much of the lesson went according to plan, until a student suggested a novel method for computing the average value, which led Jim and the class into unexpected territory for eight minutes. How Jim decided to handle this student's suggestion was consistent with his goals and beliefs, and was very much a function of his comprehensive knowledge base.
Jim's long-term *goal* for the class was to foster the development of a sense-making community. His specific content-*related* goals for this lesson were for students to have a sense of the issues related to which numbers count, how to combine the numbers, and how to think about measurement error and precision in reporting. He believed in student inquiry and placed a very high priority on pursuing *substantive* ideas raised by students over his need to follow a predetermined lesson plan. Jim had taught the content many times and was able to anticipate most student responses and how he planned to respond.

Jim engaged students in dialogic discourse making use of a questioning method he called the "reflective toss." Jim described this as "'catching' the meaning of the student's prior utterance and 'throwing' responsibility for thinking back to the students, not only to the individual student doing the talking but also to all of the students in the class" (van Zee & Minstrell, 1997, p. 227, as cited in Schoenfeld, 1998, 6.2.2 Lesson Image section ¶ 4). He applied a sequence of exchange called "interactive elicitation" (Schoenfeld, Minstrell, & van Zee, 1996, as cited in Schoenfeld, 1998), in which he asked the students to review for *him* the purpose of the discussion and to provide clarification, to which other students or Jim would provide elaboration with additional detail. Once all students had responded, Jim applied significant wait time (10 seconds) for his students to think. If the students were unable to generate the discussion he had hoped for, he suggested there was more, and offered an idea if they were unable to. Then he repeated the entire sequence again, until the topic was complete. This required Jim to
draw from a comprehensive knowledge package (Ma, 1999) of the topic in order to guide a successful discussion.

Jim’s lesson proceeded as planned with students discussing which numbers should be taken into account and why. After an extended discussion, Jim asked his generic prompt, “Anybody think of another way of giving a best value?” (Schoenfeld, 1998, Appendix B, line 163). It was at this point a student offered,

This is a little complicated but I mean it might work. If you see that 107 shows up 4 times, you give it a coefficient of 4 and then 107.5 only shows up one time, you give it a coefficient of one, you add all those up and then you divide by the number of coefficients you have (Schoenfeld, 1998, Appendix B, lines 164-167).

Applying interactive elicitation, Jim requested clarification and elaboration of one possible meaning of the student’s suggestion, which produced the following formula:

\[4(107.0) + 1(106.8) + 1(107.5) + 1(106.5) + 1(106.0) / 8.\]

Upon asking the students for their opinion, Jim heard the audible comment, “It’s the same.” He summarized, “All right. So actually it ends up being the same as the arithmetic average?” (Schoenfeld, 1998, Appendix B, lines 204-205), to which the student who offered the formula replied, “No. Because 107 gets four times the value, so the 107 counts more” (Schoenfeld, 1998, Appendix B, line 206).

Instead of responding directly to the student who was confused, Jim asked the student who originally defined “average” for an elaboration of her original definition. By working out the implications of the definition with the whole class, everyone agreed that the two methods, arithmetic average and the proposed formula as noted above, were the same. The student who offered the formula stated that she did not realize it when she proposed the example, which raised the question to Jim that he should extend the
discussion by comparing weighted and unweighted average. This prompted further discussion when the class asserted that the two formulas would produce different results. Jim pushed further and asked the students to provide an example of why the two would differ. He pursued and elaborated on an example proposed by a student in order to clarify the concept. At the end of the lesson the students were in clear agreement and had develop conceptual understanding of arithmetic average. As Lampert (1990) noted, it is important for students to see what sort of knowing mathematics involves. Therefore, the teacher must make explicit the knowledge he or she uses to carry on an argument about the usefulness of a solution strategy.

Jim was true to his goals for sense making and his beliefs about encouraging students' contributions. He drew on his extensive knowledge base by drawing from three knowledge resources. First, Jim drew on his subject matter content knowledge when he recognized the student’s formula as dividing a weighted sum by the sum of the weights, a standard way to represent the average of a collection of numbers. Because Jim was able to recognize immediately what the student was doing, he was able to proceed smoothly with the lesson. Second, Jim drew on his knowledge of pedagogy with his regard to classroom organization, supportive interaction, and informal tone. Finally, Jim’s deep pedagogical content knowledge was useful as he knew ways that students were likely to understand or have difficulties with the topic at hand.

Chazan and Ball (1995) cited similar lessons that could have unraveled had the teacher researchers not had the wealth of knowledge from which to draw. In Chazan’s Algebra I lesson, students were also computing average. They were arguing over
whether zero dollars was a bonus that should be counted since it described money the workers did not get. At the point the lesson was headed for a yes/no debate, Chazan stepped into the discussion and refocused it away from the zero debate to draw students' attention to the question of the meaning of the number that they got as a result of “taking the average” (p. 6).

In Ball’s (Chazan & Ball, 1995) third grade class, students were having difficulty determining fractional pieces on a number line that had been marked into eighths. Students agreed there were seven pieces between zero and one, which was mathematically incorrect. In order to challenge their thinking, Ball inserted her voice into the conversation. She reminded them of the previous day’s lesson when Sean, a student in the class suggested they make one less line than the number of pieces they wanted. When students still failed to make the connection, she referred to the number line to clarify their counting.

Leinhardt and Steele (2005) provided a detailed analysis of a series of lessons in which teacher researcher, Magdalene Lampert, led her fifth-grade class to understand the relationship between functions and graphs. In the second of ten lessons, Lampert followed her students as they explored navigational directions for locating X on the overhead. One student suggested that the X could be located by using geographic directions in which the location would be Northwest. Another student suggested that degrees, as in compass degrees, might be helpful. Instead of dismissing the student’s suggestion, Lampert followed it and proceeded to teach the convention of degrees. She knew this would provide a useful analogy later when the students graphed their functions
on a coordinate plane using ordered pairs. Among the many salient features Leinhardt and Steele (2005) noted, Lampert drew on her subject matter content knowledge base of mathematics (e.g., what a graph is, what origin and axis are, ways in which rules for mapping can be expressed, and the knowledge about computations with different kinds of numbers), her knowledge of what the students' experiences in mathematics were thus far, and connections to their social studies unit in order to follow the students' thinking.

Teacher researchers conducted these exemplary lessons recorded in the research. Unfortunately, when classroom teachers attempted to engage students in rich dialogic discussions in ordinary classrooms, many were unable to facilitate the mathematical discussions as well as reformers had hoped (Berry & Kim, 2008; Chazan & Ball, 1995; Hamm & Perry, 2002; Nathan & Knuth, 2003; Sherin, 2002; Wallach & Even, 2005). It appeared reformers presented a vision of the ideal classroom without providing the necessary supports for teachers and students to be successful (Ball, 1996). This is not surprising in light of research in the field of professional development. Transfer to classroom practice is non-existent when teachers are presented with theory-only treatments (lectures, discussions, readings; Joyce & Showers, 2002).

Structuring and supervising student interactions so that students can make progress on the problems, learn from each other, and know when they need more expert advice, is very hard. When these things are done well, students can learn a great deal. When superficial aspects of reform are implemented without the underlying substance, students may not learn much at all. (Schoenfeld, 2004, p. 272).

One reason that teachers have difficulty is that they are more likely to "step out" of the conversations entirely. One rationale for this behavior is that teachers are responding to exhortations "not to tell" (Chazan & Ball, 1995) thinking that in order for
discussions to be focused on student thinking, they must avoid providing any substantive
guidance at all (Stein, Engle, Smith, & Hughes, 2008). This was the case for Ann
(Nathan & Knuth, 2003), an experienced middle school mathematics teacher who
attempted to change her classroom practice to better reflect her vision of reform-based
mathematics instruction.

Though Ann’s goals and beliefs changed little during the first year, she set out to
incorporate more student participation the second year. To accomplish this, Ann decided
to remove herself physically (e.g., to remain outside the ring of student desks) so she was
less central to the discussion and resolved to talk less. Indeed, student talk increased
from one percent of all whole-class talk the first year to nearly 33% the second year.
Without Ann participating as a mathematical authority in the discussions, a role Ball
demonstrated was essential above (Chazan & Ball, 1995), Ann’s students had no apparent
way to resolve their differences and address their confusion. In one lesson, the students
tried to decide what exactly factors and multiples were. The following exchange took
place:

Brad: [A factor is] a number multiplied by another number to get the answer.
Darias: A number that goes evenly into another number.
Kenny: I think it’s for multiplication and subtraction.
Bob: I think it’s for multiplication. No, I think it’s for everything.
Anthony: Let’s vote! (Nathan & Knuth, 2003, p. 198)

Without Ann to help clarify the students’ confusion, there was no clear authority for the
students to turn to resolve their uncertainty. Voting seemed as reasonable as any other
method for establishing the definitions of these mathematical terms.
Stein et al. (2008) provided additional examples of teachers “stepping out” of classroom discourse. Often, after students had an opportunity to construct solutions to problems, students shared their solutions with the class. Without teacher scaffolding of the presentations, the discussion did not build toward important mathematical ideas, resulting in the impression that the more ways there were to solve the problem the better. At other times, one student presentation would follow another without commentary, either by the teacher or a student, drawing connections among the methods or connections to widely shared mathematical methods and concepts. Teacher participation in the classroom discourse could have prompted richer discussions and guided students to important connections.

Another reason classroom teachers have difficulty with dialogic discourse is the challenge of overcoming old habits. On average, the six first-grade teachers in Hamm and Perry’s (2002) study continued asking known answer questions 91.5% of the time, maintained classroom control by being the sole verifier of students’ answers 96.5% of the time, and responded to students’ explanations with further probing only 18% of the time, mostly in response to an incorrect or incoherent idea or explanation. Of the five Algebra I teachers in Sherin’s (2002) study, over half used their existing content knowledge to implement a new lesson, but in doing so, they implemented the lesson differently than was intended by the curriculum designers, transforming it into a more tradition-looking lesson that was familiar to the teacher. Berry and Kim (2008) observed a first-grade inclusion classroom co-taught by a regular education teacher and a special education teacher. A student teacher and a paraprofessional also provided instruction to students.
Overall the lessons were chiefly recitational, comprised of recall-based questions. The teachers afforded little opportunity for students to provide explanations, share ideas, or assist peers.

A third difficulty teachers have is learning to listen to students, which is essential to teaching for understanding. Hearing students requires experiencing the world through another's perspective. This is not an easy task given the diversity of students' perspectives (Ball, 1996). Sometimes, the teacher has difficulty "hearing" what the students are saying when their own understanding of ways to solve a problem gets in the way.

Ruth (Wallach & Even, 2005), an experienced elementary school teacher observed two students, Sigal and Ore, attempt to solve a problem, "Shirts and Numbers," in which the students were to "\[d\]ivide 15 children in a line into two groups, so that in one group there are 4 players less than in the other group" (p. 394). Interestingly, this problem has no solution, to which the girls had to provide a reason and explain why not. The second part of the problem was to change the numbers to so there would be a solution. Obviously, there were multiple ways to solve the second part of the problem.

Ruth proceeded to provide a detailed description of her observation of the students' thinking which included describing, explaining, assessing, and justifying their thinking, but was troubled that the students did not solve the problem in the same way she would. While Ruth's analysis demonstrated the complexity of the way she heard her students, her own understanding of the solution, which did not correspond to the
students' suggestion, seemed to hamper her hearing. Children's frames of reference are often incongruent to the teacher's ways of thinking (Ball, 1993).

Teachers' choice of task also impacts the amount and type of discourse in the mathematics classroom. Hiebert and Wierne (1993) observed six second-grade classrooms studying place value and addition and subtraction of whole numbers. In the classrooms where teachers worked more problems, which involved only symbol manipulation, less time was spent on each problem, the teachers asked few questions other than recall of facts and procedures, and the students responded in five words or less. In contrast, classrooms where the teachers provided open-ended problems to explore resulted in students working fewer problems, but spent more time on each one. They responded to questions that required them to describe, generate, explain, and analyze strategies to solve problems, resulting in longer descriptions and explanations. Students in these classrooms demonstrated higher levels of mathematical understanding and performance.

As discussed with the exemplar teachers (e.g., Minstrell, Chazen, Ball, Lampert), having a strong knowledge base is critical. Also essential is that teachers have clear and correct language with which to discuss the mathematics. Ball (1990) conducted a large study of 252 pre-service teacher candidates regarding their understanding of dividing fractions. No elementary candidates and less than half of the secondary candidates were successful in generating an appropriate representation of $1 \frac{3}{4} \div \frac{1}{2}$ that would provide a useful explanation if they were to teach the concept. They confounded everyday language and mathematical language in their explanations, confusing the concept of
dividing by one-half ($\frac{1}{2}$) with dividing in half ($\div 2$). Faced with the complex language of mathematics (Adams, 2003; Kane et al., 1974; Pimm, 1987; Wakefield, 2000), "teachers need clear and correct mathematical words to describe problem situations, to question students' unreasoned statements in mathematics, and to encourage students' further research and reading in mathematics" (Bruner, 1973, as cited in Capps & Pickreign, 1993). This is especially critical for low achieving students and English Language Learners (Baker, Gersten, & Lee, 2002; Baxter, Woodward, & Olson, 2001; Carter & Dean, 2006; Hudson, Miller, & Butler, 2006; Martiniello, 2008; NMAP, 2008).

Given the number of challenges facing teachers as they implement reform-based practices, researchers are beginning offer suggestions to assist inexperienced teachers as they orchestrate productive discussions. Stein et al. (2008) recommended the following: anticipate student responses to cognitively demanding mathematical tasks, monitor students' response to tasks during partner and group time; select particular students to present their mathematical responses by purposefully sequencing the student responses that will be shared; and help the class make mathematical connections between different students' responses and between students' responses and key ideas.

Chapin et al. (2009) introduced five teacher talk moves found to be effective in making progress toward achieving the instructional goal of supporting mathematical thinking and learning. Although they acknowledged these were not an exhaustive set, these talk moves provided the framework for analyzing Maggie's talk in this study.

**Revoicing.** Sometimes it is difficult to understand what students are saying when they talk about mathematics. Even if their reasoning is sound, they may not be able to
express their thinking in a logical, coherent manner. While this is problematic for the teacher, it becomes even more so for other students. In order to engage all students in the mathematics discourse, it is important to include students whose contributions may be especially unclear. Deep thinking and powerful reasoning do not always correlate with clear verbal expression. Revoicing provides a way for the teacher to deal with the inevitable lack of clarity that will help them continue to interact with a student in a way that will help him or her clarify his or her reasoning and help other students follow along in the face of confusion (Chapin et al., 2009).

Krusi (2009) noted that she often used revoicing with her middle-school students to connect ideas, affirm, amplify, and clarify student thinking, and to add a layer of mathematical vocabulary onto a statement in order to help move them to use more “official mathematical language” (Herbel-Eisenmann, 2002, p. 104). She also used revoicing to prompt a student to reevaluate an original statement. Upon reflection, Krusi (2009) advised using the talk move judiciously. As she found herself being more purposeful in her use of revoicing and increasing wait time, she found that students seemed to step into the discussion to share their ideas and reasoning more often. They attempted to explain their thinking so others could understand and listen to each other more. Student turns revealed a deepening understanding of mathematical content, demonstrating how they thought about mathematical ideas. It was important for her to balance the use of the revoicing move and use it when it could help students develop mathematically and refrain when doing so would empower students to take greater
ownership of the mathematical conversation and provide them with more opportunities for their own learning.

**Repeating: Asking students to restate someone else’s reasoning.** The teacher extends the revoicing move to students by asking one student to repeat or rephrase what another student has said, and then immediately follows up with the first student. This move is beneficial for two reasons. First, it gives the rest of the class another version of the first student’s contribution providing them more time to process, increasing the likelihood they will follow the conversation and understand the point being made. Second, it provides evidence that the students could and did hear what the first student said. For students to participate in mathematical discussions, they need to be heard and know their ideas are taken seriously (Chapin et al., 2009).

**Reasoning: Asking students to apply their own reasoning to someone else’s reasoning.** In this move, the teacher asks students to apply their reasoning to someone else’s reasoning. One purpose of this move is to elicit respectful discussion of ideas as students argue and challenge ideas, which is critical to support students’ mathematical learning (Chapin et al., 2009) and an essential component of the communication standard (NCTM, 2000).

Gronewold (2009) found it was important to provide her middle-school students with wait time prior to asking them to reason. This allowed other students to determine whether they agreed or disagreed with another student’s response and what thought they wanted to add to the discussion. She also found some students would self-correct once
they heard their own thinking aloud, if they had time to think about what they heard themselves *say*.

Shindelar (2009) found that it was helpful to use this particular talk move when her middle-school students stated two opposing viewpoints when neither was correct. By asking students to agree or disagree, it allowed students to grapple with the reasoning and determine if either conjecture seemed reasonable. This helped promote student-to-student interaction within her classroom and helped her students realize they had a responsibility of making sense of the concepts that their fellow classmates were considering. She found this move was particularly useful when students worked in small groups, extending the critical thinking that occurred and increasing the level of student engagement.

Gronewold (2009) candidly shared that when she was initially learning how to ask students to reason, she found herself trying to steer the conversation to her predetermined way of thinking and to the “right answer.” It was only after much practice, that she learned to listen carefully to the reasoning her students provided.

**Adding on: Prompting students for further participation.** This teacher talk move is used to increase participation in the discussion and to elaborate student thinking. Prompting students for more input on previous statements will, over time, *result in* students showing more willingness to weigh in on what the group is considering (Chapin et al., 2009).

Shindelar (2009) found that this *move helped her students understand that they were expected to make sense of the problem and work together and that she, as the*
teacher, would not be the one evaluating their contributions for correctness. Rather, they were responsible for making sense of one another's reasoning and reaching conclusions together.

Waiting: Using wait time. Wait time is used after asking a question before calling on someone to answer in order to provide time for the students to think. It also comes into play after a student has been called on in order to organize his or her thoughts. Five seconds is a minimal recommended wait time. This move is difficult for most teachers to implement consistently, because teachers are uncomfortable with silence, feeling as though they are putting the student on the spot. Few students, however, are able to organize their thoughts quickly. Without wait time, students eventually give up and fail to participate, knowing they cannot beat the clock.

Summary

It is apparent that the quantity and quality of teacher talk is dependent on the teacher's goals, beliefs, and knowledge structure. The uni-dimensional communication pattern is used when teachers want to transmit knowledge through a deductive approach. This interaction pattern is common in many classrooms and is recognized by the initiation-reply-evaluation exchange where teachers control two-thirds of the talk. Thus, this type of communication produces high volumes of teacher talk punctuated by students' responses that function as a springboard to the teacher's explanation. An extended example was provided of a teacher engaging in the IRE communication pattern and salient features of the resulting discourse were highlighted. Due to the teacher's limited knowledge structure, he was unable to follow student's thinking and resorted to
telling the answer when the students were unable to provide him with the response he was looking for.

In contrast, the dialogical communication pattern is seen in reform-based mathematics classrooms, where the purpose for communication is for teachers and students to co-construct meaning through an inductive approach. It is seen as a give-and-take communication pattern that increases student participation and enhances the quality of the exchange. An extended example was provided of a teacher engaging in the interactive elicitation communication pattern where he regularly used the reflective toss to engage, clarify, and connect students thinking to mathematical ideas. Due to his extensive knowledge structure, he was able to do so with ease. Additional examples from the research provided other instances where students’ thinking was clarified and elaborated on through the use of dialogic discourse.

While there are many advantages to implementing dialogic discourse in the mathematics classroom, many classroom teachers are challenged to do so. Often, they struggle with their own lack of content knowledge. Others, in order to increase student participation, step out of the conversation entirely, leaving students to flounder. Some resort to old teaching habits, or they have difficulty “hearing” students’ thinking and instead focus their listening for the correct answer. They find it difficult to choose tasks that yield reasons to talk, and struggle with their own imprecise use of mathematical language.

In order to assist inexperienced teachers, researchers offered suggestions that enhanced classroom discourse and specific teacher talk moves that increased the
likelihood of success. Five teacher talk moves were described (e.g., revoicing, repeating, reasoning, adding on, waiting) and found to be effective in increasing student engagement in mathematical discourse. These teacher talk moves were important to the structural design of this study and became a primary context by which whole-class discussion was observed.

While teachers' use of language in the mathematics classroom is critical component of successful lessons, students' use of language is increasingly important in dialogic reform-based discourse as students clarify their thinking and learn from others through discussion. We will now turn our focus to students' use of language during mathematics lessons.

Students' Use of Language in the Mathematics Classroom

"When students are given the opportunity to communicate about mathematics, they engage thinking skills and processes that are crucial in developing mathematical literacy" (Pugalee, 2001, p. 296). The *Curriculum and Evaluation Standards for School Mathematics*, (NCTM, 1989) asserted that mathematics and literacy are interconnected and interdependent:

Mathematics can be thought of as a language that must be meaningful if students are to communicate mathematically and apply mathematics productively. Communication plays an important role in helping children construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics; it also plays a key role in helping children make important connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas. (p. 26)

In order to achieve this goal, it is not enough just to have students talk more in the mathematics classroom. They need to be talking mathematically in order to advance their
conceptual understanding of the domain (Sfard, Nesher, Streefland, Cobb, & Mason, 1998). In the next section I will address two issues. The first has to do with the quantity of students' mathematics talk needed within the classroom so that students develop facility, fluency, and ownership of the language. The second has to do with the quality of students' mathematics talk as they grapple with the precision of language needed to express their intended mathematical ideas, the vagueness inherent in mathematical language and in students' conjectures, and the ways students use language to signal sense-making in the process of discovering ideas, seeing relationships, and advancing conceptual understanding.

**Quantity of Student Talk in the Mathematics Classroom**

In order to strengthen mathematical discourse, students must develop a command of the language and possess deep conceptual understanding of word meanings. When a student knows a word well, is able to explain it, and uses it in discourse easily and appropriately, the student develops “word ownership.” He or she takes pleasure and pride in using words independently. During this process, he or she also develops “word awareness,” drawing attention to new words in a more general way. Through increased participation in mathematical discourse, students increase their engagement, and subsequently, increase their use of new words (Beck, McKowen, & Kucan, 2002).

The sheer volume of exposure to the language may be the most important factor accounting for the differences in the contribution of learning from context to vocabulary growth, whether oral (Hart & Risley, 1995; 2003) or written language (Nagy, R. C. Anderson & Herman, 1987). R. C. Anderson, Wilson, and Fielding (1988) studied the
reading habits of fifth graders outside of school. They found that voracious readers spent 200 times as many minutes engaged in text as poor readers, encountering as many as 4,733,000 words per year compared to 8,000 for the poorest readers.

Reading habits within school vary just as widely. Nagy and R. C. Anderson (1984) studied the number of words middle grade students read in a year. They estimated the range from 100,000 for the least motivated to 10,000,000 to 50,000,000 for the voracious reader. Readers who are exposed to more written language acquire an expanded knowledge base that probably facilitates the learning of new words from context with a greater efficiency than less able readers.

Children who possess large vocabularies find learning easy. They read widely and subsequently learn more word meanings in the process, thus improving their reading comprehension and increasing their vocabulary even more. They become active learners by selecting (e.g., by choosing friends who read or choosing reading as a leisure activity), shaping (e.g., asking for books as presents), and evoking (e.g., his or her parents notice the child reading) an environment that motivates further growth in reading and language development. Children with inadequate vocabularies, on the other hand, who read slowly and without enjoyment, read less. This results in a limited development of vocabulary knowledge, hindering further growth in reading ability. They become passive, discouraged learners who do not construct a literate environment (Stanovich, 1986).

Dubbed the “Matthew Effect” after the Biblical Gospel according to Matthew, the reciprocal relationship between vocabulary knowledge and reading ability is compounded, resulting in the linguistically “rich” getting richer and the linguistically
"poor" getting poorer (Stanovich, 1986). Moats (2001) confirmed this phenomena reporting that linguistically "poor" first graders knew 5,000 words, while linguistically "rich" counterparts knew 20,000. By high school, the lowest performing seniors had vocabularies equal to high-performing third graders, about a fourth the size of students near the top of their class (M. Smith, 1941).

The achievement gap between students of different socioeconomic levels is a formidable and persistent challenge to educators. It is acknowledged that the ecology of the lower-class child is complex (Traub, 2000), however, researchers have investigated this problem extensively and identified a limited vocabulary as an important factor in the underachievement of children from low socio-economic backgrounds (Becker, 1977; Biemiller, 2004; Chall et al., 1990; Hart & Risley, 1995, 2003; Moats, 2001; M. Smith, 1941; White, Graves, & Slater, 1990). Hart and Risley (1995, 2003) documented similar stories of children from working class and professional families learning vocabulary, but they revealed a very bleak picture for children who are born into poverty. Differences that began early in a child's life resulted in a dramatic disparity in the exposure to new words among families of different social classes (Hart & Risley, 1995; Heath, 1983). From extrapolated data, they determined that underprivileged children were exposed to 30,000,000 fewer words by age three than their affluent peers. The recorded vocabulary size of children of professional families was larger than the parents of welfare families, more than twice the size and growing at twice the rate of children in these families. Contrasting vocabulary trajectories, presented by Hart and Risley (1995, 2003), evolved into a profoundly widening achievement gap as children entered school.
One of the important benefits of reform-based mathematics classrooms is that children have the opportunity to talk. Unfortunately, not all children feel comfortable in this situation. After 34 observations across five third-grade classrooms, Baxter et al. (2001) noted only on three occasions when low achieving students volunteered to speak during whole-class discussions. When they did, they offered one-word answers or remained silent while a peer spoke, even though their teachers tried to involve them in the discussions. Similar to the passive behavior described by Stanovich (1986), these students were often observed playing quietly with small objects in their laps, staring out the window, writing on a piece of paper, or avoiding eye contact with the teacher. Despite teachers’ efforts to focus and include these students in discussions, they remained uninvolved.

Two features of the classroom discussions made them especially challenging for these low-achieving students. First, the class discussions were often difficult to follow. Ball (1993) questioned her own ability to always understand what her students were trying to explain. If a university researcher and experienced teacher had difficulty understanding students’ reasoning during class discussions, it was likely that low-achieving students struggled as well. Second, the time needed for any kind of extended explanation during whole-class sharing precluded the number of students who could discuss their ideas. Many low achievers appeared to avoid active participation because their more capable and highly verbal peers were likely to volunteer their solutions. Since there was a surplus of highly capable volunteers who did not have an opportunity to present their ideas, it was relatively easy for the low achievers to remain quiet. Low
achievers appeared to be more engaged during partner and small-group discussions, but the kind of mathematics in which they were involved was questionable. They primarily copied their partner’s work or organized materials (Baxter et al., 2001).

The goal of increasing student talk in mathematics classrooms is laudable. However, using Hart and Risley’s (1995) extrapolations, it would require providing an average welfare child 41 hours per week of out-of-home language experience, as rich in words addressed to the child growing up in an average professional home, just to provide him or her with the same language experience equal to that of an average working-class child, and those were intervention figures for a four year old. The need to ameliorate the differences in vocabulary size for school age children grows exponentially each year. Therefore, it is important that the quality of the mathematical talk is of highest priority.

Quality of Student Talk in the Mathematics Classroom

Mathematics material is so difficult to read because it contains “more concepts per word, per sentence, and per paragraph than any other area” (Schell, 1982, p. 544). Several authors (Miller, 1993, as cited in Bryant et al., 2008; Monroe & Orme, 2002,) claimed that unfamiliar vocabulary is a leading cause of mathematics difficulties. Similarly, Bryant, Bryant, and Hammill (2000, as cited in Bryant et al., 2008) asserted that difficulty with the mathematical language is a distinguishing characteristic of a mathematics learning disability.

Lexical Density. Veel (1999) noted that given the considerable technicality and hierarchy of mathematical language, it is not surprising to find that there are significant differences in the language used by students when compared to the complexity of the
language used by teachers and the language found in mathematics textbooks and tests. For example, when considering a recent fourth-grade NAEP (NCES, 2009b) mathematics assessment, students needed a well-developed bank of geometric terminology in order to understand and be able to answer the question. Nine of the fifteen words (60%) were mathematical vocabulary terms. Similarly, a question from the recent NAEP (NCES, 2009c) eighth-grade assessment carried a considerable conceptual load within the hierarchical vocabulary of the foils. (See Figure 2.)

When examining transcripts of classroom discourse, Veel (1999) observed that the language differences between students and teachers or were not due to interaction patterns, but rather the linguistic quality of the utterance. Compared to the teacher talk and to textbooks, the students’ talk was far more like everyday spoken language. Students typically did not use as many mathematical words per sentence as either teachers or the textbook did. Although the topic of the students’ talk was mathematical, the quality of their talk, as measured by lexical density, was about the same as everyday conversation. Lexical density is measured by determining the average number of content words (or lexical items) per clause.

It is important that students are able to use the language of mathematics to express mathematical ideas precisely. Attending to precision is an important focus of the NCTM Standards (2000) and is one of the Standards for Mathematical Practice (CCSSI, 2010). Precise language allows students to communicate their thinking coherently and clearly to their peers.
Grade 4

What is the shape of the shaded figure inside the star?

A. Hexagon
B. Pentagon
C. Quadrilateral
D. Triangle

Grade 8

For a school report, Luke contacted a car dealership to collect data on recent sales. He asked "What color do buyers choose most often for their car?" White was the response. What statistical measure does the response "white" represent?

A. Mean
B. Median
C. Mode
D. Range
E. Interquartile range

Figure 2. Lexical Density of Assessment Items, National Assessment of Educational Progress: Mathematics Assessment, Grades 4 and 8. (NCES, 2009b; 2009c)

Mathematical Vocabulary. Due to the sheer number of technical terms in mathematics, teaching mathematical vocabulary is often the first intervention thought of as a way to help children use language to communicate more precisely. Indeed, one's
vocabulary reveals one's social and educational background. It is a major factor in
determining what we understand, therefore, it impacts our future based on what
information we can or cannot access (Stahl & Nagy, 2006).

One of the longest, most clearly articulated lines of research in literacy education
describes the strong connection between vocabulary knowledge and comprehension
(Davis, 1942, 1944, 1968, 1972; Whipple 1925). Correlational studies, experimental
studies, and readability research all found strong and reliable relationships between the
difficulty of the words in a text and text comprehension (R. C. Anderson & Freebody,
1981). Vocabulary knowledge and reading comprehension correlate so highly (in the
0.85 to 0.95 range) that some argued they are psychometrically identical (Carver, 2003;
R. L. Thorndike, 1974). The statistical relationship between vocabulary size and
intelligence is so strong that sometimes a vocabulary test alone is used in place of a full-

In addition to intelligence, vocabulary is also closely associated with knowledge.
A person who knows more words can speak, and even think, more precisely about the
world. A person who knows the terms chartreuse and jade and amber and goldenrod can
think about colors in a different way than a person who is limited to green and yellow.
Stahl and Nagy (2006) go so far as to assert, “It may overstate the case to say that
vocabulary knowledge is central to children’s and adults’ success in school and in life,
but not by much” (p. 4). After all, vocabulary provides access to concepts (Monroe &
Panchyshyn, 1995) that would be hard to understand without words. “The more words
we have, the more complex ways we can think about the world” (Stahl & Nagy, 2006, p.
5). Deep conceptual understanding of word meanings is at the very heart of understanding (Gee, 2008).

In E. L. Thorndike’s (1917) pioneering work, he concluded,

Understanding a spoken or printed paragraph is then a matter of habits, connections, mental bonds, but these have to be selected from so many others, and given relative weights so delicately, and used together in so elaborate an organization that “to read” means “to think,” as truly as does “to evaluate” or “to invent” or “to demonstrate” or “to verify.” (p. 114)

This is an important point to emphasize: vocabulary (comprehensively understanding what words mean) is as essential for evaluating, inventing, demonstrating, and verifying mathematics as it is for reading comprehension.

Capps and Pickreign (1993) cited three important reasons to teach vocabulary in the mathematics classroom. First, mathematical context, language, and symbolism possess unique meanings not encountered in other subjects. Second, these meanings are not reinforced outside the mathematics classroom in the same way as more common language is used to communicate ideas. Third, knowledge of a large vocabulary is related to high comprehension in a subject and that vocabulary should always be learned in a meaningful context.

This is especially important because of the huge differences that exist among the vocabularies of students. Becker (1977) suggested that, once decoding skills are mastered, the primary remaining barrier to school success for children of low socio-economic background is an insufficient knowledge of word meanings. Generally, low-SES students experience less abstract or decontextualized language (Heath, 1983), and encounter fewer words of Greek or Latin origin (Corson, 1985), which account for
approximately 70% of the English language linage (Green, 2003). Knowing these morphological (meaning) connections are helpful for students who are learning mathematical vocabulary. For example, the concept of fraction is lost on many students as they try to manipulate numerators and denominators. Knowing the Latin root fract- means "to break" (e.g., fracture), helps students understand that fractions deal with breaking things into parts and manipulating those parts (Templeton, 2004).

Vocabulary is also one of the primary challenges facing students who come from non-English-speaking homes. A child may achieve fluency in conversational English in a year or so, but it may take an English language learner five or more years to understand the vocabulary of academic English (Collier, 1989; Cummins, 1994).

An assumption was made that teachers could help young learners to understand or comprehend if word meanings were simply taught. Assigning students lists of new words to look up, record, and memorize the dictionary definitions, and asking them to create sentences containing the new words became synonymous with vocabulary instruction (Beck et al., 2002). This method generated boredom, rather than learning (Beck et al., 2002; Stahl, 2005), and is considered ineffective (Beck et al., 2002; Stahl, 2005; Stahl & Nagy, 2006).

Many researchers (Blachowicz & Fisher, 2002; Beck et al., 2002; Stahl & Nagy, 2006) suggest there is a continuum of word learning. Many mathematics educators advocate an indirect method of vocabulary acquisition preferring to develop the concept prior to the introduction of formal definitions (Herbel-Eisenmann, 2002; Keiser, 2000; D. R. Thompson & Rubenstein, 2000). Herbel-Eisenmann (2002) described how the idea of
“bridging” helped students move from less to more mathematical language by encouraging multiple ways of talking about ideas. As students start to grapple with new ideas, they may “invent” vocabulary that pertains to the mathematical concept (D. R. Thompson & Rubenstein, 2000), but is idiosyncratic to the classroom in which it is generated (Herbel-Eisenmann, 2002).

For example, students first exploring the concept of slope may call it “slantiness.” As students continue to work with the idea, they begin to use transitional mathematical language. This is language that describes a location or process that is associated with a particular representation, which includes certain set phrases that are repeated often in the classrooms, but does not include a contextual reference. When referring to “slantiness,” they begin to talk about a constant pattern, what it goes up (or down) by, what is added each time, the relationship between the amount $x$ goes up compared to the amount $y$ goes up, and so forth. Eventually, students adopt the Official Mathematical Language, which is part of the mathematical register. This is when students use actual mathematical vocabulary of “slope,” “coefficient” and “rise/run,” which would be recognized by anyone in the mathematical community (Herbel-Eisenmann, 2002).

In a study of the use of scientific language in the elementary school, Cervetti et al. (2006) acknowledged that one strategy for dealing with the obscurity of technical discourse was to avoid it or at least delay its use until middle or high school. They argued, however, that just as young learners benefit from the chance to acquire the tools of inquiry-based science, so too they deserve a chance to acquire its discourse. In order to learn the complex scientific discourse styles, including the vocabulary necessary to
“share, clarify, and distribute knowledge among peers” (p. 25), they asserted that students benefited from thoughtful immersion in and exposure to the technical language early and often. Unless students “jump in” (p. 25), they are not likely to get better at doing and talking science. They found students as young as second and third grade can use the discourse as they participate firsthand in scientific inquiry. A similar argument could be made for using the discourse as students participate firsthand in mathematics inquiry, as well.

Cervetti et al. (2006) recommended a four-pronged approach to science vocabulary instruction that can be applied to mathematics: create an environment that is rich in mathematical words and linguistic structures, select vocabulary representing key concepts along with key words needed to communicate mathematical activities and ideas to others, use everyday language to introduce and build a conceptual bridge to more mathematical language, and immerse students in firsthand investigations in a way that connects the language to the activity.

As helpful as the use of context can be in developing mathematical vocabulary, it is usually not sufficient (Monroe & Orme, 2002). Some mathematical vocabulary must be taught directly (Fogelberg et al., 2008; Monroe & Panchyshyn, 1995). Baker et al. (2002) found that low-performing students responded more positively to explicit instruction. The NMAP (2008) reported that explicit methods of instruction were effective with students with learning disabilities as well as low-achieving students.

According to Paynter, Bodrova, and Doty (2005), the structure provided by direct instruction was helpful when students were learning complex vocabulary. Students
should experience at least six exposures to the new word during the initial lesson in order to develop the breadth to be able to use the new word (Capps, 1989, as cited in Monroe & Panchyshyn, 1995; Paynter et al., 2005), with at least another 30 additional exposures over the course of the following month (Capps, 1989, as cited in Monroe & Panchyshyn, 1995). Often, a mental image or symbolic representation of their new word was useful (D. R. Thompson & Rubenstein, 2000) in order to associate a nonlinguistic representation to the linguistic understanding they were developing (Paynter et al., 2005). Paynter et al. (2005) identified the following six-step approach to teaching vocabulary through direct instruction:

Step 1: The teacher identifies the new word and elicits students’ background knowledge.
Step 2: The teacher explains the meaning of the new word.
Step 3: Students generate their own explanations, and the teacher helps clear up confusion or misinformation.
Step 4: Students create a visual representation of the new word.
Step 5: Students engage in experiences that deepen their understanding of the new word.
Step 6: Students engage in vocabulary games and activities to help them remember the word and its meaning. (p. 36)

Beck et al. (2002) suggested using student-friendly explanations. These were used successfully to increase understanding of word meanings by characterizing the word and how it is typically used, followed by explanations of its meaning in everyday language. These explanations were longer than typical dictionary definitions and used language that was easily understood by students. These definitions often started with “something that...” or “somebody who...” The explanation provided students with conditional knowledge that included when to use the word and why the word was useful. Providing student-friendly explanations or scaffolding students’ learning as they derived
word meanings from instructional contexts was only the first step in helping students develop understanding of what a word meant. It was also essential that students dealt with the meaning right away by using the word in context (Beck et al., 2002).

The "Typical-to-Technical" Meaning Approach (Pearson & Johns, 1984, as cited in Blachowicz & Fisher, 2002) required students to engage in discussion in order to clarify the differences between an earlier meaning of a known word and the new meaning. This technique was found useful in a junior high mathematics class that was studying the words **acute, complementary, angle, and supplementary** (Welker, 1987, as cited in Blachowicz & Fisher, 2002). Students discussed the common meaning of the word and then the teacher introduced its technical definition. Students continued word study through matching activities and the cloze procedure.

A variation of the K-W-L (Ogle, 1986) procedure focused on vocabulary drew students' attention to new meanings for words. In the traditional K-W-L lesson, students write what they know about a topic (K), what they want to know about it (W), and what they have learned (L). When a topic is particularly vocabulary laden, the teacher selects vocabulary words from a topic and guides students to discuss what they know about the words. Students develop questions based on their knowledge and the selected vocabulary. After the students spend time learning about the topic, they categorize the words and describe what they have learned (Blachowicz & Fisher, 2002).

Cunningham (2009) and D. R. Thompson and Rubenstein (2000) suggested incorporating writing journals that focused on new mathematical vocabulary. Prompts such as "Tell the meaning of **range** in statistics, in studying functions, and in everyday
English. What ideas do all the meanings share? What is special about each of the mathematical meanings?” (D. R. Thompson & Rubenstein, 2000, p. 571, italics in original) are useful. Students peer-edit the definitions and discuss their validity and clarity.

D. R. Thompson and Rubenstein (2000) also suggested students build models of vocabulary words, such as factors or multiples using manipulatives, or a tetrahedron by folding paper. This kinesthetic approach allowed students to see and feel vertices, edges, and faces, and comprehend why height and slant height are different and provided them with a concrete example.

Mathematical Vagueness. Using technical vocabulary is a sign of a mature language user who is able to express ideas clearly and precisely, but vagueness in mathematical language also serves a pragmatic purpose. Pimm (1987) argued that vagueness stems from the dehumanized authority of traditional mathematics. By creating distance from the speaker, it communicates the idea of “mathematics as a spectator sport” (p. 71).

It is common for the pronoun we to be used instead of I in mathematical discourse. Its effect is to move away from the individual as mathematician and convince the novice that there is power in some invisible authority (Pimm, 1987; Rowland, 2000; Wagner, 2007). “What do we take from the tens column? We take a ten, don’t we?” (Pimm, 1987, p. 65). The use of “we” conveys the message that there are “in-groups” where “‘doing it right’ has solely to do with conforming to the uniform practice of this elite group” (p. 70).
Traditional mathematics is also known for its use of the passive voice (Wagner, 2007), as in the "the liquid was weighed," imperatives (Pimm, 1987; e.g., consider, suppose, define), and the ubiquitous "it" (Pimm, 1987; Rowland, 2000), which all work together to convey an impersonal mood. Pimm (1987) questioned the phrase, "Let x be the number of..." by asking, "Who is giving the permission for this to be done?" (p. 72). Taken in sum, these traditional linguistic features provide a paradox to the active social participation in the mathematics discourse the reformers hope to create.

Whether through imitation of tradition or for some other reason, vagueness found its way into students' language. As Pimm (1987) suggested, many students are so concerned with the notion of "doing it right" they use "hedges" to avoid committing to precision in their language (Rowland, 2000). There are two main types of hedges. Shields are used when students want to "shield" themselves from accusation of error. The first type, called the plausibility shield, is used to express some doubt that their conjecture will stand up to scrutiny. Students use phrases such as "I think," "maybe," or "perhaps" in order to hedge their commitment to their assertion. The attribution shield alludes to the knowledge of some third party. The student may say, "According to Sam...." or may not name the informant at all. By using a shield, it provides the student the opportunity to offer a conjecture without making a full commitment to the idea.

A second type of hedge is the "approximator." The purpose of the approximator is to modify the proposed idea in order to make it more vague. One type of approximator is the rounder, which includes the standard adverbs of estimation, such as "about," "around," and "approximately." Rounders are commonly used when discussing
measurement and quantitative data. The second type of approximator is the adaptor. These words or phrases, such as “a little bit,” “somewhat,” or “sort of” attach vagueness to nouns, verbs, or adjectives (Rowland, 2000).

Cass (2009) observed how her middle-school students used vague referents such as it, that, and this to stand for the names of mathematical objects and processes. Rowland (2000) suggested that students use these referents for pragmatic reasons, in order to be able to say what they could not say otherwise or to draw attention to the mathematical terms for which they do not know or remember. Cass (2009) found that there was often a gap between her students’ knowledge of word meanings and application of them in connected discourse, despite the fact that she modeled the use of precise vocabulary in her own use of language. Huang, Normandia, and Greer (2005) noted similar observations of high school students who were reticent to express higher-level knowledge structures in their talk even though their teacher modeled the structures for them. These results suggest that the features of language do not automatically transfer to student discourse through class discussion, but need additional support from the teacher (Cobb et al., 1993).

Rowland (2000) offered teachers two options when dealing students who use vague referents. Either conspire with the student by taking up the referent in his or her own language (i.e., “Why can’t we do it?”) or confront the vagueness (i.e., “Wait a minute, what is this ‘it’ you’re talking about?”). Huang et al. (2005) suggested that students need explicit models of how to “talk math” in order to be successful. They also suggested putting students in a position of “teaching” so that they were required to
demonstrate the knowledge they possessed. D. R. Thompson and Rubenstein (2000) supported the notion of clarity. "...[T]he vocabulary issues ... are the 'surface structures' used to transmit ideas as we engage students in discussions that lead to the 'deep structures' of mathematical concepts" (p. 568).

Rowland (2000) suggested that teachers create a "Zone of Conjectural Neutrality" (ZCN) where students can tentatively offer conjectures that can be tested at the cognitive level rather than at the affective level. The solution can be argued as "true" or "false" rather than the student being judged as "right" or "wrong." When students offer their ideas through the use of shields (e.g., "I think...") "Perhaps...") and approximators ("It is about..."), they are able to express and own ideas, but present them in a way that makes the conjecture nearly unfalsifiable. By presenting ideas as fallible and possibly in need of modification, it opens students to think critically about the argument being presented and feel safe in offering ideas to counter or elaborate on the line of reasoning.

Connecting and Generalizing. When students connect ideas and concepts, they may signal this through overt language (i.e., "This reminds me of...", "Now I get it...", or "Wow, I just learned..."); Harvey & Goudvis, 2007). At other times, the language students use to signal these moments of conceptual understanding are much more subtle. Rowland (2000) contends that the pronoun "you" has a particularly significant role in mathematical conversation.

"You" is usually used to address the person to whom one is speaking. Rowland (2000) noted that students use the pronoun "you" to refer to another student. "Craig, you've got sixty-one now" (p. 109), but rarely addresses the teacher as "you." He
described the pronoun “you” as a linguistic devise that points to another. In the adult-child relationship, it is considered rude for children to point at adults, so the use of the pronoun “you” as a means of address can be interpreted as a message of power. Teachers use the word “you” with students, but rarely the other way around.

Rowland (2000) cited his findings in which students typically use the pronoun “you” to point to general ways of doing things, as when providing explanations. He provided following example: Simon began by using the pronoun “I” to indicate how he would calculate a particular triangular number, “And to do that, I times it by... so I do forty-eight times – no, I do forty-nine times half of forty-eight, which is twenty-four” (p. 110). But then Simon shifted to the vague generalizer “you” when he said,

> Because to work out a triangular number, you get the first and the last, and the second and that...” [T: and multiply it by how much?] “Um, the num... a half of the number... of ... half the number of numbers you’ve got. So it’s like from nought to forty-eight, so half of that, ’cos you’ve only got half the numbers to work out. (p. 111)

Rowland (2000) noted that Simon persisted with the personal “I” to describe how he would calculate a particular triangular number. He shifted to the pronoun “you” when he formulated a *general procedure* for the calculation and explained why it worked.

Rowland (2000) cited multiple examples where students signaled they had realized a generalization through the pronoun shift from “I” to “you.” He asserted that “the pronoun ‘you’ is an effective pointer to a *quality of thinking* [emphasis added] involving generality; the shift from ‘I’ to ‘you’ commonly signifies reference to a mathematical generalization” (p. 113).
Summary

There is strong evidence that the quantity and quality of students’ mathematical language varies widely. Children raised in homes where they hear an abundance of rich vocabulary and read widely, grow up to possess rich vocabularies themselves. Those who are not talked to often, or who do not access and acquire the language through reading, find themselves lagging behind. It is estimated that the gap between linguistically rich and poor students runs in the millions of words, by some estimates up to 30,000,000 by age three, widening exponentially each year.

Fortunately, student talk is valued in the reform-based mathematics classroom. Unfortunately, not all students are talking proportionate amounts. Low-achieving students, in particular, found it difficult to follow the logic of their peers. They often avoided being called on while their highly verbal peers eagerly volunteered. According to the research, the students who need the most practice with mathematical talk received the fewest opportunities to do so. There is a critical need to ameliorate this discrepancy.

While it is important to increase the opportunity for all students to talk in the mathematics classroom, research is clear that students need to be talking mathematically in order to be able develop facility with the language needed to express themselves clearly, articulate conjectures, and communicate logical reasoning. Mathematical educators often develop a students’ understanding of a new concept prior to attaching the word label to it by using “bridging” language with students as they move from less to more technical mathematical language.
Others recognized that some vocabulary needed to be taught directly. Researchers offered specific vocabulary strategies to help students learn new words and develop word ownership. These techniques helped students learn technical vocabulary, but without rich discussions, the vocabulary often did not transfer to daily use. In these situations, students resorted to vague referents. These referents served a pragmatic purpose by permitting the student to talk about mathematical ideas for which they had not yet learned the technical vocabulary or could not remember the more precise term.

Researchers also found that students used hedges such as shields and approximators to purposely infuse vagueness into their language. When their conjectures were offered as fallible, they were less likely to be criticized for faulty thinking. One suggestion was to provide students the Zone of Conjectural Neutrality where they could feel safe in offering ideas to counter or elaborate on the line of reasoning.

Finally, students were found to signal conceptual understanding by announcement or through the use of pronoun shifts. The pronoun “you” played a significant role in this process as it proved to be an effective pointer to the quality of thinking involving generality. When students were heard shifting their use of pronouns from “I” to “you” during a mathematical explanation, it paralleled the “aha” moment that occurred when their thinking shifted from individual problem-solving to a generalization of learning.

**The Reform-based Classroom Environment**

In the traditional mathematics classroom, teachers plan relatively detailed lessons in advance and then attempt to carry out these plans (Sherin, 2002) in an instructional sequence that is adopted by most math teachers: review homework, introduce a new
concept or topic, demonstrate how the new concept or topic is used in solving math problem, have students practice, and assign homework (Huang et al., 2005). Lessons are taught to the whole class except for the time when students practice the new procedure independently. Students used to the traditional IRE whole-class discourse expect to have to infer what the teacher has in mind and assumes the teacher will publicly evaluate the correctness or incorrectness of their responses (Cobb et al., 1993).

In contrast, lessons in the reform-based classroom environment look and sound very different. Instead of focusing only on procedures, students also focus on mathematical concepts, multiple representations of those concepts, and connections among them. Teachers attend to the ideas that students raise in class through an adaptive style of inquiry, discovery, and improvisation. Lessons are immersed in a context of discourse where teachers pose appropriate questions for students to consider, listen to students' ideas, and help students explain and justify their ideas in class. These classroom discussions are also a time for teachers to insert mathematical ideas and explanations where appropriate (Sherin, 2002).

Three Lesson Models

A typical reform-based mathematics lesson often proceeds in three phases, (Sherin, 2002; Stein et al., 2008). It begins with the "launch" phase with the teacher launching a mathematical problem that embodies important mathematical ideas and can be solved in multiple ways. During this phase, the teacher sets the stage for the lesson by introducing students to the problem and engaging them in preliminary thinking about the lesson. The teacher asks probing questions to quickly assess students' prior knowledge,
intermingled with lively discussion that foreshadows the big ideas they will encounter and piques their interest. The teacher familiarizes students with the tools that are available for working on the problem, and the nature of the products the students will be expected to produce.

The next phase is the “explore” phase, in which the students work on the problem often discussing it in pairs or small groups. In this phase, students are actively engaged in discussion while they work on the problem. The teacher moves among the groups asking probing questions to check for conceptual and procedural understanding while helping teams with key points and clarifying misunderstandings. Students are encouraged to solve the problem in whatever way makes sense to them, and must be prepared to explain their approach to others in the class.

The lesson concludes with the “discuss and summarize” phase, sometimes also called the “share and summarize phase.” The teacher engages the students in a whole-class discussion of various student-generated approaches to solving the problem. Students share a variety of approaches to the problem, which are displayed for the whole class to view and discuss. At this time, important connections are highlighted and conceptual understanding is advanced. At the end of the lesson, students are invited to individually and collectively reflect upon and summarize their learning (Cobb, Boufi, McClain, & Whitenack, 1997).

Three variations of the reform-based mathematics lesson were observed in this study, including The Guided Discovery Lesson, The Open-Ended Exploration Lesson, and The Integrating Direct Instruction with Guided Discovery Lesson (Rubenstein et al.,
2004). Each of these variations incorporated the three-phase core lesson format, which included the launch, explore, and discuss and summarize phases.

The guided discovery lesson. Guided-discovery lessons allow students to explore and develop ideas through a careful sequence of tasks and questions. The teacher knows what results are desired and guides students toward these results (Rubenstein et al., 2004).

The open-ended exploration lesson. Open-ended exploration lessons provide students opportunities to explore mathematics without preconceived notions of the paths or results that might be obtained. The teacher anticipates students’ responses, including understandings and misunderstandings, but realizes that the investigation might lead down unanticipated routes. These lessons, in particular, require teachers to possess a strong understanding of the mathematics and the confidence to follow students along undiscovered paths (Rubenstein et al., 2004).

The integrating direct instruction with guided discovery lesson. Integrating direct instruction with guided discovery lessons typically begin with the teacher providing direct instruction when the students need to learn to use some tool, learn vocabulary, or become proficient in some procedure that they are not likely to discover on their own. Effective direct instruction is integrated with guided discovery in order to provide students the opportunity to explore relationships using the information provided in the direct instruction portion of the lesson (Rubenstein et al., 2004).
Three Talk Formats

Teachers in reform-based classrooms consider different formats when configuring classroom interactions for instruction. Three talk formats, whole-class discussion, partner talk, and small-group discussion were observed in the study. These talk formats have been found to be particularly supportive of maximizing opportunities for mathematical learning by all students (Chapin et al., 2009).

Whole-class discussion. In this class format, the teacher and students are engaged in a singular conversation. The Whole-Class Discussion talk format is usually seen in the “launch” and “discuss and summarize” phases. The teacher is in charge of the class, however he or she is not primarily engaged in delivering the information or quizzing students. Rather, the teacher has students share their thinking, explain the logic in their reasoning, and build upon one another’s contributions. Whole-Class Discussions give students the chance to engage in a sustained discussion that builds a coherent line of reasoning. The teacher facilitates and actively guides the discourse, but does not focus on providing answers directly. The focus in Whole-Class Discussion is on the students’ thinking.

Invariably, these discussions reveal many examples of faulty reasoning, mistakes in computation, and misunderstandings. These confusions, however, are the raw material with which teachers can work to guide students’ mathematical learning. In the process, students become more confident in their ability to persevere as they grapple with making sense of concepts, skills, and problems. Explaining their reasoning is important for all students as it helps them to cement and extend their thinking (Chapin et al., 2009).
**Partner talk.** In this talk format, the teacher asks a question and then gives students a short time, perhaps a minute or two at the most, to put their thoughts into words with their nearest neighbor. The Partner Talk format may be used at any point of the lesson. This format has several benefits. Students who are keeping up with the lesson but are hesitant about voicing their thoughts have a chance to practice their contribution with just one conversational partner. Students who have not understood the lesson completely thus far can bring up their question with their partner, and perhaps formulate a way to ask their question to the whole class. This two-minute aside is invaluable for many students, particularly those who are learning English as a second language. Students can emerge from the Partner Talk ready to participate in the Whole-Class Discussion. Chapin et al. (2009) found that when the teacher asked a question and no one responded, it helped to change the format to Partner Talk for two minutes. After students had a chance to think aloud with another peer, they were more ready to share their thinking within the large group.

**Small-group discussion.** In the Small-Group Discussion talk format, the teacher typically gives students a question to discuss among themselves, in groups of three to six. This talk format is often seen in the “explore” phase of the lesson. The teacher circulates among groups as they discuss the topic at hand. He or she observes and sometimes interjects when appropriate. Since the teacher necessarily plays a diminished role, he or she is unable to ensure that the talk is productive. Students may spend time on off-task talk, and there is no guarantee that students will treat one another in an equitable manner (Chapin et al., 2009).
Summary

This section described the changes to the mathematics lesson and group configuration found in the reform-based mathematics classroom environment. Traditional mathematics lessons involved detailed lesson plans focused on covering content. The whole-class lesson began with checking the previous night's homework assignment, followed with a demonstration that would prepare students to complete the next night's homework assignment. These lessons incorporated the traditional IRE unidirectional discourse pattern. Students' participation was limited to inferring what the teacher had in mind with their contributions publicly evaluated.

In contrast, inquiry-based lessons have less predictable paths to student learning than their traditional counterparts as they focus on conceptual learning instead of rote procedures. These lessons are comprised of a three-phase lesson format, which includes launch, explore, and discuss and summarize phases. Three variations on the three-phase lesson were observed in this study. The Guided Discovery Lesson, The Open-Ended Exploration Lesson, and The Integrating Direct Instruction with Guided Discovery Lesson were important to the structural design of this study and became one way by which observations of student talk were analyzed.

Students are arranged in a variety of talk formats during inquiry-based lessons as they engage in the mathematics discourse. Students are situated in Whole-Class, Partner Talk, and Small-Group Discussions according to the purpose of the lesson phase. While time does not permit every student to share during Whole-Class Discussions, the use of Partner Talk and Small-Group Discussions increases the likelihood that every student
will have a voice in the mathematics classroom. These student talk formats were important to the structural design of this study and became one way by which observations of student talk were analyzed.

**Summary of the Literature**

This chapter focused on unpacking the overall purpose statement of the study: to explore how teachers and students use language to advance students' conceptual understanding of mathematics in a reform-based mathematics classroom. Each of the five sections of this chapter addressed one aspect of this statement: mathematical language, conceptual understanding, teachers' use of language, students' use of language, and the environment of the reform-based classroom.

Throughout the review of literature, the value of discourse in the mathematics classroom cannot be underestimated if students are to fully realize the elements of the NCTM (2000) Communication Standard. When students are challenged to think and reason about mathematics and to communicate the solutions of their thinking to others, they learn to be clear and convincing. Through communication, ideas are reflected upon, refined, discussed, revised, and new connections are made. Students reap dual benefits as they communicate to learn mathematics and they learn to communicate mathematically.

This is a challenging goal for many students due to the complexity of mathematical language. We explored the reasons for its difficulty as we discovered that its vertical and horizontal knowledge structures require students to not only climb the hierarchy of technical vocabulary, but to traverse the range of vocabulary across mathematical strands in order to become a fluent language user.
We also determined that each mathematical term carries with it a full spectrum of concepts. In order to fully understand these concepts, students must develop deep and broad knowledge packages as they make connections to their own personal use, within and across the domain, and outside to the world around them. We found that when students develop conceptual understanding, they reap the benefits of more efficient, effective, learning.

Teachers support students' communication according to their goals, beliefs, and knowledge structures, which affect the quantity and quality of teacher talk in the mathematical classroom. When teachers of reform-based mathematics classrooms engage students in dialogic discourse, communication becomes a two-way give-and-take as together they co-construct meaning. This process can be challenging for teachers for various reasons. Fortunately, researchers provided teachers with suggestions to increase success. Five teacher talk moves (e.g., revoicing, repeating, reasoning, adding on, waiting) were important to the structural design of this study and became a primary context by which whole-class discussion was observed.

Student talk varies significantly in quantity and quality. The resulting vocabularies correlate directly to student achievement. Fortunately, engaging students in productive mathematical talk can address these discrepancies. Many mathematical terms are learned consequently through exposure in class, while others must be taught directly. In lieu of precise language, students resort to vague language to indicate their tentative commitment to a conjecture. Students also signal when they make new connections or generalize new learning. Student talk was the primary focus of this study. It was
analyzed within the context of teacher talk moves across the dimensions of lesson design and talk formats.

Finally, we compared the classroom environment of the traditional whole-class lesson format with the three-phase inquiry-based lesson design found in reform-based classrooms. During the launch, explore, and discuss and summarize phases, students engage in Whole-Class Discussion, Partner Talk, and Small-Group Discussion talk formats. Three variations on this lesson design (*Guided Discovery, Open-Ended Exploration, Integrating Direct Instruction with Guided Discovery*) together with the three talk formats (Whole-Class, Partner Talk, and Small-Group Discussion) were important to the structural design of this study and became important ways by which observations of student talk were analyzed.
I turn now to Maggie, where the challenges of mathematics reform and the complexities of research on mathematical language, conceptual understanding, teacher talk, student talk, lesson models, and talk formats converged as Maggie engaged her eighth-grade mathematics students in a unit of study on the topic of data and statistics. While the purpose of this study was to explore how Maggie and her students used language to advance their conceptual understanding of mathematics in a reform-based mathematics classroom, I specifically sought to describe how the reform-based mathematics classroom environment impacted the quantity and quality of her students’ mathematical language. Since Maggie was the designer and facilitator of this environment, it was important to get to know her as a teacher, understand what she valued, how she organized her lessons, and what she said to ensure all her students were valued members of the learning community.

The methods I used for conducting interviews, observing the classroom discourse, and taking field notes were completed as described by Bogdan and Biklen (2003). What follows was synthesized from three audio-recorded semi-structured interviews with Maggie held before, during, and after she taught the data and statistics unit to her students; e-mails; personal communication; Maggie’s vita; her lesson plans; handouts she distributed to students; a PowerPoint, websites, and textbook she used as teaching resources; 14 hours of video-recordings from each of 10 Flip Video™ camcorders located strategically around the classroom; and 14 hours of audio-recordings from each of seven
digital voice recorders placed between every two students that captured all classroom talk that occurred during classroom observations over fourteen 52-minute lessons from March 21, 2011 to April 8, 2011. Field notes supplemented the video- and audio-recordings with additional information pertaining to the students’ utterance patterns during mathematics instruction. I also considered relevant information regarding the physical setting, classroom climate, pre- and post-mathematics activities, observations of non-speaking students, and additional comments I wanted to remember. Prior to this research, I had completed a pilot study during March 2010 with one classroom in which I became familiar with the electronic equipment, began exploring possible categories of student talk, and developed questions for further inquiry.

I want to note that the limited number of observations might not represent the entire repertoire of teacher talk moves that Maggie expresses, thus generalizations of teacher talk found in this study to all eighth-grade mathematics teachers in reform-based mathematics classrooms should be avoided. Also, the limited number of observations of lessons models and talk formats might prevent a pattern of instruction from being established for Maggie’s classroom, thus environmental factors found in this study should not be generalizations to all reform-based mathematics classrooms.

In order to describe the reform-based mathematics environment as completely and accurately as possible, I include multiple excerpts and vignettes from the interview and lesson transcripts. I wanted to listen carefully and honor Maggie’s voice as it now entered the Discourse. I also wanted to provide a rich context for the experience that she created. Examples excerpted from the transcriptions include line numbers that refer to
speech turns from the original transcripts. Citations refer to the sequence of the interview or lesson from which the talk was excerpted. All names are pseudonyms to protect the identity of the teacher and her students.

Maggie

I want to introduce Maggie, the participant teacher of this study. In this first section, I provide a brief summary of her teaching career and what she taught. I also describe her school and present the learning community.

Maggie: The Teacher

At 39, this was Maggie’s 16th year as a teacher of mathematics, joining the faculty of her current school 12 years ago. Maggie earned both an undergraduate and a master’s degree in secondary mathematics, is a Nationally Board Certified Teacher in the area of adolescent mathematics, and is certified by the state to teach secondary mathematics and Spanish. She is married and a mom to three elementary school-aged daughters.

This year Maggie taught 8th and 10th graders, though her teaching assignment varied from 6th to 12th grade depending on the year. Maggie often taught students at a nearby public university who enrolled in courses that ranged from exploratory mathematics to advanced statistics and calculus. Since her school was located near the university, she frequently mentored a number of university undergraduate and graduate students who visited her classroom to observe and practice their teaching. Depending on the day, there were between one and five university students in her classroom during the time I conducted the study.
Maggie was a frequent speaker and presenter at state, regional, and national conferences. She both wrote the curriculum and was a facilitator for university sponsored professional development workshops to practicing teachers. She consulted for the state department of education and co-authored several articles and a book chapter on the topic of discourse. She was highly regarded among her peers who recognized her service through distinguished awards and honors at the state and local level.

Maggie was selected to participate in the study through purposive criterion sampling due to her expertise in both the areas of mathematics and discourse and her self-avowed belief in reform-based mathematics. She was graciously willing to take part.

**Maggie’s Math Class: Samples and Population Unit**

Maggie’s first period mathematics class met daily, with each class period 52 minutes in length. During the study, the class was engaged in the *Samples and Populations* unit, from the *Connected Mathematics 2* series, (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). This curriculum was developed with funding from NSF in 1991-1996, and in 2000-2006, the Connected Mathematics Project (CMP) developed a complete mathematics curriculum for middle school teachers and students. It reflected a social constructivism philosophy, especially about the influence of discourse on learning, and utilized the three-phase lesson model of launch, explore, and summarize (CMP, n.d.). This was the first time Maggie taught the unit.

Maggie used the published CMP materials as a framework, but supplemented many of the examples that she thought would be more relevant to her students, such as utilizing data from their spring break experiences, newspapers, magazines, and websites.
She also engaged the students in different activities if she thought they were more meaningful, such as discussing how school administrators sampled various populations during a recent poll to determine attitudes about changing the school colors. Knowing that her students would be comparing bivariate data toward the end of the unit, and that this concept would be a bridge to an upcoming algebra unit, Maggie also incorporated the Meaningful Distributed Instruction (Rathmell, 2010) strategy to review slope. She knew that her students would need to remember the concept of slope in order to make sense of the least squares regression line and correlation. Meaningful Distributed Instruction lessons were short five-minute lessons in which concepts or procedures were previewed or reviewed on a daily basis prior to upcoming instruction.

Maggie’s School: A Center for Innovative Teaching

Built during the 1950s, the publicly-funded school where Maggie taught was located in a mid-sized Midwest university town. At the time, it was touted as a premier facility in which to provide the best education for students who attended there. Maggie shared the spacious classroom in which she taught with other instructors throughout the day, so each class period she brought any needed teaching materials with her. She had an office next door to the classroom where she had a teacher’s desk and kept her personal belongings.

Faced with crises in the 1970s, 1980s, and again in the 2000s that threatened the school’s existence, few updates have been made to the building and it showed its age. The classroom had new carpet and a new interactive whiteboard that quit working the first day of the unit, but drab hallways, sagging window blinds, and marred wooden
student tables etched with years of graffiti showed the effects of declining budgets. During the weeks I observed, one of the student chairs fell apart. Another was in need of a hammer to pound in a nail that had worked its way out of the crossbars, scratching a student’s leg. Both caused brief distractions from the lesson, which Maggie quickly took care of as a matter of routine.

Despite the needed updates to the physical condition of the building, new life was breathed back into the school as the administrators and teachers were rallying to restore the school to its former prestige. Innovative teaching and learning practices were studied both as formal research and through continuous professional development opportunities. Others around the state looked to the faculty for guidance. Students open enrolled to receive a different education than they could receive at their neighborhood schools.

**Maggie and Her Students: The Learning Community**

Maggie’s classroom was an inter-generational learning community that learned math with and from each other. Maggie taught 14 eighth graders during her first period mathematics class. Thirteen participated in this study. Neither Maggie nor the other students were aware that this student was not participating in the study, and was treated the same as any other student who was participating. Nine of Maggie’s students were males and four were females. All were Caucasian except for one male who was African-American. All were native English speakers. Many lived in the neighborhood, however, a number of her students transferred to the school through open enrollment as seventh graders.
Prior to the start of the study during my first interview with her, Maggie shared that her students were an academically diverse group. Table 1 describes the class scores on two mathematics subtests of the Iowa Tests of Basic Skills in the areas of problem solving and concepts and estimation.

Table 1.

<table>
<thead>
<tr>
<th>Sub-Test</th>
<th>M</th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving</td>
<td>72.2</td>
<td>13</td>
<td>54</td>
<td>72.5</td>
<td>93</td>
<td>99</td>
</tr>
<tr>
<td>Concepts and Estimation</td>
<td>69.9</td>
<td>13</td>
<td>55</td>
<td>74.5</td>
<td>86</td>
<td>99</td>
</tr>
</tbody>
</table>

With the range of the class spanning the 13th to the 99th national percentile, it confirmed the heterogeneity of the group. The class distributions were similar for both mathematical areas, and the achievement of the class, as measured by the tests, clustered above average.

Although none of Maggie’s students were identified with an individualized educational program (IEP), several of her students required additional support. One student had difficulty focusing which impacted all academic areas including math, one struggled to decode the written word, one student was highly anxious and identified as at-risk, and a fourth had difficulty with basic fact/recall memory and was currently undergoing evaluation at the regional hospital school. Maggie identified three of her students as high achievers. The rest performed somewhere in between adolescent “cool” and “cute.”
Maggie also taught between one and four undergraduate students and one master's level graduate student, depending on the day. Two undergraduate students were male and two were female. The graduate student was female. All were Caucasian and native English speakers. All were enrolled in the teacher education program at the nearby university and were secondary mathematics education majors. Although the university students were not the focus of the study, two of them taught two lessons each during Lessons 3, 4, 6, and 8. All interacted freely with the eighth graders during Partner Talk and Small-Group Discussion, unless asked by Maggie to simply observe in order that the middle school students would have more opportunity to lead the discussion. In addition, I was an observer of the classroom, but talked with students only before and after class so as not to alter the natural flow of communication during the lesson.

Besides being the teacher, Maggie was a life-long learner. She demonstrated her own continuous learning of the content and refinement of her teaching practices by leading and engaging in numerous professional development opportunities. These will be highlighted in a later section. However, the best professional development, she acknowledged, came from the numbers of university students who observed her teaching and wrote reflective journals about what they saw.

_The Best Professional Development, Interview 3_

004 Maggie: Probably the best thing has been 10 years of people being in my room and writing journals about what they see. And me going, "Oh!" This is what they're seeing and sometimes remembering that and sometimes not. But that if you're given a gift of people reflecting in your classroom repeatedly, .... I mean college students are pretty honest about whether somebody's understanding something or not. Not that that's the best lens to improve from, but when
you’re trying to help other people become a teacher, that’s helpful to yourself. So that’s been pretty good PD.

Maggie loved her students and was not afraid to tell them so. One day, while recounting a particular student whose height and shoe size was relevant to an introductory lesson on correlation, Chloe was surprised by the fact that her teacher could remember former students from so many years ago.

_ I Love My Students, Lesson 14_

201 Chloe: You remember his name?

Maggie’s students loved her, too. One day, while Evie was working with her partner, Dan, she called her teacher to her desk to ask for some help. During the conversation that ensued, Evie complemented her teacher.

_You’re the Best, Lesson 3, Partner Talk_

242g Evie: Actually, yesterday was a very easy to listen to day.

And then she added,

242t Evie: You’re the best!

Maggie established this level of rapport by honoring her students’ capabilities and caring that they succeeded in life, as well as math. These dual notions of student success and responsibility were foundational to her belief system about her students. She went out of her way to support them, which was reflected every day through her actions.

_What Maggie Values_

Maggie valued students and she valued mathematics. Both were inherently important to her. The beliefs she held about students and the way math should be taught
influenced every aspect of her teaching, including the way she used instructional time, organized and enacted her lessons, and interacted with her students through her actions and use of language. In the next section, I explore Maggie’s beliefs about her students, her beliefs about mathematics education, her long- and short-term goals for her students, and her own content knowledge. I also provide examples where Maggie enacted these beliefs, goals, and knowledge in her classroom.

Maggie’s Beliefs About Her Students: All Are Capable of Learning

Maggie believed that all of her students were capable of learning. She recognized, however, that they were part of a system that decided time was a critical factor in this process and that did not work for all of them. Given their individual learning rates, and the different connections they made, it resulted in a struggle for some of them. To help her students acquire the learning she knew they were capable of, Maggie enacted two practices that maximized her students’ learning potential. First, she maximized her use of instructional time scheduled for her to teach mathematics, also known as the class period. Second, she supported her students by being available for them, setting high expectations for them, and encouraging them to do their best at all times.

Maggie’s use of instructional time. Maggie’s classroom was alive with energy from the moment class began until students transitioned to their next class 52 minutes later. Because she was part of a system that held high learning expectations for students as they passed from grade to grade and from teacher to teacher, she realized it was important that she set a particular pace.
When I analyzed her instructional time on task, Maggie used every available minute and then some. If all had gone according to plan, she would have 728 minutes of instructional time over 14 days (14 lessons times a 52-minute class period). But things do not always go according to plan. On one occasion, a school administrator needed to talk to students about a school issue, so class started seven minutes late. I recalculated Maggie’s available instructional time as 721 minutes remaining to instruct her students (728 total minutes – 7 minutes of administrative delay = 721 instructional minutes remaining). Maggie, however, was able to make up all but one of those minutes by extending class a half-minute here and a minute there. By the time the unit ended, 727 minutes were accounted for, which didn’t include the time she often worked with students on review problems at the board as soon as they walked in the door. Table 2 illustrates Maggie’s use of instructional time.

Table 2.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Minutes</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Non-instructional activities which include the following:</td>
<td>27</td>
<td>4%</td>
</tr>
<tr>
<td>Administrative delay (1 minute)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lesson 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time not regained throughout the unit</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Announcements (17 minutes)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons, 4, 13, 14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weekend update (7 minutes)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lesson 10 (school initiative)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thank you party (2 minutes)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lesson 14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mathematics instruction</td>
<td>701</td>
<td>96%</td>
</tr>
<tr>
<td>Total Available Instructional Time</td>
<td>728</td>
<td>100%</td>
</tr>
<tr>
<td>52-minute class period X 14 Lessons</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Class announcements, at the beginning of Lessons 4, 13, and 14, took a total of 17 minutes. Weekend updates, a school initiative to build positive relationships with middle school students, conducted at the beginning of Lesson 10 took seven minutes. I brought breakfast as a thank you party for students on the last day of the unit, which used two minutes of class time. Typical classroom routines, such as taking attendance, consumed no instructional time as Maggie completed the task over her cell phone. When totaled, there were 27 minutes used for class business, which accounted for nearly 4% of time. The remaining 701 minutes, over 96% of the time, were spent in mathematics instruction.

Maggie communicated to her students that time was precious. When a student let her know that few minutes remained before class was over, she replied,

*Time is Golden, Lesson 3*

521 Maggie: No, I know. We’re going to use them, too, because that’s three minutes of golden time.

Maggie maximized every minute of class. She reminded her students often to work and move quickly.

*Really Quickly, Lessons 1-14*

364 Maggie: *OK, quickly,* talk to your neighbor. What kind of graph is this? Is this good information or not good information; be critical about it. Go. Talk to your neighbor about it. (Lesson 1)

213 Maggie: *Really quickly.* Tell me one difference between these two data sets. One very clear difference. Dan? (Lesson 5)

008 Maggie: So *really quickly.* Really quickly on your blue sheet. Listen, I need to see you writing right away, OK? (Lesson 6)
020 Maggie: How did you calculate the slope? *Really quickly.* (Lesson 6)

340 Maggie: We need to decide on an interval that makes sense so we all have the same interval. If we don’t have the same interval, it’s going to be really hard to compare. OK? So, movies watched. What’s the smallest of our movies watched? Look at data *really quickly.* (Lesson 7)

243 Maggie: Can you go up and show us *really quickly,* Dianna? Thanks. (Lesson 14)

Maggie also wanted the class period to be longer. She shared her wishes regularly.

*I Don’t Want Class To End,* Lessons 2, 4, and 8

280 Maggie: I don’t want this class to end. I know you guys may not feel that way about it. I want this class to be about ten minutes longer. (Lesson 2)

396 Maggie: Connor, is it time to go? Is that what are you trying to tell me? That’s very sad for me. (Lesson 4)

272 Maggie: Everyday, I want this class to be longer. OK. You don’t, but I do. (Lesson 8)

Every minute in Maggie’s classroom was accounted for. Nearly all was spent on mathematics. This attention to and responsibility for her content were foundational to her belief system about mathematics, and were reflected in her everyday actions.

**Maggie’s availability, high expectations, and encouragement.** To accommodate different learning needs for students who needed more assistance than could be provided during class, Maggie made herself available both in person and virtually to anyone seeking help. She arrived at school early, usually by 7:00 a.m., to assist students with their questions. She met with them during their study hall, if it aligned with her planning time, and made herself available many nights after school. Maggie posted assignments
on the class Moodle website and on Facebook. She accepted students’ phone calls and freely provided them with two e-mail addresses. In return, she expected them to complete their homework.

Maggie set high expectations for her students. On one occasion, Maggie acknowledged she was assigning challenging homework problems that would prepare them for the types of problems they would eventually face on college entrance exams. She spoke as if college were a given. She wanted them to be prepared and do well.

*Challenging Homework, Lesson 3*

506 Maggie: Number 31 and 32 are not problems we’ve talked about in class. I know that. I need you to know that I know that. OK. We’ve definitely not talked about this in class. They are that information that you probably have enough knowledge to solve all ready. Let me tell you, they are the two... they are frequently found on *Iowa Tests of Basic Skills* test or Iowa.... They are... these will be on your ACT. These will be on any standardized test. These will be on college entrance exams. These kind of problems. I don’t... I’m going to just tell you, I don’t necessarily like them, but you need to know how to do them for.... Not because you’re ever going to do this in real life, because it will be on some standardized event.

The following day, she provided support by offering precious class time to help them figure out the puzzling problems. Note that inaudible text is recorded for the non-participating student (NPS).

*Test People Like It, Lesson 4*

043 Maggie: OK. Give me one problem that you weren’t quite sure about or you think, you know what? This is the one I’m going to just tell you, I don’t necessarily like them, but you need to know how to do them for.... Not because you’re ever going to do this in real life, because it will be on some standardized event.

044 NPS (inaudible)
045 Maggie: Thirty-one? And oh, that’s the one I said, “Hey, this one, most people don’t actually do but test people like it.” OK. So if I were to read this problem, first I have to read it carefully. So...

Maggie offered encouragement when students needed it most. If individual students fell behind, Maggie talked to them privately to make arrangements to help them get caught up. Evie was particularly busy practicing for a dance recital most evenings, arriving home at 10:30 p.m. She was falling further behind with her homework and her frustration began to spill out in class. Maggie pulled Evie aside and offered to help her with some time management skills so that she could come to class prepared.

*Let’s Get It Done, Lesson 8, Small-Group Discussion*

271 Maggie: ...So I’m just noting the difference in you that I want to help specifically this. I would like you to consider coming in right after school, because you don’t have anything right after school. It’s getting it done before you leave here. And then you don’t have to come in here and say I’m not done so then uh, uh, uh, OK? So would you consider that? So what are you doing after school today?

271w Maggie: ...You want that dance award. So let’s get it done before. Does that sound good? OK.

In addition to providing support to struggling students, Maggie reached out to those who wanted to learn more. To enrich students’ mathematical and science learning, Maggie and the science teacher co-hosted a weekly “STEM in the Afternoon” experience for third through eighth graders involving high school students, pre-service teachers, and other school employees.

Maggie also extended support to her university students. One student, Preston, worked the night shift to pay his tuition. He wasn’t supposed to arrive until second hour,
but he asked Maggie if he could come straight from work to her classroom. He was afraid to go to his apartment first, because then he would want to sleep. Since he didn't have any time to sleep, he preferred to come to her first period class even though it doubled the time he needed to observe according to his university requirements. Although Maggie had several other mathematics education students scheduled to observe her class at that time, she was empathetic to his schedule and didn’t have the heart to tell him not to come. So Preston participated in her classroom every day of the unit.

Despite her best efforts to offer a variety of opportunities to support her students and be as accessible as she could, Maggie found it difficult to wrestle with the time constraints that affected their success.

That's Difficult, Interview 1

002 Maggie: I offer opportunities in particular ways and then that affects the success of my students. And I try to vary that and to make that as, you know, as accessible as I can, but that’s difficult. I mean that’s difficult. But I think that my students can do math and often do do math. But they are also very typical students. It’s not necessarily their favorite thing to do.

Perhaps math wasn’t their favorite thing to do, but if the smiles on their faces were any indication of their true feelings, Maggie’s class was about as good as it could get for her typical eighth grade students, short of a good baseball game or trip to the mall. And when a college student forewent sleep to be in her class, it spoke volumes.

Maggie’s Beliefs About Mathematics Education: “If I Could Rule the World”

Maggie held three overarching beliefs about mathematics education. First, she believed students should actually do mathematics.” By do, Maggie referred to the
Mathematical Tasks Framework (Stein & Smith, 1998), which differentiated problems or tasks based upon the cognitive complexity needed to solve them. Stein and Smith (1998) defined a “doing mathematics” task as a portion of a classroom activity devoted to the development of a particular mathematical idea that could take anywhere from twenty minutes to an entire class period to solve. These tasks required students to think conceptually and make connections, which resulted in students doing mathematics. They provided students with very different types of thinking compared to those problems that asked students to perform memorized routine procedures. Over time, the cumulative effect of the types of classroom-based tasks students were asked to do influenced their implicit ideas about the nature of mathematics, whether they could personally make sense of it, and how long and how hard they should have to work to do so.

Second, Maggie believed that communication in the classroom was important in order for students to be able to talk about mathematics. This belief aligned with NCTM’s (2000) communication standard, which at grades six through eight asks teachers to use oral and written communication in mathematics to give students opportunities to—think through problems; formulate explanations; try out new vocabulary or notation; experiment with forms of argumentation; justify conjectures; critique justifications; [and] reflect on their own understanding and on the ideas of others. (p. 272)

Third, Maggie believed the NCTM process standards (2000), or the Standards for Mathematical Practice (CCSSI, 2010) were the most important, outweighing the importance over individual facts, or algorithms, or anything else. The Standards for Mathematical Practice describe a variety of expertise that mathematics educators should seek to develop in their students. They are based on NCTM’s (2000) process standards
and the strands of mathematical proficiency specified in *Adding It Up* (NRC, 2001). The eight Standards for Mathematical Practice (CCSSI, 2010) include:

1. Make *sense of problems* and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning. (pp. 6-8)

I asked Maggie to clarify her suggestion that the process standards were in some sort of hierarchy, or took precedence over content standards.

*If I Ruled the World, Interview 1*

001 Maggie: I think in theory I do, but in practice, it’s hard to maintain that, because... I think that the way we assess mathematics puts the content over the process standards. I think I truly believe the process standards are more important, but it’s hard for me to continually enact that belief in my classroom because other things are realities for our kids. Yeah, *if I could rule the world of how math was always taught, which I get to rule some things but not everything, then I would flip that. Because, I think if you have those process standards in place or the Standards for Mathematical Practice, which ever ones you want to reference... they’re slightly different, but ... mean similar things, then you’re going to be able to learn the content that comes to you later.*

I wondered if she felt this focus on content was driven wholly by the role assessment played as a result of NCLB (2002), but this was not the case.

*The Content Is Easier To Teach, Interview 1*

001 Maggie: It is about progression, because you go from one teacher to the next and you want to learn some specific progression of content when they get there. And so there is that expectation simply within our math department. So, no, it’s not just about the exterior assessment.... But it’s also
the need to... we have a system where you go from teacher to teacher, and there is something that says, “Where’s the pre-knowledge here?” “What do you know?” So the content becomes... the content is easier in many ways to teach than to teach kids how to critically... well, like one of the standards for Mathematical Practice says to critique arguments of others and to persevere in problem solving. Those things are hard and long... They’re not a check-off box.

Teachers’ beliefs are influenced by their own personal and professional histories.

It was apparent that Maggie drew from her rich professional history.

Beliefs Have Instructional Implications, Interview 1

001 Maggie: So, I think that those beliefs have instructional implications in my classroom.

Maggie enacted each of these beliefs in her classroom. In the following example, Maggie asked her students to do mathematics, talk mathematically, and engage with the Standards for Mathematical Practice, Standard 1: Make sense of problems and persevere in solving them, and Standard 3: Construct viable arguments and critique the reasoning of others, in a single lesson.

Maggie provided her students with Problem Based Instructional Tasks (PBITs) to grapple with mathematical ideas related to the unit. One task, from the Connected Mathematics 2: Samples and Populations (Lappan et al., 2006, pp. 47-50) unit asked students to estimate the time period during which each of two archeological sites were inhabited based on the arrowheads found there. They were provided with two sample sets of data based on the length, width, and neck width of the arrows, that they compared to arrowhead data sets from two known sites. Students worked in small groups to
complete the task with very little guidance from Maggie or the university participants regarding how they should approach or solve the problem.

_Think for Yourselves, Lesson 12_

048 Maggie: You have to think for yourselves. You have to come up with it. You’ve got to figure it out. So Miss Becker you may not tell them. OK. I’m not going to tell them. Mr. Dunn isn’t going to tell you. You get to figure it out.

Maggie expected her students to work cooperatively in their groups, to talk, and solve the problem. She clearly communicated her expectations and acknowledged that the success of the group was based on the interdependence and contribution of each member.

_Be A Contributing Group Member, Lesson 12_

063 Maggie: You have to be a contributing group member.... because your group is going to need you.

082 Maggie: And so, go ahead and get yourself settled wherever you want to be in the room.... And you need to get your papers and get yourself organized. At like 8:14, I should see you really functioning as a group.

Maggie knew the problem would be challenging, but she wanted her students to persevere in solving it. She discussed what this might look like in action and provided strategies to focus their effort.

_Standard 1: Make Sense of Problems and Persevere In Solving Them, Lesson 12_

063 Maggie: If you are persevering, we might say, “I don’t understand” but you are not going to immediately go to the (raises hand) hand. You might read it again. Think about it again. You might ask your group member.
After twenty minutes of work time, Maggie paused the groups to acknowledge their work. She named their experience so that her students knew what it felt like to do mathematics.

*Doing Mathematics, Lesson 12*

084 Maggie: I want you to stop for a second and give me your attention. Stop for a second and give me your attention. I have to tell you that there has not been a single person in here that is not *doing* some form of mathematics right now and that’s awesome. OK. I think I’m right about that. There have been maybe little lulls here and there, but for the most part, you guys are *doing*. That’s exactly... that’s good. So that shows some perseverance... see you’re not even stopping.

Then, Maggie asked her students to engage in another of the Standards for Mathematical Practice. She challenged them to think more deeply about the conversations they were having and the reasoning they were offering for determining their solutions.

*Standard 3: Construct Viable Arguments and Critique the Reasoning Of Others, Lesson 12*

084 Maggie: Here’s the second thing I want you to work on today. One of the other math practices that you need to become good at, and some of you are already awesome at this. Some of you are quiet in this class, but you actually have a lot going on in your head that you could contribute. And here’s the second thing. Critique the reasoning of others. This is another one of the Standards. Critique the reasoning of others. So you... a lot of you are going to be moving on to this is why we’re choosing site one, this is why we’re choosing site two. So I want you to make sure when you’re writing those, that you don’t just go OK, that’s fine. OK, yeah, that’s fine. That’s not really critiquing the reasoning of others. And not critique in a bad way, but sort of push. Well, have you thought about this? Or why should we say that? Or, so I want to make sure that you’re also doing that. So that’s what I’m going to be listening for for the second part of class. OK? But you guys are doing a great job.
Beliefs determine which goals and actions have high priority (Ball, 1996), and in Maggie’s classroom, doing math, talking about math, and enacting the standards for mathematical practice were omnipresent. Her beliefs about mathematics education also shaped the long-term goals she had for her students.

**Maggie’s Long-Term Mathematics Goals: Learning How To Learn**

Maggie loved mathematics and her students knew it. When she asked them how she thought about events in the world, they were quick to respond as Chloe did in the following example. Note that inaudible text is recorded for the non-participating student (NPS).

**I Think Mathematically, Lesson 8**

142 Maggie: ...of course, how was I thinking about it?
143 NPS: (inaudible)
144 Chloe: Mathematically!
145 Maggie: Mathematically!! (laughs) I was!! I was!! I was!! I can’t help myself! OK. So I was thinking about it mathematically and I was thinking about what we were studying...

While Maggie would have loved for her students to like math as much as she, realistically she knew they would not all become mathematicians. As a compromise, her first long-term goal for her students was that she wanted them to learn how to learn math. And if they weren’t going to love math, her second goal was that she hoped that at least they wouldn’t be fearful of it.

**Don’t Worry, Learn How To Learn, Interview 1**

003 Maggie: Really, I want them to be able to learn how to learn math and to not have that fear factor of, “Oh my gosh! I’m not doing this because there is math in it.” I’m not there yet,
and neither are they. You’ll hear them talk, and it’s really, “Maybe I won’t do that because it has a lot of math in it!”

Maggie’s concern about math phobia was not unfounded. A simple Google search on the term “math fear” yielded 44 million hits in a tenth of a second. In an AP-AOL News poll (Lester, 2005) of 1,000 adults, nearly 40% indicated they hated math and twice as many said they hated math as said that about any other school subject.

A third goal of Maggie’s was that she hoped her students appreciated structure and beauty of math and would recognize its application in the world. This goal combined the aesthetic, logic, and practical aspects of mathematics.

*Math is Beautiful, Interview 1*

003 Maggie: I would love for them to see the interconnectedness of math, and the, you know, the inherent beauty and structure of math, but we’re just sort of looking for piques of interest and to see how it’s played out in multiple places within their world and our world.

It was not hard for Maggie to find real-world application of math to share with her students. There were plenty of examples in the daily news.

*Throwing Statistics Around, Interview 1*

004 Maggie: Think about all the stats that have been going on since Friday about the [Japan] earthquake to try to bring about an understanding... like being able to understand the size of what happened. People are just throwing mathematical statistics out over and over. And you wonder, “Well, do we understand even what they’re saying? Do we even understand the size of an 8.9?” I don’t think people get that because the numbering system of the Richter Scale numbering system is logarithmic instead of... anyway.

Maggie enacted these long-term goals throughout her teaching. She continually prompted her students to think about ways they could be problem solvers and figure out
solutions, even if the answers weren't apparent to them from the beginning. She was helping them learn how to learn.

*Figure It Out, Lessons 1-14*

268 Maggie: What happens if I make the interval size smaller? Diana, *what did you figure out?* (Lesson 2)

394 Maggie: So, *how would you figure out* how much... *how could you figure out...* disprove this result? How would you go about *trying to figure out* today? 2011. March 2011. How many hours, on average, do teens listen to rock music? *How could you figure that out?* Talk to the person next to you first. *How could you figure that out?* (Lesson 5)

110 Maggie: Well, *you could figure it out,* right? (Lesson 8)

63 Maggie: So all you need to do for yourself is ask yourself that question, "What's the value of \( x \)? *Let me figure out* what \( y \) is." OK? So can... Um, Connor, what's another point on this line? (Lesson 11)

Maggie knew if she were patient with her students and helped to scaffold their thinking, they could be successful learners. During a Meaningful Distributed Instruction lesson on slope, Maggie asked students to determine two points on a line given the linear equation \( y = 7x + 2 \). Maggie had provided Partner Talk time for students to discuss and work out the problem. Then she proceeded to *discuss solutions* with the whole class. The intonation of Landon's voice, his gestures, and the number of questions he asked revealed his lack of confidence. Maggie posed a single question, which enabled him to develop and implement a plan of action. Planning is the most conscious part of being strategic (Johnston, 2004). Once Landon was strategic, he could no longer be helpless. Maggie helped Landon and his classmates learn that math was not a topic to be afraid of.
You Just Put It In the Equation? Lesson 11

47 Maggie: Well, if I told you a value for x, could you figure out what y is? Landon? Gavin? Anybody else?

48 Landon: I think so.

49 Maggie: If I told you a value for x, could you figure out what y is?

50 Landon: (nods)

51 Maggie: Yeah? So you give me a value for x.

52 Landon: Um, two.

53 Maggie: Two? So if x is two, how do I figure out what y is?

54 Landon: (shrugs shoulders) You just...

55 Maggie: (simultaneously) What's... Go ahead.

56 Landon: ... put it in the equation?

57 Maggie: Yeah. And what do you get?

58 Landon: Um, 16?

59 Maggie: Yeah. Sixteen. OK. So, what I just did there, was supplying one question, “Can you tell me a value of x?” You really knew sort of the rest of it, right?

60 Landon: (smiles and nods)

Maggie also found ways to insert the structure of mathematics into her lessons anytime she could. In the following excerpt, Maggie summarized for her students that the linear equation for slope they just described was a descriptor for an infinite number of points on that line.

All These Points Describe That Equation, Lesson 8

132 Maggie: ... So then I'd say, “Ooh! Is that going to...” Look! I'm going to go over three... to three. Seven. This point, three, seven [writes the ordered pair (3,7) on whiteboard next to point] is on my line. OK? So this point (points to “1” in the equation y = 2x +1) is like a descriptor. All the points on this line. All the points on that line. Which is pretty cool.

133 Dylan: (nods) Yeah.

134 Maggie: OK. So of course it’s cool to me, but look at all these points all describe that equation. Way cool. OK.

Maggie reiterated this concept several days later. This time, students were to determine points on a line when given the equation y = 7x + 2.
That Equation Describes All These Points, Lesson 11

Maggie: This is a pretty important concept (points to y = 7x + 2 written on the whiteboard). That... that little bitty thing describes an infinite number of points. All right, that little equation is really telling you all these infinite number of points. And sometimes I picture it similar to like your name. OK. Or whatever we call you. Like, Diana. OK, so we'll say “Diana.” Diana is really... there's a whole bunch of DNA... I don't know... this is a very loose analogy... but really some things about Diana... We can have a little name and it really is describing a huge amount of things. But we just say, “Hey, that’s Diana.” We don't really say all of the things. Sorry, Diana, for choosing you. OK. That was a little longer than I was planning on spending... that's OK.

Maggie used examples from the real world as the basis for the discussions in her classroom. In this final example, Maggie capitalized on the on-going discussion of changing the school colors that was taking place as the school was in the process of changing its image. In addition to evoking a lot of emotion, this became a lesson on the concept of “population.” Maggie helped her students understand that math was everywhere in the world and it was relevant to them.

Who Cares About Changing the School Colors? Lesson 5

Maggie: How many people answered the question if the school colors should be changed?
Chloe: I did!
Maggie: Did you all answer that?
Connor: Is that going to happen?
Maggie: OK. Tell me who cares about... who would care about that? What population of people would care about the change of our school colors?
Ss: (general chatter)
Chloe: Our population of people.
Maggie: What population would care about if we changed the school colors or not? Matt? Wait. Let's make sure we're listening.
Probably students and teachers.

Students. Teachers. Are there any other populations that would actually care about the changes in student colors?

But I care.

I’m not asking if you, independently or individually, care. We’re saying populations. Tony?

Alumni.

We have teachers. We have alumni. Chloe?

Companies that we order jerseys from?

Companies we order jerseys from. Very good. Hey, they might want us to change our school colors, right?

Yeah. Make more money.

Yeah. OK. Hold up. Sh...

Orders to make more money now.

And we’d have to change all these signs and stuff.

Maggie’s long-term goals for mathematics were powerful. Implicitly, they conveyed the message to her students that math was something they could figure out, they needn’t fear it, it made sense, and it was relevant to them on personal level. By enacting her goals throughout her teaching, students received daily doses of these messages, which impacted their confidence and attitudes.

Maggie’s Short Term Goals for the Unit: Becoming Wise Creators and Wise Consumers of Data

Maggie believed that data and statistics was probably one of the most important and practical areas of mathematics that students needed to understand. Unfortunately, there wasn’t time allotted in the curriculum to study it to the depth she would have liked.

We Don’t Have Enough Time For How Much I Think It’s Important, Interview 2

I could study this the rest of the year and I would be happy! I really would! I think it’s really important and I don’t think we have enough time for how much I think it’s important.
Maggie knew that her students were growing up in an age of information explosion. According to a recent review (Gantz & Rainsel, 2011), the digital universe cracked the zettabyte (1,000,000,000,000,000,000,000 bytes) barrier last year, and has already reached 1.8 zettabytes of digital information in 2011. This is nearly as many bits of information as there are stars in the physical universe, and more than doubling every two years. Maggie felt it was critical that her students were prepared to deal with it.

*Thinking Critically About Information, Interview 2*

004 Maggie: I think we’re, as a society, we’re bombarded with information and have to process that and are asked to make real snap judgments about things based on statistics and the more we can have it ingrained in our head, to be critical of that information, and those stats, the better... the better decision-makers we are.

I mean for the average citizen who doesn’t have to use a lot of math, I think this is the math they are asked to do something with frequently. And more so than writing a linear equation, even. I mean I think that’s important, don’t get me wrong. I wouldn’t not teach that. But you know, I’m going to teach the equation of circle and I’m going to tell you that for a whole lot of people, they are not going to use the equation of a circle again. But few people won’t use the idea of thinking critically about a survey again, or a sample, or having to read... interpret some graph, you know, statistical graph. I think most people will have to do that again.

Because there are so many ways that data can be represented and because there are so many misconceptions about how statistical sampling should be carried out,

Maggie’s primary goal for the upcoming unit was for her students to become wise creators and wise consumers of this information. She also wanted them to be able to
think about the connections between data and be able to represent those relationships appropriately.

*Intentionally Craft the Message, Interview 1*

004 Maggie: And again, the overarching is how do you read that and really know what it’s telling you and how do you create it so you’re actually crafting the message mathematically that you want to craft.

Maggie clearly communicated these goals to students at the beginning of the unit. She established these expectations in Lessons 1 and 2.

*Goal: To Be a Wise Consumer, Lesson 1*

242 Maggie: If you were being really, really critical and reading it very carefully, what would you look at so you were like an informed consumer of information? So, just take a minute to look at this quietly. I want you to come up with one thing by yourself. First of all, what type of graph is it? And something you can tell me that the graph... some information it gives you. So type of graph and information it gives you. And one thing you’d want to look critically at if you were trying to be a real critical consumer of information, which some of you already are and we’re going to make you more.

294 Maggie: One of the things that I want us to be careful of when we’re reading the graphs is to take the information here, try to make logical connections so we... that was actually an example of reading beyond the data... is taking this information then making some judgment about it, right? But we’ve got to be careful about what judgments we actually make.

*Goal: To Be a Wise Creator, Lesson 2*

206 Maggie: Let’s talk a little bit about when you’re making graphs. When you’re making graphs, you guys get to make a lot of decisions. And the decisions you... you make when you create the graph actually affects the way somebody reads it.
All right. People that create statistics have a lot of power because they make choices that influence the reader.

One of the ways Maggie enacted this goal was to highlight real-life examples where the perceived message was deceiving. She hoped that by sharing these examples and explicitly talking about the deception, it would increase her students’ level of critical thinking. The following vignette was excerpted from Lesson 8, when the class was learning about statistical sampling. Maggie shared a clip from a recent American Idol episode and led the class in a discussion about the importance of being a critical consumer of data. Note that inaudible text is recorded for the non-participating student (NPS).

American Idol, Lesson 8

160 Video: Stefano is safe.
161 Maggie: Oh, Stefano is safe.
162 Video: And now let’s hear it for Karen and Thea. Come on down, girls. After the nationwide vote, Thea, you are safe.
163 Dylan: What? That bothered you?
164 Maggie: OK. What bothers me about that? That was it!
165 Connor: He said, “You were safe.”
166 Maggie: No. Yeah. I was sort of bothered by that because I thought, “She’s not my favorite.” But that’s OK. I’m sure she’s a nice person. But he said something that was really bothersome to me. Who knows what it is? Try to figure it out.
167 Dylan: It’s something about math, but I don’t know.
168 Chloe: Nationwide vote?
169 Maggie: After the...
170 NPS: (inaudible)
171 Maggie: Nationwide vote. So what do they want you to think in your head?
172 Dylan: Hold it! I didn’t vote!
173 NPS: (inaudible)
174 Chole: I didn’t vote!
175 Ava: I didn’t vote!
176 Maggie: Yeah! I didn’t even vote and I watch the show!
Connor: I voted! I voted!
Tyler: I didn’t. Yeah.
Maggie: After the nationwide vote. And they say it like over and over and I want to say, ‘’Stop causing our people in our nation to have statistical illiteracy!!’’ OK. So. This is the deal, is they are implying by that comment what?
Chloe: That everybody in the nation voted.
Ava: That everybody voted.
Maggie: Or something like that. Or they sampled everybody in the nation. Did they sample everybody in the nation?
Chole: No.
Maggie: Well, wait. Did they do a sample? Yes or no.
Ss: Yeah.
Maggie: Tell me more about their sample.
Landon: They...
Maggie: What kind of sample?
Landon: ...did like a voluntary response.
Maggie: They did a voluntary response...
Landon: Yeah.
Maggie: ...sample. All right?
Dylan: Probably online.
Maggie: They did a voluntary response sample. And who is... Who did they want us to believe is the population that they’re sampling?
Dylan: The American Idol watchers.
Tyler: Everybody in the nation.
Maggie: Would you say that again?
Tyler: Everybody in the nation.
Maggie: Everybody in the nation. But really who are they probably more and likely sampling, Dylan?
Dylan: Like celebrities.
Maggie: No, you said it... you said it right before. That’s why it went to you.
Maggie: American Idol watchers. They’re population is probably American Idol watchers. So their population is American Idol watchers. That’s a little bit of an assumption, but probably better than the entire nation. OK? And then from the American Idol watchers, they are sampling out of a voluntary response method. Right? And by the way, you can vote 50 times online or whatever else. OK.
To achieve the overarching goal of being wise creators and consumers of information, there were a number of individual skills that students needed to acquire during the unit. Maggie wrestled with the balance between writing good questions and representing the data. Once again, it was a time issue.

*A Time Choice, Interview 1*

005 Maggie: Now here’s a dilemma, as a math teacher, is that I know that the questions, all the set-up stuff is as important as the collecting of the data and all that. We’re not going to have the time to be as thorough about that. That’s something they’re going to become more articulate later on. So if you look at, “Why is she letting them do that?” It’s about a choice. You know, a time choice. It’s about a time choice and I’m focusing on representation and being able to read what you have. We know that the quality of what you get is based on what you ask for, but we’ll get to that depth later.

Maggie decided to focus her students’ attention on representing and interpreting the data. In order to represent and interpret univariate (single variable) data, students needed to be able to accurately read, interpret, and create histograms, box-and-whiskers plots, and line plots. In order to represent and interpret bivariate (two variables) data, students needed to be able to accurately read, interpret, and create scatterplots. Maggie drew on her rich theoretical background using Curcio’s (1987) language to differentiate the types of comprehension in which she asked her students to engage when reading and interpreting graphs.

*Making Sense of Graphs, Lesson 1*

242 Maggie: And we’ve talked about reading the data, reading between the data, reading beyond the data.
Reading the data required students to conduct a literal reading of the data, title, or axis label. Reading between the data required students to compare data. Reading beyond the data required students to extend, predict, or infer based on the data.

When Maggie's students were engaged in the type of thinking that fulfilled these goals, she made it transparent for them by acknowledging and naming it. In this process, she fostered their self-awareness of what was going well, which subsequently cultivated self-efficacy. There is power in naming. “Once we start noticing certain things, it is difficult not to notice them again; the knowledge actually influences our perceptual systems” (Harre & Gillet, 1994, as cited in Johnston, 2004, p. 11).

The following vignette illustrates one example of how Maggie enacted her goal of increasing her students' ability to read and interpret graphs. Maggie used a variety of graphs obtained from the USA TODAY Snapshots website (USA TODAY, n.d). Maggie displayed a graph via a PowerPoint presentation on the screen. (See Figure 3.)

![Figure 3. Kids Who Say Reading Books Not Required for School Is Important.](image-url)
Students were eager to share their interpretation of the graph and challenge their classmates’ thinking. Each had his or her viewpoint, which quickly escalated into a passionate discussion. Maggie decided it was a line of reasoning to follow, so she took time away from the primary discussion of analyzing the attributes of the graph to determine if it were a histogram or bar graph. She acknowledged and encouraged her students’ reasoning, facilitated the discussion with purposeful questions, revoiced their thoughts to add clarity, and then concluded the side bar by informing them that they were “reading beyond the data” (Curcio, 1987).

*Reading Beyond the Data, Lesson 2*

191 Dylan: They say you have to read every night. And you go and then you’re like, “Why am I listening to this? I’ll just go play video games.” And then at twelve to fourteen, you start playing video games and stop.

192 Evie: It’s not that, Dylan. There are probably more outside activities.

193 Maggie: Hold up. Hold up. Only some of you are listening to you right now. So if you’re not listening to Dylan right now, if you’re not listening to Dylan right now, I want you to actually listen. We’re going to take a little moment to have a side bar about this because it is interesting to me, as well. OK. Say what you’re thinking, Dylan.

194 Dylan: OK. I was just saying...

195 Maggie: Wait. Wait. You’re going to show him that you’re respecting him by... Yes, you know what to do.

196 Dylan: I was just going to say that I think it’s funny as like the kids get like a little bit older they start thinking, “Why am I listening to my teachers who are telling me to read everyday? I can just do what I want.” And so then they start playing video games and slacking off and that’s just what happens.

197 Evie: Dylan!

198 Maggie: You think that’s just what happens?

199 Evie: No. Wait. Wait. Wait. Dylan. But this is outside of school when they’re not forced to read.

200 Maggie: Evie, say what you said again louder so I can hear.
201  Evie: Well, it’s like, yeah...  
202  Gavin: They’re never forced to read.  
203  Dylan: Mr. Johnson forces me to read, but I don’t do it anyway.  
204  Gavin: See, then he doesn’t force you to read, then.  
205  Evie: It’s saying, it’s not saying that you’re... It’s saying the opposite of what you’re saying. The teachers aren’t making them read. This is as important outside of school and the teachers aren’t making you read.  
206  Chloe: Yeah, but he’s looking at the 15% (inaudible).  
207  Maggie: Well, now could this be a personal opinion? Do you think it would vary person-by person? Yeah. OK.  
208  Gavin: There’s a difference from six to eighteen, too.  
209  Maggie: What were you saying?  
210  Gavin: Nothing.  
211  Maggie: Are you sure? Were you making a good observation? I think you were.  
212  Gavin: No, I wasn’t.  
213  Maggie: All right. OK. I’ll trust you on that one. So, hold up. So, Evie, your point was, this is really about people just reading to themselves outside of school.  
214  Evie: But like also it kind of makes sense that a job... So you do more activities outside of school and you have more homework than you would and stuff.  
215  Maggie: So you’re thinking that this age group just might have less time...  
216  Evie: Yeah.  
217  Maggie: ...to actually engage in reading for leisure? Would that make sense? And Dylan, you might say, some people don’t enjoy it as much, to read. And they replace it with doing some other activity that is they have more control of. OK. Yeah. All those could be. Right there, what you guys just did, is you didn’t read just the data straight from here (points to graph), right, you didn’t read the combined data. You sort of looked at it and made inferences to your own world about what might happen. You read beyond the data.  

Maggie had two additional goals that were conveyed through the title of the unit, *Samples and Population*. First, she wanted her students to learn about various sampling methods, specifically convenience, voluntary response, systematic, and random sampling, including basic methodology and advantages and disadvantages for each. Second, she
wanted them to develop an understanding of how they could determine a reasonable sample size from which they could be fairly certain to draw conclusions.

One way Maggie enacted the goal of learning about the four sampling methods was to have students determine which was used in each of four examples and tell whether they thought it would give a sample that let them make accurate predictions about the population. Most of the students had some experience with which they could connect in order to demonstrate a basic understanding of three methods, convenience, voluntary response, and systematic. Understanding what it meant to take a random sample was fraught with misconceptions, however. In the following vignette, students discussed the second example: “Dan suggests putting 320 white beans and 30 red beans in a bag. Each student would draw a bean as he or she enters the auditorium for tomorrow’s assembly. The 30 students who draw red beans will be surveyed.” While correctly identifying this as an example of random sampling was one aspect of her goal, Maggie was more interested in eliciting students’ reasoning, which was central to developing the concept of random. Although this was an introductory lesson, students raised the notions of equal chance, blind selection, and the nuances of probability.

*The Red Bean, Lesson 7*

187 Maggie: All right, Matt.
188 Matt: I think it’s random.
189 Maggie: You think it’s random. *Why do you think it’s random?*
190 Matt: *Because...*
191 Evie: Oh! I forgot to do number three.
192 Matt: Because there’s only ... you have to pick beans and there’s only 30 right ones. However many there...
193 Maggie: *Does everybody have an equal chance of being selected?*
194 Connor: Yeah.
195 Matt: *(shakes head, but changes answer) Yeah.*
196 Maggie: Everybody... What is everybody's chance of being selected?

197 Connor: One out of 30.

198 Maggie: One out of...

199 Chloe: One out of a hundred... I don't know.

200 Maggie: What... what do you think? Andrew, you were doing this (raising hand). What do you think?

201 Andrew: Well, if uh, somebody gets to draw before you, you might take one of the red ones out so it might not be the same.

202 Maggie: Yeah. It's actually a harder question than just a one out of 300... we want to say, well everybody... everybody's going to get a bean, right? But as soon as that first... the first person has... what's the chance for the first person that draws?

203 Chloe: One out of 350? Or one out of 320?

204 Maggie: Well, how many red... that's red, right? How many red ones are there?

205 Chloe: Oh, 30.

206 Maggie: Yeah. Thirty out of 350. And then from there, it would be dependent on whether a red bean has been drawn or not. OK. We're not going to go into calculating that probability, but probability can become complex. You really got to think carefully about it, so that's good. Evie and then Diana (hands were raised).

207 Evie: Also, like, what if somebody got... somebody picked a red bean and then somebody who wants the color better than me, “Oh I want a red bean” so they try to get a red bean? So then it could throw it off if somebody else.

208 Maggie: Yeah. If we were truly doing it randomly, Evie, though, we wouldn't give the opportunity for the person to actually know how to attempt to try to get a red bean. They would have to not see. It would have to be completely random. OK, yeah.

209 Diana: Or you could just have them give the beans back because it's not very useful to have them just keep a random bean. They're probably not going to even want it. If you say put the bean back.

210 Maggie: Oh, wait. So if somebody got a red bean could we put it back in the container?

211 Connor: Could.

212 Diana: Yeah. And then just mix it up again.

213 Maggie: And then just mix it up again. Now I want you to think about that.

214 Diana: Would be the same.
Maggie: Would that still be... are you still going to have 350 people go through and draw the beans?
Diana: Yeah.
Maggie: I mean you have 350 students, right?
Diana: (nods)
Maggie: Yeah. Is each student going to draw a bean? Are you going to have them put it back one at a time? Or are you going to going to put them all back after you’ve used them once?
Diana: You put them back after you’ve picked the one out and then put it back in.
Maggie: OK. I just want to make sure I understood. So if we come through, Chloe draws a bean. She’s going to put it back, right?
Diana: Then everyone would have an equal chance.
Maggie: Now... Wait. Wait. Would you say that again?
Diana: So then everyone would have an equal chance because there would still get the same amount of beans for everyone. And there wouldn’t be like one bean left for the last person and they wouldn’t have a chance.
Maggie: OK. I want you to hold on because there’s one... there’s some little flaw in the logic there a little bit I think. But I’m thinking through it myself, also, to make sure that I’ve got it. Yeah. Maybe Ava can help us.
Ava: Wouldn’t there be more than 30 people with the red bean though?
Chloe: Yeah.
Ava: Because they would keep putting it back?
Maggie: What... what do you think? What do you think, Diana? Would more than 30 people?
Chloe: Or less even.
Diana: Yeah, but.
Maggie: Or less even?
Evie: She just said that.
Maggie: Huh?
Evie: Oh.
Maggie: Yeah. More or less. Yeah, Andrew?
Andrew: Wouldn’t you have to, like, keep a tally of who got the red one and who got the white one?
Maggie: Yeah. You would definitely want to keep a tally... or you’d want to have that. So, I think if you wanted to ensure that you’re going to have 30, 30 people, then yeah. But um, and Andrew, your question about this person’s probability change or not change... if we distribute them all at one
time, we still might want to think about how that is. So the probability is tricky, but I think it's really trying to give everybody an equal playing field. Everybody gets a chance to get a red bean, right? Um, and we go from there. We're going to worry about some of the innuendos of probability, which is a whole... another topic that goes hand-in-hand with this. And the reason why I say it goes hand-in-hand with this is if you look at the advantages under "random." Go to Plan 4 and look at advantage under "random," I put one that was bold. We just started to talk about that naturally here. Is that [reading] "...most preferred by statisticians because the results can be analyzed with probability tools." So that right there, what you naturally started to talk about, is exactly what statisticians start to do with that also. So random has a lot of things in favor, but we need to be thoughtful about how we do the random... randomly generated data or random samples. And we need to know how to look at our data once we get a sample or more than one sample.

To address the second goal, determining a reasonable sample size from which confident conclusions could be drawn, Maggie guided her students' discovery through a PBIT in which they, in small groups of three, were provided with a data set of the number of hours 100 children slept on average per night and the number of movies those same children viewed during a week. They were to obtain their own random sample using a random number generator on their calculator for a sample size of 30. Once they represented their data using a line plot and box plot, they would compare their plots with their group members' plots. They were asked to compare the variability of the three line plots and three box plots, and then draw conclusions based on the data. Finally, they were to determine whether they would have drawn a different conclusion had they only had one sample set of 30 available to them instead of three.
When comparing their sample size of 30, Diana and Evie were not convinced that generating random numbers was a necessary process in order to gather their samples of 30 data points. In the following excerpt, they discussed the following question, “Would you decide the same thing based on your partners’ sample of 30? Or would you decide something different about your population based on your partners’ sample of 30?”

**Sample Size of 30, Lesson 10, Small-Group Discussion**

219s Diana: I think that it would not be good if you made a prediction using different people’s graphs because they’re basically all the same.

219t Evie: I agree. Because looking at this, we still got close to the same answers even though it was supposed to be “random.” So... my conclusion is that it wasn’t random. It was random, but it wasn’t because it somehow didn’t matter anyways. So we could have done something different and way easier that would take way less time and probably get the same answer. We could just like randomly just point... point... choose people. That works!

As Maggie circulated around the room, she stopped to listen to their Small-Group Discussion. She helped them to clarify their confusions and posed questions that helped them look at their data in a different way.

219cm Maggie: So all three samples are almost identical? Right? But are the movies watched almost identical? I mean... when... if you wanted to describe the number of movies watched, you wouldn’t say everybody watches the same number of movies, would you?

219cn Ava: No.

219co Evie: No. It’s very similar.

219cp Maggie: OK. I think either I’m not understanding something or I think you’re confusing two points. So I am really asking you to do two things. Ready, Evie? I’m asking you to compare these three samples. And that’s where I think what you’re saying is that all three samples are very similar. Right? And I’m asking you to describe the variability of just movies in general. OK? So...
They're all clustered at like five and under and there's some middle ones that...
All right. So if you were to describe the population, of 100 people, right. So what would you say about that in general? Of that population of 100?
Most people watch under five movies.
Right.

Through Maggie’s careful questioning, the group was able to draw an accurate conclusion and then evaluate it. She continued to revisit this topic in subsequent lessons, having students generate samples of 20, 10, 5 and others, using Fathom® Dynamic Data software to decide how small was too small.

Maggie’s short-term goals, or learning objectives, for the data and statistics unit influenced her day-to-day decisions. Excerpts and vignettes from her classroom illustrate how she carefully chose specific tasks and activities that were directly aligned to these learning outcomes. These goals impacted the minute-to-minute decisions Maggie made regarding the questions she posed and the comments she made to students as they engaged in these activities.

Maggie’s Expertise: Content Knowledge For Teaching

Maggie was a student of mathematics. As noted earlier, she held both an undergraduate and graduate degree in the field of secondary mathematics education. But as H. C. Hill, Rowan, and Ball (2005) note, the mathematical knowledge that teachers need in order to be successful in the classroom go beyond mathematics courses taken or basic mathematical skills. Teachers of mathematics not only need to know how to calculate correctly, but also must be able to “use pictures or diagrams to represent mathematics concepts and procedures to students, provide students with explanations for
common rules and mathematical procedures, and analyze student solutions and explanations” (p. 372).

During the post-unit interview, Maggie highlighted several experiences that shaped her as a teacher. One opportunity that had a “huge impact” was her work as a field-tester for Course 1 Core Plus Mathematics 2nd edition. According to the Western Michigan University Core-Plus website (Core-Plus Mathematics, n.d.), Core-Plus Mathematics, funded by the National Science Foundation, is a four-year high school curriculum that replaces the traditional Algebra-Geometry-Advanced Algebra/Trigonometry-Precalculus sequence. Instead, interwoven strands of algebra and functions, statistics and probability, geometry and trigonometry, and discrete mathematics are featured in each course.

*I Thought I Knew It*, Interview 3

004 Maggie: To be able to um, really have to say, “You’re going to try this out whether I...” and then come back and you have to articulate what worked, what didn’t work. What understanding did you see come out of this? What didn’t? That was... that was huge and that was over five years. It was, yeah. That was... and it was a lot of mathematics I didn’t know before. I mean, even though I thought I knew it, I didn’t really know it.

A second professional development experience that stood out for Maggie was Teaching the TEKS through Core, held in San Antonio, Texas. She related that it was about formative assessment and how to get inside the heads of students and open up the classroom environment to allow them to express, think, and conjecture.
He Never Said Wrong Or Right, Interview 3

004 Maggie: That was really good and I almost wish I could go back through that training because I remember him facilitating the whole week and him never saying wrong or right. But you were learning accurately.

Maggie also cited how her experience as a professional development provider helped her to grow professionally, and how several colleagues provided inspiration and were critical friends with whom she could discuss teaching practices honestly and openly.

Because of Maggie’s extensive content knowledge for teaching, she was able to navigate students’ misconceptions with ease, as illustrated in the following example.

During a five-minute Meaningful Distributed Instruction lesson on slope, Maggie asked students to name two points on the line given the equation \( y = 7x + 2 \). She provided Partner Talk time for students to discuss and solve the problem with a partner. Then she proceeded to discuss solutions with the whole class. Landon established that \((2, 16)\) was one point on the line (see Maggie’s Long-Term Mathematics Goals: Learning How To Learn section: You Just Put It In the Equation? Lesson 11 for a transcript of this exchange). Following Landon’s turn, Connor figured out that \((6, 44)\) was another point that would fall on the line. At this point, Evie spoke up. Not only did she have an error in her multiplication, she applied a faulty strategy for discovering another point.

Direct Variation, Lesson 11

091 Evie: Wait. Aw... I must have calculated wrong because I got four and 41 for one of those...
092 Maggie: You had four and 41?
093 Evie: For one of those points.
094 Maggie: Let’s try it out. So if \( x \) is equal to four...
095 Evie: Ooh! Let me recalculate.
Maggie did not get sidetracked with management issues or inaccurate calculations, but kept the focus of the lesson on her goal of helping her students learn how to learn. Maggie honored Evie’s thinking, helped her identify where her logic went awry, provided her a way to self-check and self-correct her answer by testing her suggested point (4, 41) against the equation $y = 7x + 2$, and quickly got the lesson back on track. Several other situations surfaced during the lessons led by the novice university students where Maggie stepped in to assist in order to untangle misconceptions or
missteps in the lesson before it unraveled. Maggie was able to do this because of her
wealth of content knowledge for teaching.

How Maggie Organizes Her Lessons

Maggie taught using the three-phase “launch, explore, summarize” lesson model
described by Sherin (2002) and Stein et al. (2008), which she divided between Whole-
Class Discussions, Partner Talk, and Small-Group Discussion talk formats (Chapin et al.,
2009). Table 3 provides a summary of how she allocated her 701 minutes of time used
for mathematics instruction.

Table 3.

<table>
<thead>
<tr>
<th>Talk Format</th>
<th>Minutes</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole-Class Discussion</td>
<td>520</td>
<td>74%</td>
</tr>
<tr>
<td>Lessons 1-14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Partner Talk</td>
<td>104</td>
<td>15%</td>
</tr>
<tr>
<td>Lessons 1-6; 8-11; 14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small-Group Discussion</td>
<td>77</td>
<td>11%</td>
</tr>
<tr>
<td>Lessons 7-8; 10; 12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>701</td>
<td>100%</td>
</tr>
</tbody>
</table>

Whole-Class Discussion

Maggie taught primarily using Whole-Class Discussion, which was found in
every lesson. She spent 520 out of 701 minutes (74%) of the time devoted to
mathematics instruction in this talk format. Maggie used Whole-Class Discussion at the
beginning of the “launch” phase, at the beginning of the “explore” phase, and at the
beginning and end of the “summarize” phase. During this time, she was typically in
charge of the class, although she did step out (Rittenhouse, 1998) of the lesson during the summarize phase of Lesson 6 to observe a 17-minute student-led Whole-Class Discussion as students shared advantages and disadvantages of four sampling methods (convenience, voluntary, systematic, and random) and their reasoning for each.

During Whole-Class Discussion, Maggie facilitated students' understanding as she guided their discovery of new concepts, provided her students with open-ended exploratory opportunities, and presented direct instruction regarding procedural conventions. Each of these lesson models, The Guided Discovery Lesson, The Open-Ended Exploration Lesson, and the Integrating Direct Instruction with Guided Discovery Lesson (Rubenstein et al., 2004) were observed and were variations of the three-phase launch, explore, summarize lesson format. Because the Integrating Direct Instruction with Guided Discovery combined two lesson models, these were separated for analysis purposes.

Maggie explained that she typically opened a new unit with more of a guided discovery format. She encouraged students with comments such as, "You can do this" and "Hey, let me listen to you." She asked questions such as "Are you really thinking about this?" or "Have you thought about this?" As the unit progressed, she addressed the areas of vocabulary. Her comments became more direct when she used a direct instruction approach to teach procedural knowledge. Then at the end of the unit, she began to question again, but at this stage, the purpose of the questioning was to deepen students' knowledge that had been gained throughout the unit, by having them restate or
summarize, “This is what I know.” Table 4 describes the various ways Maggie used Whole-Class Discussion in her classroom.

Table 4.

| Eighth Grade Mathematics Instruction: Use of Whole-Class Discussion |
|---------------------------------|---------|-----|-----|-----|-----|-----|-----|-----|
| Whole-Class Discussion Model    | Minutes | %   | M   | Min | Q1  | Med | Q3  | Max |
| Direct Instruction              | 260     | 50% | 21:42| 6:10| 14:13| 18:02| 27:50| 45:55|
| Lessons 3-14                    |         |     | 21:42| 6:10| 14:13| 18:02| 27:50| 45:55|
| Guided Discovery                | 243     | 47% | 22:06| 4:41| 18:47| 20:55| 23:37| 48:00|
| Lessons 1-3; 5-11, 14           |         |     | 22:06| 4:41| 18:47| 20:55| 23:37| 48:00|
| Open-Ended Exploration Lesson 6 | 17      | 3%  | 17:07| 17:07| 17:07| 17:07| 17:07| 17:07|
| Total                           | 520     | 100%|      |     |     |     |     |     |

The direct instruction lesson. The most-frequently used lesson model was the Direct Instruction Lesson employed half of the time. This model was imbedded within the entire Whole-Class Discussion in 12 of the lessons. This model was not used in Lesson 1 and 2, when Maggie introduced the unit. In those first two lessons, she pre-assessed what her students knew about data and statistics, and guided them to make connections with their prior knowledge. The time devoted to The Direct Instruction Lesson model ranged from six to 46 minutes.

When the only way students can access mathematical knowledge is from an external source, such as a book, the teacher, or another student, The Direct Instruction Lesson is appropriate. This includes learning about mathematical symbols, mathematical terminology, and mathematical conventions (Chapin et al., 2009).
Maggie used The Direct Instruction Lesson to explain procedures, such as how to make a box plot, how to make a histogram, how to generate a random number using the random-number generator on the calculator, and how to use Fathom® Dynamic Data software to sample a population. She used this lesson format to explain “SCRAP the Gap” strategy, an original strategy that Maggie and her colleague co-developed to help students write better mathematical descriptions. She also had them read handouts and from the textbook during this time.

The guided discovery lesson. The second most frequent lesson model was The Guided Discovery Lesson, which Maggie used 47% of the time. This model was imbedded within the entire Whole-Class Discussion in 11 of the lessons. This model was not used in Lessons 4 or 12, when The Direct Instruction Lesson model was used, or in Lesson 13, when the students were formally assessed. The time devoted to this type of lesson model ranged from nearly five to 48 minutes.

The Guided Discovery Lesson is appropriate lesson model to use when mathematical concepts and skills can be reasoned through logic and the goal of the lesson is for students to make sense of the mathematics and create new understanding. The teacher guides student thinking as they process information, apply reasoning, listen to others’ thinking, and connect new thinking to their prior knowledge (Chapin et al., 2009).

Maggie used The Guided Discovery Lesson to prompt student thinking and challenge student reasoning. She incorporated teacher talk moves, discussed in a later section of this chapter, as a way to engage the class in a single discussion. She posed
questions and provided experiences that guided students to a deeper understanding of mathematical ideas.

The open-ended exploration lesson. The Open-Ending Exploration Lesson was the least-used lesson model, as Maggie used this approach 3% of the time she devoted to mathematics instruction. This single lesson was imbedded within the entire Whole-Class Discussion during Lesson 6. The time devoted to this type of lesson model was 17 minutes.

The purpose of this lesson type is similar to the Guided Discovery lesson model, in that the goal of the lesson is for students to make sense of mathematical ideas and deepen their understanding of new concepts. The difference in the Open-Ending Exploration Lesson is that this lesson model provides students with opportunities to explore mathematics without guidance from the teacher regarding the course the students might take, and has the possibility of leading down unanticipated routes. The teacher must have a strong understanding of mathematics and the confidence to follow students wherever they may venture (Rubenstein et al., 2004).

Maggie used the Open-Ending Exploration Lesson model when she removed herself as the center of Whole-Class Discussion. Because she was a good word processor, students chose Chloe to lead and record the discussion at the computer while Maggie retreated to the back of the room to become an observer of her students' thinking. During the 17-minute student-led discussion, students shared advantages and disadvantages of four sampling methods (convenience, voluntary, systematic, and random) and their reasoning for each.
**Purposeful decisions.** Maggie was purposeful in her choice of whole-class discussion lesson models. She was flexible in the way she organized her whole-class instruction and considered a number of factors, including the intent of her lesson, time to develop lessons, nature of the content, and student needs when choosing the particular lesson model to implement.

Maggie acknowledged that at times she removed herself from the discussion, but often, she wanted to guide and direct the talk. It depended on the intent of her lesson.

*On Stepping In, Stepping Out, Interview 1*

008 Maggie: I know a lot of articles talk about how you remove yourself from the large group discussion, and I’m definitely a believer in that, but sometimes I don’t want to remove myself because it’s real purposeful that I’m in there because I think it’s going to get to a certain place and I want to move it to that place. And sometimes I just want them to be talking with each other and I will try to remove myself. And with some groups, that’s more successful than others. I’m not always clear on what I do or I don’t do, or what they do, or what they don’t do that makes it more successful with some groups than others.

Time was also a factor that affected her choices. Sometimes the content took longer to teach than Maggie expected. Sometimes it took longer for the students to process, the technology didn’t work, or she pursued a line of learning that followed students’ thinking, instead of forging ahead with what she had planned. She found herself behind by the end of Lesson 1 and several other times throughout the unit.

*It Took A Little Longer, Lessons 1, 11, 14*

381 Maggie: We did not get to actually the main activity, which I’m... so tomorrow when you come in... (4 seconds later) Tomorrow when you come in, you need to be ready. (Lesson 1)
That was a little longer than I was planning on spending... that’s OK. (Lesson 11)

Instead of us all getting our computers out, because I think it’s going to take too long, even thought that’s what I want to do, that’s going to take longer than we have to do it. (Lesson 11)

I know you’ve been listening a lot and I have to tell you that I planned this slightly differently, but this is going to be OK because we don’t have the internet... (Lesson 14)

As she reflected on her pacing, Maggie acknowledged that she continually negotiated the need to follow what she had planned with the need to follow her students’ lead.

*I Found Myself Just Being Interested*, Interview 3

I changed [my mind] several times throughout that whole unit. Something took way longer than I imagined it to be. ... And so I thought, “Oh, we’ll have time to do this.” But we didn’t because we... Sometimes they were on the right task, or sometimes I found myself just being interested in what they had to say, so then I need to ask another question about it instead of holding my tongue and going, “OK, let’s really get to this point” because that will be good learning, too.

Maggie conceded that preparation time was another variable that played a factor in how she organized her lessons. Over the years, her units evolve as she has time to select alternate tasks and develop or adapt materials with more relevant contexts for her students.

*It’s a Time Variable*, Interview 3

I will say that setting up an exploration takes a lot more up-front time. And sometimes it’s a time variable, which is not a good excuse, but it is a good excuse.... And they sort of take a long time to even adapt or anything else to where I want to use them. And so then when... Especially when
it's a new unit that I've never taught before, then I take a baby step into that. You know, and next year it's going to... Next year I probably would then transform more of it. And more of it... But it's hard for me to do all that work up front.... And so since I hadn't taught the unit before, um, I didn't have anything developed. So anything that I developed or adapted was done within that little three-week period of time. And uh, sometimes that's a horrible reason to decide, but it is reality.

Another consideration was the nature of the content. Some content was better learned through guided discovery or open-ended exploration lesson formats. Other content, especially if it was procedural in nature, was taught using the direct instruction model.

**SCRAP the Gap, Interview 3**

002 Maggie: SCRAP the Gap was pretty direct instruction and I wasn’t planning to do that originally. But I did that. They were not really meeting what I wanted them to meet as far as description. And it's a pretty straightforward strategy. It produces results. It gives some structure to... to their descriptions. And, um, so I think that that is something that works.

In order to enhance the likeliness that her students would be successful, Maggie varied the structure of her lessons. She recognized that some students had the ability to handle open-ended exploration lessons and sometimes they needed more guidance or structure.

**Striking A Balance, Interview 3**

002 Maggie: And sometimes I'll balance with, you know, some kids can handle exploring all the time. And sometimes kids can't. And um, you can move them to be able to handle it more, and more, and more. Um, so I think all kids can handle exploring, it's just to the depths of which you ask them to
do that repeatedly. And so I think it's OK to vary... vary that.

Maggie was strategic as she selected the Whole-Class Discussion lesson model for her instruction, using The Direct Instruction Model and The Guided Discovery Model most frequently. Her decisions were based on choosing the model that would best serve her intended learning goals, the available time to prepare and implement, the appropriateness for the content, and most importantly the model that would be particularly useful in order to maximize opportunities for mathematical learning by all her students.

**Partner Talk**

Maggie integrated two other talk formats, Partner Talk and Small-Group Discussion, throughout each lesson according to what she felt would be most meaningful.

*Misleading Talk*, Interview 1

006 Maggie: The word partner can be switched out for small group depending upon the need of the lesson. If it's more meaningful to have partners, then we have partners. If it's *more meaningful* to have small group, then we have small groups. So that has been a shift for me over the years for a time, because I used way more groups earlier in my teaching. And I've gone back to more partnerships and then groupings or sometimes partners have partners like two and two (gestures to form a quad), but if you're going to be four then you guys know you're going to be the four.

Partner Talk was the second most-used talk format in Maggie's classroom. She incorporated this talk format in Lessons 1 through 6, 8 through 11, and 14. Maggie did not use Partner Talk in Lessons 7 or 12, as students were engaged in Small-Group Discussion, or in Lesson 13, when students were formally assessed. Students were
engaged in Partner Talk 104 out of 701 minutes (15%) of the time Maggie devoted to mathematics instruction. Table 5 describes the various ways Maggie used Partner Talk in her classroom.

Table 5.

<table>
<thead>
<tr>
<th>Use of Partner Talk</th>
<th>Minutes</th>
<th>%</th>
<th>M</th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discuss Questions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons 1-5; 8, 11, 14</td>
<td>20.5</td>
<td>20%</td>
<td>0:54</td>
<td>0:08</td>
<td>0:26</td>
<td>0:39</td>
<td>1:10</td>
<td>3:39</td>
</tr>
<tr>
<td>N = 24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solve Math Problems (Slope)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N = 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Create Graphical Representations</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons 2, 4, 5, 9</td>
<td>29.5</td>
<td>28%</td>
<td>7:24</td>
<td>1:49</td>
<td>2:01</td>
<td>3:23</td>
<td>8:45</td>
<td>21:02</td>
</tr>
<tr>
<td>N = 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Math Activities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons 6, 9</td>
<td>25.5</td>
<td>24%</td>
<td>12:45</td>
<td>7:59</td>
<td>10:22</td>
<td>12:45</td>
<td>15:08</td>
<td>17:32</td>
</tr>
<tr>
<td>N = 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discuss/Do Homework</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons 4, 8, 14</td>
<td>22:5</td>
<td>22%</td>
<td>7:33</td>
<td>2:03</td>
<td>3:56</td>
<td>5:50</td>
<td>10:18</td>
<td>14:47</td>
</tr>
<tr>
<td>N = 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>104</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Maggie injected the use of Partner Talk 24 times into Whole-Class Discussion by pausing the discussion for short periods of time while she posed a question for students to discuss. The time provided for students to discuss questions ranged from eight seconds to over three minutes. Maggie also used Partner Talk twice as an opportunity for students to solve mathematics problems around the topic of slope during her MDI mini-lessons. This talk time ranged from nearly three to over four minutes. Both of these uses of
Partner Talk provided time for students to collect their thoughts, practice their contributions with another peer, and express their ideas during the mathematics lesson. It was not feasible for every student to talk during the Whole-Class Discussion due to time constraints. Partner Talk provided everyone an opportunity to participate.

On four occasions, students worked with partners to create graphical representations, such as making a histogram and a box plot, or explored Fathom® Dynamic Data software, which produced all sorts of graphs and a number of other statistical calculations for them. These activities were procedural in nature, and partners helped each other with the process. The time provided for students to work on these activities ranged from seven-and-a-half to 21 minutes.

Twice, Maggie engaged students in short mathematical activities. These tasks required students to explore mathematical concepts and discuss these at greater length with their partners. These included evaluating the advantages and disadvantages of various sampling methods, and the activity, \textit{Who Would You Choose?} in which students used statistical data to determine whether they would choose Player A or Player B to be on their basketball team. Students engaged in more complex thinking during these activities and shared reasoning with each other. The range of time permitted for these activities was nearly 13 to 17 minutes.

On three occasions students discussed, completed, or started homework during class. Maggie provided from two to 15 minutes for students to work with partners on this task.
During Partner Talk, Maggie circulated the room to listen in on students’ discussions. She asked specific questions of partners if she thought it would prompt thinking, clarify confusions, or deepen thought. She also provided assistance when it was requested.

**Small-Group Discussion**

Small-Group Discussion was the least-used talk format in Maggie’s classroom. She incorporated this talk format in Lessons 7, 8, 10, and 12. Students were engaged in Partner Talk during the other lessons as noted above, except for when they were formally assessed in Lesson 13. Students were engaged in Small-Group Discussion 77 out of 701 minutes (11%) of the time Maggie devoted to mathematics instruction. Table 6 describes how Maggie used Small-Group Discussion in her classroom.

Table 6.

<table>
<thead>
<tr>
<th>Use of Small-Group Discussion</th>
<th>Minutes</th>
<th>%</th>
<th>M</th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Based Instructional Tasks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lessons 7-8, 10, 12</td>
<td>77</td>
<td>100%</td>
<td>15:24</td>
<td>5:16</td>
<td>10:08</td>
<td>12:29</td>
<td>12:33</td>
<td>36:38</td>
</tr>
</tbody>
</table>

Maggie engaged her students in two PBITs during Small-Group Discussion. The first PBIT occurred over three days during Lessons 7, 8, and 10. It began as a Whole-Class Guided Discovery Lesson in which Maggie guided the students to discover the meaning of random and how random numbers were generated. Then the lesson shifted to
a Direct Instruction model as Maggie demonstrated how to use a random number generator on her calculator. The lesson shifted once again to a Small-Group Discussion when Maggie grouped students into groups of three and one group of four. She provided them with a data set of the number of hours 100 children slept on average per night and the number of movies those same children viewed during a week. Students obtained their own random sample of 30 pairs of sleep and movie data from the population of 100 using the random number generator on their calculators. Much of the Small-Group Discussion up to this point was procedural in nature as students discussed the process for generating the random numbers and recording their data.

Once students represented their data using a line plot and box plot, they compared their plots with their group members' plots. The Small-Group Discussion shifted to reasoning when students compared the variability of the three line plots and three box plots, and then drew conclusions based on the data. Student reasoning continued when they determined whether they would have drawn a different conclusion had they only had one sample set of 30 available to them instead of three.

The second PBIT lesson began as a Guided Discovery Lesson during Whole-Class Discussion as Maggie introduced the occupation of an archeologist. The next day, during Lesson 12, students were organized into groups of three and one group of two. The talk shifted to a Small-Group Discussion as students were presented with the task of estimating the time period when two archeological sites were inhabited based on the arrowheads found there. Students were provided two sample sets of data based on the length, width, and neck width of the arrows. They compared these to arrowhead data sets
from two known sites. Just as with the Sleep/Movie PBIT, the Small-Group Discussion focused on the procedural task of constructing the box plots. Once the box plots were completed, the discussion shifted to reasoning as students shared how they used the statistics and the graphs to determine the solution. As groups finished with the original task, Maggie provided them with a question to extend their thinking. Groups were to decide if it was possible to predict the length of an arrowhead if they only had a known width. Students continued to share their reasoning with group members as they pondered the possibility.

During Small-Group Discussion, Maggie moved among groups to listen in on students' discussions. She asked specific questions of group members if she thought it would stimulate ideas, clear up misconceptions, or bring about deeper understanding. She also helped students when they asked.

Maggie enjoyed interacting with students during Partner Talk and Small-Group Discussions. She was able to differentiate her instruction based on the needs of individual students or a few students at a time. It also provided her with time to listen to students' thinking independent of her input.

*I'd Rather Mini-Teach*, Interview 1

006 Maggie I'd rather mini-teach than group teach.

As noted, these talk formats provided students time to grapple with ideas, conjecture, and reflect on the plausibility of these claims with one or two other people. They also provided students with time to hone their skill in constructing various graphical representations, which were important learning objectives for the unit.
How Maggie Implements Her Lessons: Weaving It All Together

Maggie was masterful as she wove her lessons together, moving between lesson models and talk formats as appropriate and meaningful. Usually, she followed the typical reform-based three-phase mathematics lesson, which included “launch,” “explore,” and “summarize” phases. These three-phase lessons were incorporated every day except for Lesson 13, an assessment day. Maggie also incorporated nearly an hour of her total lesson time to the Meaningful Distributed Instruction Lesson type during Lessons 6 through 11. Only 45 minutes of mathematics instruction time was used for students to work individually. These individual work times were either used for students to read short assignments (Lessons 5, 7, and 11) or complete an assessment (Lesson 13). Table 7 describes how Maggie used Lessons Types in her classroom.

Table 7.

<table>
<thead>
<tr>
<th>Eighth Grade Mathematics Instruction: Type of Lesson</th>
<th>Minutes</th>
<th>%</th>
<th>M</th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three-Phase Lesson (launch, explore, summarize)</td>
<td>602</td>
<td>86%</td>
<td>43:01</td>
<td>34:58</td>
<td>43:00</td>
<td>46:24</td>
<td>50:15</td>
<td>53:47</td>
</tr>
<tr>
<td>Lessons 1-12, 14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Meaningful Distributed Instruction</td>
<td>54</td>
<td>8%</td>
<td>8:58</td>
<td>2:27</td>
<td>6:16</td>
<td>8:47</td>
<td>12:03</td>
<td>15:16</td>
</tr>
<tr>
<td>Lessons 6-11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Independent Work</td>
<td>45</td>
<td>6%</td>
<td>11:20</td>
<td>1:04</td>
<td>2:54</td>
<td>3:39</td>
<td>12:05</td>
<td>37:00</td>
</tr>
<tr>
<td>Lesson 5, 7, 11, 13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>701</td>
<td>100%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Three-Phase Reform-based Mathematics Lesson

The three-phase reform-based mathematics lesson was the most frequent type of lesson Maggie incorporated into her instruction. She engaged her students for 602 minutes (86%) of her mathematics instruction in this type of lesson spending anywhere from 35 to nearly 54 minutes per 52-minute class period. As in many reform-based classrooms, Maggie spent more than one class period to complete the lesson sequence. In the following example, Maggie involved her students in learning about bar graphs and histograms beginning in Lesson 1 and continuing through Lesson 3.

At the beginning of the unit, Maggie clearly announced her learning goal so students were aware of the new topic. Then she launched right into a pre-assessment discussion.

*Announcement of Learning Goal, Lesson 1*

041 Maggie: We’re going to be learning about data and statistics, and I need to know a little bit about what you know about data and statistics.

During the “launch” phase, Maggie set the stage for the lesson by introducing her students to the problem using a Whole-Class Discussion talk format (Chapin et al., 2009). As she planned this phase of the lesson, she decided to engage her students in preliminary thinking about the lesson by probing their prior knowledge through purposeful questions to assess what they all ready knew about the topic.

*Planning for Instruction Think-Aloud, Interview 1*

006 Maggie: We’ll say, “Hey, what is it?” “What does it tell you?” “What do you know?”
She decided that The Guided Discovery Lesson best suit her purposes as she wanted students to listen to each other and begin the process of sense making. As Maggie launched the unit and the first lesson, she probed student’s understanding of the topic and their familiarity with various plots and graphs.

*What Do You Know?* Lesson 1

294 Maggie: What type of plot is this? What information does it tell you?

364 Maggie: What kind of graph is this? Is this good information or not good information? Be critical about it.

Maggie knew that it was important to engage her students in discussions that helped them make connections with what they might already know and foster confidence in their abilities. By accepting all ideas, Maggie provided them a safe environment to conjecture.

*Planning for Instruction Think-Aloud, Interview 1*

008 Maggie: If we’re doing the launch, I expect it to be exploratory, non-judgmental, like accepting of different ideas. That’s hard for them, but that’s what I want to be happening. OK, brainstorm-ish.

During this launch phase, Maggie began foreshadowing two important concepts students would explore over the next few lessons. She showed students examples of bar graphs and histograms and asked them to think about how to tell them apart. She wanted them to notice attributes of each type of graphical representation. One of the graphs Maggie displayed, obtained from *USA TODAY* Snapshots (*USA TODAY*, n.d.), was presented via a *PowerPoint presentation* on the screen. (See Figure 4.)
Most Americans say they need more than seven hours of sleep to be at their best but get an average of six hours, 55 minutes.

Figure 4. Average Hours of Sleep on Workdays/Weeknights.

Maggie accepted all comments during this initial phase and appreciated approximations as much as those who offered the "correct" answer. She received a variety of creative responses to the question, "Do you know what type of graph this is?"

It's a Bubble Graph, Lesson 1

306 Maggie: ... All right. Tyler, do you know what type of graph this is?
307 Tyler: It’s a bar graph.
308 Maggie: It’s a bar graph. I would not disagree because there are bars. It would be hard to disagree with Tyler that it’s a bar graph. But I would say...
309 Evie: It’s a bubble graph.
310 Maggie: Oh. Hold up.
311 Gavin: That’s not a bubble. That’s a pillow.
312 Maggie: Yeah. This is... I will say USA Today has, um, really creative graphics. So it does... it’s made to look like a pillow. It’s very creative. Ava, were you going to agree with ... that it’s a bar graph, or not?
313 Ava: I was thinking it’s a histogram.

Often these Whole-Class Discussions were punctuated with Partner Talk if Maggie felt it would be more meaningful to do so. She often used Partner Talk so that students could collect their thoughts, practice their contributions with another peer, and
express their ideas in a low-risk situation. Since it was not feasible to hear from each student during the Whole-Class Discussion, and to encourage reticent students to share their thoughts, Partner Talk was an ideal format for students to hear their own voice and share their fledgling understandings.

During this phase of the lesson, the talk was more free form where a question was posed to students and then they were asked to talk about it. The following excerpts provide examples of the typical phrases Maggie used to signal these Partner Talk discussions.

*Signaling Partner Talk, Lessons 1-6; 8-11; 14*

364 Maggie: Talk to your neighbor about it. (Lesson 1)
083 Maggie: Talk to the person next to you. (Lesson 2)
179 Maggie: Talk about it. Talk to your partner. (Lesson 3)

As the lesson progressed, Maggie planned to introduce mathematical vocabulary to students to provide them with the language to talk about their new learning.

*Planning for Instruction Think-Aloud, Interview 1*

007 Maggie: As we get a little bit further on, if it’s real specific, I mean that’s where some things with vocabulary interferes there, comes into play.

The next day, Maggie decided it was time her students developed a deeper understanding of histograms and bar graphs as she guided them to compare attributes for each. In order to discuss the attributes with greater precision, she began to introduce new vocabulary.

*Histogram or Bar Graph, Lesson 2*

007 Maggie: Histogram or bar graph?
008 Andrew: Bar graph.
Maggie: Why? Histogram or bar graph and why? Andrew, you said bar graph. Can you give me some reasons why?

Andrew: It has bars.

Maggie: It has bars. Does a histogram have bars?

Andrew: Yes.

Maggie: So if I had a histogram and a bar graph up here (refers to two chart papers on a bulletin board; one is labeled “Histogram” and the other is labeled “Bar Graph”). Would... Bars would be on both of them, correct? This would be on both of them (writes “bars” on both charts). So that wouldn’t... That’s going to help me know what’s common about them, but it wouldn’t necessarily help me know how to distinguish between the two of them.

Connor: Does it have percents?

Maggie: Would you say that again?

Connor: Does it have percents?

Maggie: Does it have percents? Um, it is... no, it does not have percents, but what would you think that would help you with?

Connor: Histograms.

Maggie: Histograms. Histograms could have something that’s called relative frequency. And we haven’t talked a lot about relative frequency (writes “relative frequency” on the Histogram chart paper), but relative frequency does use a percent. Does anybody know... what does frequency mean? Like on the list here (vocabulary word wall) we have frequency. What does that word mean, Tony?

Tony: I wasn’t here yesterday.

Maggie: Yeah, but do you know what frequency means?

Tony: We studied it in science but it was like...

Maggie: Yeah. Frequency has a lot of different meanings, doesn’t it?

Tony: Amount of waves.

Maggie: It was about waves.

Tony: Yeah, waves and uh, like...

Maggie: What did it mean in science about waves?

Tony: Waves. Like certain amount of waves and certain amount of time.

Maggie: And what happened if the frequency was... if it was more frequent, what happened?

Tony: More waves.

Maggie: More waves, right?

Tony: More, yeah.

Maggie: So frequency was about how often something occurred.
Tony: Yeah.

Maggie: And that's the same, actually, that's exactly the same in math.

Maggie continued to help students refine their thinking as she provided examples to deepen their understanding of how “relative frequency” was a different from the term “frequency” that they already knew. In this lesson, she also introduced other mathematical vocabulary, which included “discrete information,” “categorical data,” “range,” “intervals,” and “numerical data” so students were able to discuss the attributes of bar graphs and histograms more accurately.

During the “explore” phase, Maggie often began the lesson with Whole-Class Discussion and then moved to Partner Talk or Small-Group Discussion. Maggie explained this process in the following excerpt from the pre-unit interview.

*Planning for Instruction Think-Aloud, Interview 1*

Maggie: ...it’ll happen [Whole-Class Discussion] at the very beginning of the “explore” and then not at all unless we’re having major issues where I cannot get to everywhere I need to get to because I’d rather mini-teach than group teach. But then if we do have an issue that we need to discuss, then we’ll go back to the whole group. And that’s fine.

Maggie wanted her students to begin asking harder questions of each other and clarify things for each other during this second phase. Maggie explained,

*Planning for Instruction Think-Aloud, Interview 1*

Maggie: During the “explore,” I want them to be asking harder questions of each other, clarifying things for each other. I want them to be asking “what if?” questions.
As the lesson progressed, Maggie guided her students to discover how the choice of interval size affected readers’ perception of data. Students explored a computer program that enabled them to create histograms from interesting prepared data sets, such as gas mileage for the year 2000 cars, ACT scores, or percentage of body fat. Students were able to adjust the interval width and then observe what happened to the size of the bars, the appearance of gaps, and how this made a difference in how the reader interpreted the data. She guided partners’ exploration with prepared questions to discuss. Maggie uploaded the questions to Google Docs so students could record their thinking to a set of class notes that everyone could see, save, and print as class copies.

Creating Histograms, Lesson 2

226 Maggie: So you can choose whatever data you want. So for right now, let’s go with, um, “Gas Mileage for Year 2000 Cars.” All right. And this is a histogram. And I want you to just sort of see what happens here. I want you to play around with it. See what happens and then start to try to answer the questions. All right?

Students were engaged in the activity until they had technical difficulty with too many students accessing the computer program at once. As predicted, Maggie called the group together and redirected them back on task.

Creating Histograms: Technical Difficulty, Lesson 2

246 Maggie: Stop for a second and give me your attention.
247 Connor: Everyone is highlighting and messing with it.
248 Maggie: Is that the problem?
249 Connor: Yes.
250 Maggie: Is it because everybody is doing it at one time?
251 Connor: Yes.
252 Maggie: This is what I would like you to do. Stop for a second. Paper is always a fine backup. OK? So this is the deal, is that you have those questions. I want you to not worry
about Google Docs.... Take out a sheet of paper and I need you to be focusing on the histogram. OK.

During the “summarize” phase, Maggie wanted students to solidify their knowledge through restatements or summaries. Maggie explained that the way she typically enacted this phase in the classroom was to begin with Whole-Class Discussion, move to Partner Talk or Small-Group Discussion as appropriate, and then return to Whole-Class Discussion to close the lesson. Maggie used questioning to deepen the knowledge students had gained. She varied these questions depending on the student and their level of insight.

Planning for Instruction Think-Aloud, Interview 1

007 Maggie: “How about here?” or “Have you seen this here?” or “Have you thought about this?” And depending on the student, where the students are at, this would be different. So I might be talking to Tyler and be talking about, “Hey, have you thought about this?” Or “Hey, why don’t you look here?” but I might be talking to Dan still in the more real specific knowledge type of event.

As the lesson moved to the “summarize” phase during Lesson 3, Maggie helped students recap learning from the Creating Histograms activity, and returned, once again, to Whole-Class Discussion. They processed what they had discovered the day before during their Partner Talk and applied their learning to a new histogram. Maggie plotted the results of a question she had asked them at the beginning of Lesson 1 about how many minutes they played video games on the previous Sunday. In the following excerpt, Maggie questioned students about interval size and how that affected the readability of the graph. Students experimented with “friendly” (i.e., whole) numbers and different sizes of intervals to solidify their understanding that the numerical data
represented by the interval size affected the readability of a histogram. Maggie provided students with the video game data set on their computers so they could manipulate the graphs themselves. Note that inaudible text is recorded for the non-participating student (NPS).

*Choosing the Interval Size, Lesson 3*

063 Maggie: I mean, would you publish a graph like that with... well 192.98?

064 Evie: (shakes head)

065 Tony: No.

066 Maggie: What would you like as a reader of a graph? What would you prefer as a reader of a graph? (Chloe, Tony, and non-participating student [NPS] raise hands) NPS?

067 NPS: (inaudible)

068 Maggie: When you say rounded number, what do you mean? Give me an example.

069 NPS: (inaudible)

070 Maggie: Like 200 instead of 192? Something like that? OK. So we might need to change our... Our interval size is 22.553 or twenty-two and five hundred fifty three thousandths. Who wants to go up by that if were to do that on paper? Do you like marking that off?

071 Ss: (shake heads)

072 Maggie: I do not. All right. So let’s try a different interval size. Talk to the person next to you. What would be the *interval* size you would choose and why? Try it and then we’ll go *from* there. So what would be the interval size and why?

073 Ss: Partner Talk: Total time 20 seconds

074 Maggie: Diana and Ava, what do you think?

075 Ava: 25.

076 Maggie: Diana and Ava said 25. Let’s check what happens.

077 Ss: (chatter)

078 Maggie: OK. Stop for a second. Before I push “update interval,” I want you to think about what’s going to happen? How’s it going to change the graph? Make a prediction. Don’t say it out loud. Predict it in your own head. How’s this going to change the graph? I’m going to push 25 instead of 22.553. What’s one thing that’s going to happen? Evie, what’s one thing that’s going to happen?

079 Evie: Each bar’s going to expand.
Each bar’s going to get wider? Is that what you mean?

Yeah. There’s going to be more ranges of numbers in one bar.

Would you say the last part again?

More, like, there’s a bigger range for each bar instead of...

More... more data could fit inside of each bar?

Because there’s 25 minutes as they go up, right, because they talk about minutes per video game? OK. Anything else going to happen? Will the heights of the bars change at all?

No.

Maybe.

No. Yes. They may. No they won’t.

Yes? No?

We’re not changing that (moves hand up and down).

Oh, what did you say about this (moves hand up and down), Chloe?

We’re not changing the...

Y.

... the y or the (inaudible).

Well let’s see if it automatically changes it or not. But right now, it’s going up by threes right? And it’s up by 12. So just let’s see.

I remember seeing it change. But it didn’t make sense if it did change.

OK, so let’s see. But we actually have control over this. So we’re going to look at it. OK, ready? Quietly look and you’re going to look at the differences.

What if some... what if some bar says twelve?

I know you’re talking about it so that I like that. (refreshed screen, then 5 seconds later) Is that better?

Whoa!

Well, it’s better in a way that it’s easier to read the data, but it’s not better in a way because we lost a bar. Well we didn’t lose a bar. No we didn’t. Never mind. We gained a bar.

So let’s make sure we understand what happened here.

Maggie continued to use teacher talk moves and questioned students in order to help them focus on the important concepts of the lesson and summarize their thinking. She
varied the talk formats between Whole-Class Discussion and Partner Talk. In all, the
"summarize" phase of this lesson went on for another 137 turns concluding with:

240 Maggie: That one (interval size of 8) really shows the differences more. So your choices when you create a histogram today make a big difference in how the reader reads your histogram.

As Lesson 3 continued, Maggie wanted the students to use this information to construct their own histogram for the first time. She decided to accomplish this by shifting the lesson from a Guided Discovery to a Direct Instruction Lesson model for the rest of the lesson. Maggie used the Direct Instruction model whenever she wanted to model or explain a procedure with students. Prepared with Maggie’s oversight, Miss Becker, one of the university undergraduate students, directed students on how to make a histogram using the top 100 most-viewed YouTube videos during Lesson 3. Chloe was confused when Miss Becker constructed her graph and labeled the first interval 100 million on her x-axis, and the rest of the intervals, which were spaced evenly, by 10 million. (See Figure 5.)

Students in Maggie’s classroom were encouraged to critique the reasoning of others and challenge respectfully no matter who was making the claim.

*Question More, Lesson 7*

083 Maggie: You can question each other more.

In order to clarify her thinking, Chloe questioned Miss Becker about the inconsistent scale she used as she labeled the x-axis of her histogram. As a novice teacher, Miss Becker was focused on keeping her lesson moving, and missed the point Chloe tried to make. Maggie interjected in order to clear up any confusion that Chloe or Miss Becker
The following excerpt highlights Chloe’s thinking and Maggie’s support. Miss Becker was not a participant in the study, so her turns were omitted.

*Figure 5. Top 100 Most-Viewed YouTube Videos*

**A Consistent Scale, Lesson 3**

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

**Count**

100 million 110 million 120 million 130 million 140 million 150 million 160 million 170 million 180 million

327 Chloe: Wait.
331 Chloe: Wait. But wait. We can't have a hundred *right there*, *though*. Because it would be the same amount of gap as from zero to a hundred, right?
333 Chloe: Well yeah, but if they're the same size, and you say you're going ten, and not a hundred.
337 Maggie: I... I have a question that might for... *that might*... I think that you might not be... Chloe, I think you were saying that the interval size has to stay consistent.
338 Chloe: Right.
339 Maggie: Is that true? So she was worried that the next one would be 110, right?
As Maggie concluded the lesson for the day, she typically assigned homework and previewed the up-coming lesson. At then end of Lesson 3, Maggie assigned students homework from their text, asking them to analyze different histograms by comparing the data and drawing conclusions from them. The assignment reinforced the day’s learning and supported Maggie’s goals of being a wise consumer and a wise creator of information as they read and interpreted graphs.

*Reinforcing the Learning, Supporting the Goals, Lesson 3*

Page 20, 19-25, and 30-32. This is going to have another opportunity for you to sort of work independently on some of these ideas. I need you to know this. Are you ready? I have to tell you a couple of things about these problems. A lot of these are going to ask you to start doing some comparison. Like *if you’re given this histogram* and *if you’re given this histogram*.* What do you know about the difference between those two... those two items? So a lot of them are going to be about comparison.

When you come tomorrow, you may not come with a blank sheet. You need to have all the questions you knew how to do, done. All the questions you know how to do, done on 19 through 25 and 30 through 32. You may come with the
ones that you're not sure how to do. Push in your chairs. Have a very good day.

Meaningful Distributed Instruction

At the beginning of each of Lessons 6 through 11, Maggie used Meaningful Distributed Instruction (Rathmell, 2010) to review the topic of slope with her students and preview upcoming work. She engaged her students for 54 minutes (8%) of her mathematics instruction in this type of lesson. The lessons ranged in length from over two minutes to over 15 minutes and incorporated both Whole-Class Discussion and Partner Talk.

Meaningful Distributed Instruction (MDI) lessons provide short, consistent, and repeated opportunities for students to make sense of a mathematical idea throughout the school year. The tasks are chosen, with a variety of problem structures and contexts, to enhance the understandings of different representations and reasoning strategies related to the concept. A classroom routine is established where students are expected to create and explain their solutions, listen to others’ explanations, and ask for clarifications when needed. Over time, as students use these representations and reasoning strategies, they become more flexible and fluent in their use (Rathmell, 2010).

Maggie planned to transition from the topic of data and statistics to an algebra unit on linear regression. She wanted students to have a firm understanding of slope prior to the algebra unit so that her students could focus on new concepts and not be sidetracked trying to remember how to calculate it. These lessons were meant to be short, but extended at times when students expressed confusion. The following vignette
provides an example of an MDI slope task. Note that inaudible text is recorded for the
non-participating student (NPS).

*Slope: How Would You Get the Answer? Lesson 9*

015 Maggie: OK. So, Tyler ran at a consistent pace. Here is a snapshot. After 10 minutes, he'd run a mile and a half. After 15 minutes, he'd run two and a quarter miles, or two point two five. Two and twenty-five hundredths. After... after 30 minutes, four and a half miles. After 35 minutes, six miles.

016 NPS: (inaudible)

017 Maggie: OK. So NPS says she knows how to do it. That's what I want you to do. I want you to right now, you don't actually have to get the answer, but I want to know, do you know what you would do to get the answer?

018 Ss: Yes.

019 Maggie: OK. Say, "Yes?"

020 Tony: I'm trying to figure it out.

021 NPS: (inaudible)

022 Maggie: I know. I know.

023 Tony: We're finding out speed? So, oh, yeah! I know how to do that.

024 Maggie: OK. Good. Good. Good. You should see an overlap. She... OK. Give everybody another couple seconds to think about it. How would you find it? It's OK if you don't know the answer.

(7 seconds later) Yeah, Andrew?

025 Andrew: What?

026 Maggie: Were you raising your hand or just stretching?

027 Andrew: I was going to raise my hand.

028 Maggie: You were going to raise your hand.

029 Andrew: Yeah.

030 Maggie: You actually were raising your hand. OK. Go for it.

031 Andrew: Um, would it be OK to say he was running at nine miles an hour?

032 Maggie: Nine miles an hour? How did you get nine miles an hour?

033 Andrew: Uh, well in thirty minutes he ran four point five miles.

034 Maggie: Ah. So thirty minutes is four point five miles (writes "30 min → 4.5 miles" on the whiteboard). So in one hour, I'm getting a little nervous actually, that I made a mistake, but we'll see. Nine miles (writes "1 hr. → 9 miles" on the whiteboard). That's pretty nice. Especially if it was consistent. Very nice. Is that... Anybody else?
The lesson progressed with a discussion of decimal point placement, the use of equivalent fractions to determine a consistent pace, linear function, slope, and the formula $\Delta d/\Delta t$. By the time Maggie came to Lesson 14, her students were ready to plot the arrowhead neck width and width from Lesson 12 on a scatterplot and locate the least squares regression line to analyze the correlation. Because of Maggie’s six-day review of slope using the MDI lesson, the connection to the linear equation was easy to make.

The following excerpt is not part of the MDI sequence of lessons, but rather provides “the rest of the story.” It illustrates how Maggie’s foresight in planning provided her students the opportunity for an “aha” moment when they made a connection between the slope of the least squares regression line, the linear equation that represented...
it, and the correlation of points on the scatterplot. Lesson 14 was the day to pull it all together.

*How Did It Find the Equation? Lesson 14*

125 Maggie: I want you to look at something that it shows you. OK. When you’re doing this least squares, um, line. *In the very bottom, it’s still... it’s still probably a little hard to see. But at the very bottom here, we have the brown and the greenish color. And it tells you two pieces of information. Somebody in the front, Landon. Landon, can you read... can you read this down here?*

126 Landon: *Um, kind of, yeah.*

127 Maggie: Yeah? Do you know what that’s telling us?

128 Landon: *Um, is it telling us like where it is on the graph? I don’t really know.*

129 Maggie: OK. *What’s this? One point... Actually this right here, is the green one.*

130 Landon: It looks like coordinates.

131 Maggie: Would you say that again?

132 Landon: It looks like coordinates. Because there’s like neck width and then there’s regular width.

133 Maggie: Yeah. *If I were to... it actually writes this all out. It says neck width is equal to 0.496 times the width plus one point three.*

134 Tony: How did it figure that out?!

135 Maggie: What the heck does that mean? What does that mean?

136 Tony: How did it find the equation?!

Maggie followed up to Tony’s “aha” question with a brief explanation of how the least squares regression line was calculated. Then she provided students Partner Talk time to decide which number in the equation was slope and asked them to make a prediction about what the equation told them about neck width and width. The Whole-Class Discussion resumed with the following excerpt.

*Because the Equation is mx + b, Lesson 14*

145 Maggie: All right. Which one is the slope? And why? Chloe, which one is the slope?
146 Chloe: The zero point four nine six.
147 Maggie: This is the... (writes “N = 0.496” on the whiteboard). How do you know that’s the slope?
148 Chloe: Because the equation is mx + b, so that m is the slope, which is that.
149 Maggie: Um...hm.
150 Chloe: And then you just add... it’s mx plus b.
151 Maggie: All right. Anybody else? How do you know it’s the slope? Is that... yeah, Landon?
152 Landon: Um, it’s like the neck width, which is what’s going upwards, basically.
153 Maggie: Oh, it’s going...
154 Landon: So, I’d say yeah, basically.
155 Maggie: Is our slope positive? Is our slope positive?
156 Tony: Yes.
157 Maggie: Yes. And why is that nice? Because when we look at our line, how do we know the line is positive?
158 Tony: It’s going up.
159 Maggie: It’s going up and to the right. Can anybody tell me what... what’s the relationship between width and neck width if we use this, you know, the line of prediction a little bit? Can you tell me that at all? If the neck width is bigger, if the neck width happens to be bigger by one... Did I say neck width?
160 Ss: Yes.
161 Maggie: That’s unfortunate. Let me try that again. OK. If the width is bigger by one, what’s going to happen to the neck width?
162 Tony: It’s going to get bigger.

Maggie’s own understanding of the conceptual connections (Ma, 1999) between statistics and algebra became a roadmap that enabled her to prepare her students so that when they arrived at the point of need in the progression of lessons, they were able to make conceptual connections, too. By using MDI lessons on a consistent basis to review the concept of slope, she ensured that her students were fluent in its use prior to having to apply it.
Independent Work

Maggie used Independent Work the least frequently of any type of lesson she incorporated into her instruction. She engaged her students for 45 minutes (6%) of her Whole-Class mathematics instruction in this type of lesson. It was the only time Maggie did not permit students to talk. During Lessons 5, 7, and 11, Maggie assigned students to read handouts or the textbook for short periods of time that ranged between one and nearly four minutes. During Lesson 13, Maggie provided 37 minutes for students to complete an individual assessment. Maggie spoke of her little use of individual work during the pre-unit interview.

Planning for Instruction Think-Aloud, Interview 1

006 Maggie: I’m trying to think... sometimes we do do individual work. That’s probably the thing we don’t do the most, but I do think it’s important.

I make note of Maggie’s use of independent work time in contrast to the use of independent work in the traditional mathematics classroom where independent practice time consumes a large portion of the class period (Huang et al., 2005).

Clearly, Maggie’s beliefs and goals for students and mathematics instruction underscored the way she implemented her lessons. Each lesson was carefully crafted from strategically selected lesson models and talk formats, woven together using the three-phase reform-based lesson, supplemented with Meaningful Distributed Instruction and minimal use of Independent Work. Highlights of lesson vignettes were shared to provide a glimpse into how Maggie orchestrated the complexities of these various aspects in order to provide powerful lessons for her students.
What Maggie Says

We’ve been listening in on Maggie’s teaching throughout the chapter. So far, we have explored how Maggie’s beliefs and goals formed a foundation for her teaching. Then we delved into the lesson models and talk formats that were woven together into the structure of her lesson types. In this next section, we will discuss another decision-making process Maggie employed as we explore the patterns of teacher talk she used to communicate important messages to her students, inviting them to become valued members of the Discourse community and encouraging them to think deeply about mathematics. As Johnston (2004) declared, “... the language that teachers (and their students) use in classroom is a big deal” (p. 10). It determines, to a large extent, what students learn (Fennema & Franke, 1992).

This Is For Everybody

The belief that all students were capable of learning meant that Maggie invited and expected everybody to participate in the learning. Her classroom was inclusive; whatever she said and whatever individual students’ said were important contributions to be valued and respected by the entire learning community. Whole-Class Discussion was just that, a discussion for the whole class. Therefore, Maggie communicated that everybody needed to attend in order to learn and out of respect for others, everybody needed to think about the question on the floor, everybody had something to contribute, so that everybody was able to understand the concept being taught.
Maggie’s dialogic style of interacting with students appeared as though each individual student felt as if he or she were having a personal conversation with Maggie about mathematics.

Conversation is a process of coming to an understanding. Thus it belongs to every true conversation that each person opens himself to the other, truly accepts his point of view as valid and transposes himself into the other to such an extent that he understands not the particular individual but what he says. (Gadamer, 2004, p. 387)

Students rarely raised their hands to be called on during Whole-Class Discussion unless Maggie used Wait Time. Instead they spoke when they had something to say and Maggie stopped to listen to them, no matter where in the room they were sitting or how loudly or softly they called out her name. If a student spoke during Whole-Class Discussion, she expected everyone else to attend, as well.

Maggie had a keen ear tuned to students’ needs during Partner Talk and Small-Group Discussion also. It didn’t matter if Maggie was busy helping a student in the front of the room. If another student was sitting with a small group in the back of the room and asked a question at a regular conversational level, Maggie acknowledged that she heard the student and attended to him or her as soon as she finished with the first. Of the hours of transcripts, I found only one time that Maggie missed hearing a student ask for help. This interactive style of teaching was something that Maggie worked on and enjoyed. The following excerpts illustrate how she fostered a learning community.
Everybody Listens, Lessons 2, 5, 7, and 14

193 Maggie: Hold up. Only some of you are listening to you right now. So if you’re not listening to Dylan right now, if you’re not listening to Dylan right now, I want you to actually listen. (Lesson 2)

195 Maggie: Wait. Wait. You’re going to show him that you respect him by... Yes, you know what to do. (Lesson 2)

263 Maggie: Give your attention and respect to Evie. So if you’re chatting amongst yourselves, you’re going to listen to Evie. Yeah, Evie? (Lesson 5)

279 Maggie: Um, Chloe just asked a great question. I want to make sure everybody heard it. (Lesson 7)

237 Maggie: All right. Sh... (whispers) Now, can you hear me? I think most people can hear me. I need everybody to be able to hear me. (Lesson 14)

Everybody Watches, Lesson 10 and 11

365 Maggie: OK. So this is what I need you to do. Give me your attention up here. Thank you for those people that are watching but I need everybody to be looking. (Lesson 10)

275 Maggie: Look up here or you won’t get the point. If you are just listening, it’s not enough. (Lesson 11)

Everybody Thinks, Lesson 1, 2, 5, 6, 9, and 10

331 Maggie: I’m glad Evie and Chloe are thinking about this. Good. I want everybody to think about how... (Lesson1)

180 Maggie: Does everybody agree that it’s a histogram? (Lesson 2)

163 Maggie: OK. Hold. Just let everybody do this so it’s a... Give everybody a chance to think about it. (Lesson 4)

068 Maggie: I want everybody to focus on this very quick question. (Lesson 5)
079 Maggie: Do not say it out loud. Give everybody a chance to think about it. (Lesson 5)

093 Maggie: ...so give everybody 30 seconds to quietly look at the four different kinds of samplings. (Lesson 6)

024 Maggie: Give everybody another couple seconds to think about it. (Lesson 9)

334 Maggie: I want everybody to think about that question. (Lesson 10)

*Everybody Contributes, Lesson 7, 9, and 12*

083 Maggie: By the way, a lot of you did a fantastic job of contributing and working together to join that in there. So I appreciate that. I think almost everybody contributed something. And there was one person that didn’t but they actually did, it just wasn’t heard, I think. And so, um, and then somebody else picked it up for that person is what we decided. So that was... that was good. (Lesson 7)

192 Maggie: Each partnership needs to contribute... (Lesson 9)

202 Maggie: OK. So right there, stop for a second. Somebody can contribute to this... (Lesson 9)

084 Maggie: Some of you are quiet in this class, but you actually have a lot going on in your head that you could contribute. (Lesson 12)

*Everybody Understands, Lessons 3, 7, 10, and 11*

056 Maggie: Just waiting to make sure everybody’s got it. (Lesson 3)

103 Maggie: So let’s make sure we understand what happened here. (Lesson 3)

136 Maggie: OK. Did everybody get that? Cause I think that we just went through that just quickly. We’re going to pause, because that’s really important. (Lesson 7)

212 Maggie: So the next ten minutes, we need to be intensely focused on making sure we understand the purpose of that and get the lesson. (Lesson 10)
Maggie held high expectations for all her students. They developed an individual ownership for their learning and collective responsibility for the learning of others.

**What Do You Think? Why?**

Maggie was genuinely interested in what her students were thinking. She asked them at least 36 times, “What do you think?” This gave students a way to access what they knew, and provided them an opportunity to articulate it in a way that made sense to others. It helped her students understand that making sense was not a matter of getting the right answer, because they quickly found that their classmates made different, yet similar sense (Johnston, 2004). Having her students organize and consolidate their thinking and communicate that thinking coherently and clearly to peers, to her as their teacher, and to others were key components of the Communication Standard (NCTM, 2000).

Maggie wanted to make sure that students were doing more than mere guessing when they offered an idea. It was important that her students knew how and why they knew what they did. She followed up by asking her students “Why?” at least 75 times. Asking “why” is the essence of inquiry, and is the basis for developing students’ persuasion and argumentation abilities, and logical thinking (Johnston, 2004). Constructing a viable argument is one of the Standards for Mathematical Practice (CCSSI, 2010) and one of Maggie’s core beliefs about mathematics education. Just any
reason, however, was not sufficient for Maggie. She expected students to support their reasoning mathematically.

*Mathematical Reasons*, Lesson 9

158 Maggie: And you need to give me mathematical reasons why. You can't give me anything else. You need to give me *mathematical reasons why*.

168 Maggie: You're going to write out why. But it has to be mathematical reasons why.

How Do You Know? How Could You Check?

When students were asked to verify answers with sources of information or logic, it fostered independence and confidence in their construction of knowledge. They were able to figure out problems on their own, rather than relying on someone to tell them (Johnston, 2004). Maggie fostered knowing in her students by asking them, “How do you know?” and “How can you check?” The following excerpts illustrate her use of these questions.

*How Do You Know?* Lessons 1, 4, 5, and 14

314 Maggie: How do you know... can you articulate why it's a histogram and not a bar graph? (Lesson 1)

175 Maggie: Tyler, how do you know the answer is going to be less than 79.5? (Lesson 4)

088 Maggie: How do you know it's less? (9 seconds later) He's right. How do we know that he's right? (Lesson 5)

147 Maggie: How do you know that it's the slope? (Lesson 14)

When Maggie asked the question, “How do you know?” she assumed her students would provide an intelligent attempt or comment, even if they weren’t quite sure this time. By
shifting the pronoun from “you” to “we” in line 088 of Lesson 5, she moved the burden of justification to the group rather than the individual. As the group collectively thought about and shared reasoning for the idea, then Connor, the student who first offered the answer had the opportunity to acquire the thinking process of the group. As Johnston (2004) noted, “It’s this assumption of a knowledgeable and agentive person that is the important message” (p. 58). Taking seriously how students know what they claim to know is an important aspect of critical mathematical literacy.

“How could you check?” placed Maggie’s students in the position of knowledge production, with all the responsibilities that go with it. This required that Maggie’s students had to cross-check their claims against sources to determine the credibility of their claims (Johnston, 2004).

_How Could You Check, Lessons 3 and 9_

<table>
<thead>
<tr>
<th>Time</th>
<th>Maggie:</th>
<th>Dialogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>076</td>
<td>Maggie:</td>
<td>Diana and Ava said 25. Let’s check what happens.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Lesson 3)</td>
</tr>
<tr>
<td>068</td>
<td>Maggie:</td>
<td>Now. How could you check to see if he was running a consistent pace?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>What would have to be true?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Lesson 9)</td>
</tr>
<tr>
<td>084</td>
<td>Maggie:</td>
<td>That was the easier one to just see. And you’d have to check that all the</td>
</tr>
<tr>
<td></td>
<td></td>
<td>way through. Right?</td>
</tr>
</tbody>
</table>

At times, Maggie modeled self-checking so that her students knew proficient mathematicians used this strategy in order to be confident of their solution.

_I’m Going To Check This, Lessons 4 and 11_

<table>
<thead>
<tr>
<th>Time</th>
<th>Maggie:</th>
<th>Dialogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>157</td>
<td>Maggie:</td>
<td>Did I do that right? OK. Eighty-three and 91. 174. I’m pretty sure about</td>
</tr>
<tr>
<td></td>
<td></td>
<td>that guy. (Lesson 4)</td>
</tr>
<tr>
<td>117</td>
<td>Maggie:</td>
<td>When x is seven (writes x = 7 on the whiteboard)... I’m going to check</td>
</tr>
<tr>
<td></td>
<td></td>
<td>this. Y = seven times seven... (writes y = 7)</td>
</tr>
</tbody>
</table>
• 7 + 2 on the whiteboard). Yeah, 51 (writes y = 51 on the whiteboard). Good. Yes! Beautiful!! (Lesson 11)

By using “How could you check?” repeatedly, Maggie expected students to begin questioning themselves. Maggie noted during the post-unit interview that students often self-corrected as they processed their thinking aloud.

_They’re going to Self-Correct Themselves, Interview 3_

003 Maggie: You’re hoping that some student is going to come back to it. You know, they’re going to self-correct themselves as they go through their process of talking. Just like when we go through our process that we’re here and we come back to it and we refine it.... So, usually I’ll wait to see... No, not usually. Sometimes I just correct them on the spot, sometimes I ask a question, and sometimes I think this will this will work itself out. And that will be more powerful than have me work it out.

Throughout the process, students gained confidence in their abilities. Maggie wanted to make sure her students noticed the change in their behavior, as well.

**How Confident Are You?**

Maggie frequently took a pulse of students’ confidence levels in order to determine how solid their understanding of a concept was and as a way that students could begin self-evaluating their own degree of certainty. Maggie shared the following reflection regarding her use of the confidence question.

_It’s A Form of Student Self-Evaluation, Personal Communication, July 19, 2011_

Maggie: I want my students to self-evaluate their learning and the degree to which they truly understand. I think confidence can be one indicator of how solid a person's understanding of a concept is. So often I might hear a student say, "I think I get it" and then move on without giving "it" any more thought. My hope is students begin to realize that they need to move on from that stage to the point where
they are confident. It is mostly, though, one more form of student self-evaluation.

The following excerpts highlight ways Maggie assessed her students' confidence, as well as opportunities she provided her students to consider how sure they were of their response. She also prompted students with ways they could increase their level of confidence.

*Im Completely Confident*, Lessons 7, 8, 10, and 14

160 Maggie: Now I want you to give me a confidence... How confident are you on your answer? (thumbs up) Total confidence. (thumbs sideways) I think I got it. (thumbs down) I'm not so confident. (Lesson 7)

120 Maggie: Are you confident in your response? (Lesson 8)

310 Maggie: OK. So if you wanted to feel confident about it, you might need to take it [sample] more than once. (Lesson 10)

067 Maggie: I want to know which would one you'd be more confident making a prediction about? Would you be more confident if you knew the width, you could predict the neck width? Or if you knew the width, could you predict the length? Which one would you be more confident? (Lesson 14)

Other times, students were not sure of their response at all. Being able to distinguish between what is sure and what is puzzling helped students to focus their problem solving on the unsolved part, making the problem more focused and easier to deal with (Johnston, 2004). The following excerpts illustrate examples of Maggie's accepting attitude.

*It's OK Not To Be Sure*, Lessons 2, 6, and 11

067 Maggie: You're not sure. That's OK to not be sure. It's completely OK to not be sure. Anybody want to help? Yeah. (Lesson 2)
Maggie: Tyler is going to tell us whether he thinks it’s a bar graph or a histogram and if he’s not sure, it’s OK. (Lesson 2)

Maggie: And you’re not the only one that is not quite sure... I’m going to say that I get a sense that a lot of people... So that’s why we’re trying to build a common knowledge of the difference. Go ahead. (Lesson 2)

Maggie: You’re not sure? Can you read it again? Read Plan 2 for me. And think about it. (Lesson 6)

Maggie: Not sure? (Lesson 11)

The notion of confidence is important to students’ success in school and in life. When students have a strong belief in themselves, they work harder, focus their attention better, are more interested in school, and are less likely to give up when they encounter challenge. Students who doubt their competence set low goals, choose easy tasks, and plan poorly. When faced with difficulty, they become confused and lose concentration. Over time, these students disengage, decrease effort, generate fewer ideas, and become passive and discouraged. The behaviors of both types of students become cyclical in nature. The students who feel competent plan well, choose more challenging tasks, and set higher goals. Those who don’t are limited in both their personal experience and potential learning path. Performance differences for students with and without confidence continually diverge, particularly from fifth grade on (Johnston, 2004).

Teacher Talk Moves

Once Maggie’s students presented their ideas, then others could build on that thinking. She engaged her students by using five teacher talk moves (Chapin et al., 2009) that supported mathematical thinking and learning. Maggie encouraged her students to
express multiple perspectives and flexibly apply their new learning. These talk moves were used in nearly all of her lessons during Whole-Class Discussion.

**Revoicing.** Maggie used revoicing for multiple purposes in every lesson except Lesson 12, when students were engaged in the Small-Group Discussion Archeology Mystery PBIT, and Lesson 13, which was an assessment day. One reason Maggie used the revoicing talk move was to add clarity to students’ fledging ideas in order to determine if she were understanding the student correctly and so that others could understand what was being offered for consideration. This talk move provided an opportunity to open the possibility to reflect on, modify, or challenge what has been said (Johnston, 2004). Another reason was to provide more “thinking space” to help all students track what was going on mathematically (Chapin et al., 2009). Finally, Maggie used revoicing as a way of translating what the students said into mathematical language in order to provide them with the vocabulary they needed to discuss new ideas and concepts more articulately. The following excerpts provide examples of these various uses. Note that inaudible text is recorded for the non-participating student (NPS).

*Adding Clarity, Lesson 1*

<table>
<thead>
<tr>
<th>Time</th>
<th>Maggie:</th>
<th>Diana:</th>
</tr>
</thead>
<tbody>
<tr>
<td>063</td>
<td>OK. Diana, give me one. What kind [of plot] do you already know about?</td>
<td></td>
</tr>
<tr>
<td>064</td>
<td>Like plotting on a graph.</td>
<td></td>
</tr>
<tr>
<td>065</td>
<td>Plotting on a graph. OK. So... so that’s one. Most of the time it’s... <em>I think you’re thinking about a scatterplot. Is that like where you have an x, y pair? And you say, 3, 4 and you go over 3 and up 4?</em> (Revoicing)</td>
<td></td>
</tr>
<tr>
<td>066</td>
<td>Yeah.</td>
<td></td>
</tr>
</tbody>
</table>
Thinking Space, Lesson 3

078 Maggie: I'm going to push 25 instead of 22.553. What's one thing that's going to happen? Evie, what's one thing that's going to happen?

079 Evie: Each bar's going to expand.

080 Maggie: Each bar's going to get wider? Is that what you mean? (Revoicing)

081 Evie: Yeah. There's going to be more ranges of numbers in one bar.

082 Maggie: Would you say the last part again?

083 Evie: More, like, there's a bigger range for each bar instead of...

084 Maggie: More... more data could fit inside of each bar? (Revoicing)

085 Evie: (nods)

Translating, Lesson 1

346 Evie: Well, like when we were discussing like what job they had and stuff, I think it’s important that we know that because, um, if they’re like a construction worker, they’re going to be working really hard all day in different weather and stuff. They’re going to be really tired and maybe sleep more. Or if they have a time they need to wake up, it’s gonna change, but if they don’t have a time they need to wake up they might sleep more.

347 Maggie: And what you’re talking about, to me, like if I were to translate that in my little statistical mind, 'cause you guys know how I walk around this Earth. All right. I'd be like, "Oh! Evie thinks it's really important that we take an appropriate sample out of a population of people."

(Revoicing)

Mathematical Reasons, Lesson 9

207 Gavin: Wait. Can we do a reason not to...

208 Maggie: Yeah, go ahead.

209 Gavin: ...choose somebody? OK. A reason not to choose, uh, A is because he's like all over the place. He does have a higher maximum, or whatever you want to say, but he's also got a minimum of two.

210 Maggie: OK. All right. So he’s all over the place. And what I like about your response, Gavin, is it included the... his high and his low. What was... His high to his low is his range,
right? His min to his max? Range? All right. What’s his range? What is Player A’s range? Huh?

(Inaudible)

Twenty-six. Player B’s range? Ten. That’s is a ... that’s... So the difference between Gavin, you saying, “He’s all over the place” and me saying, “I’m wanting you to give some mathematical reasons” is you’d say, “He’s all over the place because he has a range of 26 while the other one has a range of 10.” Wait. So do you guys see the difference? What I want you to understand about describing something mathematically. Did you see the difference of how we might say it in normal words versus mathematical explanation? Are you hearing...? OK.

(Revoicing)

Repeating: Asking students to restate someone else’s reasoning. Maggie used the Repeating talk move only once throughout the data and statistics unit. While she frequently asked students to repeat what they said, the only time she asked students to repeat someone else’s reasoning is highlighted in the following excerpt from Lesson 2.

Who Can Say It In a Different Way? Lesson 2

What did it mean by “interval size?” What did that word mean? “Interval size?” Yeah.

Kind of like the two, like the two numbers that the data was in between or like the interval of the data.

OK. Who can say it in a different way? I think you’ve sort of got it. Chloe? (Repeating)

The distance in between the two numbers.

Repeating was a beneficial talk move as it gave the class another version of Evie’s contribution and added to the likelihood that they followed the conversation and understood the point. It also provided evidence to Evie that her input was taken seriously.

Reasoning: Asking students to apply their own reasoning to someone else’s reasoning. Maggie used the reasoning talk move for several purposes in every lesson.
except Lesson 12, when students were engaged in the Small-Group Discussion Archeology Mystery PBIT, and Lesson 13, which was an assessment day. When Maggie asked students to apply their own reasoning to someone else’s reasoning she was inviting them to refine their logic and to apply Standard 3 of the Standards for Mathematical Practice (CCSSI, 2010), in which they constructed viable arguments and critiqued the reasoning of others. This talk move helped students understand that it is acceptable to disagree and that people have legitimate opinions that differ. Cognitive change takes place when learners confront and manage conflicting viewpoints. Disagreement, more than agreement, moves students’ thinking forward. When students experience their own conceptual growth, they start to learn that this process is beneficial to them personally; especially when teachers help them notice it is happening (Johnston, 2004).

Maggie used the reasoning talk move when she wanted students to think critically about a mathematical choice of representation or the reasonableness of their solution. She also used the it when she wanted students to think critically about claims she suggested and to decide whether these could be generalized to multiple problem situations. The following excerpts highlight the various ways Maggie used the reasoning talk move with her students.

*Do You Agree or Disagree? Lesson 6*

<table>
<thead>
<tr>
<th>281</th>
<th>Chloe:</th>
<th>OK. Who has an advantage or disadvantages for Plan 4? Evie?</th>
</tr>
</thead>
<tbody>
<tr>
<td>282</td>
<td>Evie:</td>
<td>OK. It is random and everyone has a fair chance. Only problem...</td>
</tr>
<tr>
<td>283</td>
<td>Chloe:</td>
<td>(types “It is random and everyone has a fair chance” on computer which is projected onto screen)</td>
</tr>
</tbody>
</table>
| 284 | Maggie: | Wait. Hold up. Hold up. What does the word “fair” mean when she just used it? She said, “Everybody has a fair
Is This a Good Choice Or Not? Lesson 1

276 Maggie: ...in this case, there's really not much differentiation. I mean this is not a lot of interesting information so you'd say, well, is this really the best way to present it? Is a bar graph a good choice or not? Because we're going to talk about choices. All right. Hm... OK. Something else you'd want if you're being really critical? You guys are doing a good job of being critical. (Reasoning)

What Would Be Reasonable? Lesson 4

135 Maggie: OK. If I were adding, I could just drop it [decimal point] down. So you do have one of those things locked in. But I'm not adding. But in this case, it does work, almost. It looks like you could just drop it down. But what would be reasonable? Would 4,770 be a reasonable answer? (Reasoning)

Is That a Rule? Lesson 4

179 Maggie: [understanding mean] Let's say I have a different set. OK. It happens that three of the numbers are over 79 and a half. So I totally agree with that that gives you a good indication. But is that a rule that like three out of the five numbers would have to be over it? Or is there something... (Reasoning)

Adding on: Prompting students for further participation. The adding on talk move was implemented in every lesson except Lesson 12, when students were engaged in the Small-Group Discussion Archeology Mystery PBIT, and Lesson 13, which was an assessment day. When Maggie encouraged students to offer multiple perspectives, the alternative perspectives often helped students arrive at a better, more nuanced understanding of the concept or solution to the problem. When faced with different
perspectives, students were required to articulate their own position more fully (Johnston, 2004).

Maggie used adding on to elicit multiple perspectives, support a student who wasn’t sure, get additional ideas in the open for discussion, and ensure that everybody had a chance to contribute who wanted to. The following excerpts highlight the uses of the adding on talk move.

*The Difference Between a Bar Graph and a Histogram, Lesson 1*

198 Andrew: So the minutes would be on the class list and you’d have each bar going up like how many minutes each person played and each bar is representative of each person.

199 Maggie: So each bar is going to be a different person and then you have the how long. That’s good. *Could you do it a different way? What do you think, Ava?* (Adding On)

200 Ava: You’d have like how many minutes on the bottom and how many people going up. You could have how many people did each that amount of minutes.

*Chloe Wants to Help You, Lesson 9*

051 Tony: Is it point one five?
052 Maggie: Yeah. How do you say that number?
053 Tony: Um, fifteen hundredths?
054 Maggie: Yeah. Fifteen hundredths (writes $r = 0.15$ on the whiteboard). *How did you know that, Tony?*

055 Tony: Because, I know that with hundredths, it’s not going to be like hundredths with ten. Wait. No, I’m not going to say that. With like... I don’t know.

056 Maggie: You want... *Chloe wants to help you out. Is that OK with you?* (Adding On)

057 Tony: Yeah, go ahead, Chloe.
058 Maggie: OK.
059 Chloe: So when you divide by ten you can just move the decimal point over.
Are There Any Other Populations? Lesson 5

246 Maggie: What population would care about if we changed the school colors or not? Matt? Wait. Let’s make sure we’re listening.

247 Matt: Probably students and teachers.

248 Maggie: Students. Teachers. Are there any other populations that would actually care about the changes in student colors? (Adding On)


263 Maggie: Yeah, parents. Parents? Are there any other populations that care? (Adding On)

Anything To Add From This Group? Lesson 10

300 Maggie: All right. Um, anything to add from this group? (Adding On)

344 Maggie: What else? (Adding On)

362 Maggie: OK. All right. Anything else from this group that we haven’t already talked about? (Adding On)

Waiting: Using wait time. Maggie used the waiting talk move for several purposes in every lesson except Lesson 12, when students were engaged in the Small-Group Discussion Archeology Mystery PBIT, and Lesson 13, which was an assessment day. When Maggie incorporated waiting, she caused the discussion to slow down so students had time to think and reflect. Wait time, also known as “thinking time” is positively related to more student talk, more sustained talk, and more higher order thinking. It also communicates the message that the student is expected to figure out something or self-correct and opens the possibility of changing the conventional
initiation-response-evaluation (IRE) teacher-student discourse pattern to that of initiation-
response-response (IRR; Johnston, 2004).

Maggie used various prompts to signal waiting. These included modeling thinking, prompting think, or merely putting the discussion on hold for a number of seconds. Maggie used this last form of waiting 26 times with the pause ranging from two to 39 seconds with a mean of eight-and-a-half seconds. The following excerpts highlight these situations.

**Modeling Thinking, Lesson 2**

237 Maggie: *Wait. I'm thinking still. I remembered. I got it.* (Waiting)

**Prompting Thinking, Lesson 4**

163 Maggie: *OK. Hold. Just let everybody do this so it's a... give everybody a chance to think about it.* (Waiting)

**Pausing the Discussion, Lesson 11**

034 Maggie: Seven. Do you all agree with seven? What's the "two" in this equation? (Chloe, Tony, Ava, NPS raise hands) (4 seconds later; Waiting)

Anybody? Anybody? (10 seconds later; Waiting)

What is the "two" in this equation?

The following excerpt provides an example that illustrates how multiple student responses followed a Maggie-initiated question when waiting was used. Even when Maggie finally shared the correct answer, students continued to share their thinking.

Note that inaudible text is recorded for the non-participating student (NPS).

**IRIRRIRIRRIRRIRR**, Lesson 11

304 Maggie: Skewed to the left or skewed to the right. Really hard for me to remember. OK. Because... don't say it out loud.
Everybody’s not saying it out loud, right? I just want you to say, do you think this one would be skewed to the left or skewed to the right?

(9 seconds later; Waiting)
(whispers) Just say it quietly. Skewed to the...

305 Landon: (whispers) Left.
306 Maggie: What do you guys think?
307 Landon: Left.
308 Ava: Left.
309 Connor: Left.
310 Maggie: Everybody thinks left?
311 Dylan: I think right.
312 Maggie: I was going to say I’m the only one that’s weird. Like I really want this to be skewed to the right.
313 Evie: No, because if you turn away...
314 Connor: But it’s like smaller...
315 NPS (inaudible)
316 Maggie: But it’s not. It’s skewed to the left.
317 Evie: It’s because you’re facing it. If you turn away from it, it is left. So, just saying.
318 Maggie: Is that what helps you?
319 Evie: Yeah.
320 Chloe: Yeah, that’s what helps me.

Maggie’s use of the five teacher talk moves effectively increased student participation in a singular, focused Whole-Class Discussion. Maggie also posed questions to engage students in Partner Talk and Small-Group Discussion or provided students with tasks that prompted higher-order thinking.

Higher Order Thinking

In order to engage students in productive conversations, Maggie prompted them with a question prior to their engagement in Partner Talk or used this talk format to engage them in a mathematical activity. Partner Talk was incorporated into every lesson except Lessons 7 and 12, when students were engaged in the Small-Group Discussion Archeology Mystery PBIT, and Lesson 13, which was an assessment day. Maggie also
engaged students in *doing* mathematics by providing them with complex PBITs during Small-Group Discussion and processed these tasks using questions that prompted Small-Group Discussions. Small-Group Discussion was incorporated into Lessons 7, 8, 10, and 12. These questions and tasks spanned Bloom’s Taxonomy (L. W. Anderson & Krathwohl, 2001) of cognitive domain. Table 8 describes the cognitive complexity of the questions Maggie posed to prompt student talk and the cognitive complexity of the mathematical tasks in which she engaged her students during Partner Talk and Small-Group Discussion.

Table 8.

<table>
<thead>
<tr>
<th>Cognitive Complexity</th>
<th>Quantity</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1: Remembering</td>
<td>PT Lessons 1 &amp; 11</td>
<td>3</td>
</tr>
<tr>
<td>Level 2: Understanding</td>
<td>PT Lessons 4, 5, 10, 14, S-G Lessons 8 &amp; 10</td>
<td>8</td>
</tr>
<tr>
<td>Level 3: Applying</td>
<td>PT Lessons 5, 8, 14, S-G Lesson 7</td>
<td>6</td>
</tr>
<tr>
<td>Level 4: Analyzing</td>
<td>PT Lessons 1, S-G Lessons 10 &amp; 14</td>
<td>8</td>
</tr>
<tr>
<td>Level 5: Evaluating</td>
<td>PT Lessons 1-3, 6, 9, 11, 14, S-G Lessons 10 &amp; 14</td>
<td>19</td>
</tr>
<tr>
<td>Level 6: Creating</td>
<td>PT Lesson 1, 5</td>
<td>4</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>48</td>
</tr>
</tbody>
</table>

Note: PT = Partner Talk; S-G = Small-Group Discussion
Forty percent of the questions and tasks that Maggie posed required her students to evaluate with less than a quarter of the questions and tasks focused at the lowest two levels of thinking. It should also be noted that Maggie did not sequence her questions and tasks in any type of hierarchy. Students responded to higher-level questions beginning in Lesson 1. The following excerpts illustrate how she varied her questions and provided students opportunities to engage in productive talk that caused them to think.

**Remembering, Lesson 11**

020 Maggie: What is the slope? Write it down. *And give me any two points you want out of the infinite number of points that are described by that equation* \( y = 7x + 2 \).

**Understanding, Lesson 5**

209 Maggie: What are some differences between the amount of sleep that you got on Sunday evening *and the amount you... of sleep* you got on Saturday night? OK. Talk with each other really quickly. What are some differences and similarities you could say?

**Applying, Lesson 8**

222 Maggie: [referring to Android Survey Results on website] Talk to the person next to you and decide. Who's their population and how did they sample?

**Analyzing, Lesson 2**

83 Maggie: Talk to the person next to you.... Bar graph or histogram and what features, *what characteristics* make it one or the other?

**Evaluating, Lesson 9**

158 Maggie: These are basketball players. Player A and Player B. OK? And, um, you're going to get to choose just one of them....
[to be on your basketball team] Now, if you want, I want you to decide with the person next to you. Who are you going to choose between Player A and Player B? And you need to give the mathematical reasons why.

Creating, Lesson 5

394 Maggie: How would you go about trying to figure out today. 2011. March 2011. How many hours, on average, do teens listen to rock music? How could you figure that out? Talk to the person next to you first. How could you figure that out?

Talking Mathy

Maggie provided a language-rich environment where students were exposed to a plethora of mathematical vocabulary. Maggie believed that communication in the classroom was important and provided many opportunities for students to be able to talk about mathematics during Whole-Class Discussion, Partner Talk, and Small-Group Discussion. She stressed the importance of using mathematical language. She expected her students to use mathematical vocabulary and support their claims with mathematical reasoning.

Talking Mathematically, Lessons 9 and 11

168 Maggie: You’re going to write out why. But it has to be mathematical reasons why. (Lesson 9)

212 Maggie: What I want you to understand about describing something mathematically. Did you see the difference of how we might say it in normal words versus mathematical explanation? (Lesson 9)

180 Maggie: That is what we are doing with this but it’s with math words. OK? (Lesson 11)

Maggie supported language learning in the classroom with wall charts that were developed during class. One chart listed the attributes of bar graphs and histograms so
students could learn to differentiate between the two. Another chart listed new mathematical vocabulary as it was introduced. New vocabulary was presented daily, except for Lesson 13, when students were assessed. Table 9 lists the new terminology that was presented in the data and statistics unit according to the lesson in which Maggie first introduced it. In all, students were introduced to 86 new mathematical terms during the 14-lesson unit.

Maggie modeled the use of mathematical language and frequently translated students’ regular talk into “mathy” language. The follow excerpt provides an example of this translation.

Translating, Lesson 11

<table>
<thead>
<tr>
<th>Time</th>
<th>Maggie:</th>
<th>Dylan:</th>
<th>Maggie:</th>
</tr>
</thead>
<tbody>
<tr>
<td>216</td>
<td>And the mode. A lot of you did this. You’d say, “A lot of people watched three movies.” That’s true. But if you were to like tell me that all mathy, you would have said, “The mode is three.” OK.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>217</td>
<td>The mode is three?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>218</td>
<td>Yeah. I can translate it like, “A lot of people watched three movies. The mode is three.” Or whatever it is.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It was important to Maggie that students conceptually understood what words meant. It wasn’t enough for them to simply provide an answer or a definition. She actually preferred when they didn’t. She wanted them to explain the concept using their own connections in their own words.

That Helps Me Know They Understand It, Interview 1

<table>
<thead>
<tr>
<th>Time</th>
<th>Maggie:</th>
</tr>
</thead>
</table>
| 010  | If they can just repeat back to me what the book, or what the phrase, or what I said, then that’s something but that’s usually not “it.” So it has to be usually internalized in some manner and said back in some of their... with them involved... with their personality.... That helps me know that they’ve actually “got it.” ....Yeah, isn’t it sad when it’s...
### Table 9.

**Eighth Grade Mathematics Instruction: Mathematical Vocabulary First Introduced by Teacher During Data and Statistics Unit**

<table>
<thead>
<tr>
<th>Lesson in which word was first introduced</th>
<th>Vocabulary</th>
</tr>
</thead>
<tbody>
<tr>
<td>data</td>
<td>statistics</td>
</tr>
<tr>
<td>plot (noun)</td>
<td>graph</td>
</tr>
<tr>
<td>box-and-whiskers plot</td>
<td>pie chart</td>
</tr>
<tr>
<td>histogram</td>
<td>fine plot</td>
</tr>
<tr>
<td>bar graph</td>
<td>dot plot</td>
</tr>
<tr>
<td>circle graph</td>
<td>relative frequency plot</td>
</tr>
<tr>
<td>&quot;read between the data&quot;</td>
<td>&quot;read the data&quot;</td>
</tr>
<tr>
<td>interval</td>
<td>&quot;read beyond the data&quot;</td>
</tr>
<tr>
<td>discrete data</td>
<td>range</td>
</tr>
<tr>
<td>numerical data</td>
<td>relative frequency</td>
</tr>
<tr>
<td>categorical data</td>
<td>frequency</td>
</tr>
<tr>
<td>interval size</td>
<td>median</td>
</tr>
<tr>
<td>gap</td>
<td>mode</td>
</tr>
<tr>
<td>average</td>
<td>scale</td>
</tr>
<tr>
<td>mean</td>
<td>third quartile</td>
</tr>
<tr>
<td>five-number summary</td>
<td>maximum (max)</td>
</tr>
<tr>
<td>first quartile</td>
<td>&quot;box&quot; in box-and-whiskers plot</td>
</tr>
<tr>
<td>minimum (min)</td>
<td>variability</td>
</tr>
<tr>
<td>outlier</td>
<td>sample</td>
</tr>
<tr>
<td>percentile</td>
<td>y-intercept</td>
</tr>
<tr>
<td>interquartile range (IQR)</td>
<td>sampling methods</td>
</tr>
<tr>
<td>population</td>
<td>systematic sampling</td>
</tr>
<tr>
<td>linear equation</td>
<td>random sampling</td>
</tr>
<tr>
<td>slope</td>
<td>probability</td>
</tr>
<tr>
<td>convenience sampling</td>
<td>rise over run</td>
</tr>
<tr>
<td>voluntary response sampling</td>
<td>Δy/Δx (delta)</td>
</tr>
<tr>
<td>fair</td>
<td>random number generator</td>
</tr>
<tr>
<td>rate of change</td>
<td>integer</td>
</tr>
<tr>
<td>y = mx + b</td>
<td>peak</td>
</tr>
<tr>
<td>constant rate</td>
<td>meaning of equation:</td>
</tr>
<tr>
<td>chance</td>
<td>y = mx + b as a representation of all points on a line</td>
</tr>
<tr>
<td>distribution</td>
<td>equivalent</td>
</tr>
<tr>
<td>ways to write multiplication: 2x, 2*x, 2(2), 2(x)</td>
<td>measure of center</td>
</tr>
<tr>
<td>r = d/t</td>
<td>direct variation</td>
</tr>
<tr>
<td>linear function</td>
<td>census</td>
</tr>
<tr>
<td>sample size</td>
<td>symmetrical</td>
</tr>
<tr>
<td>points</td>
<td></td>
</tr>
<tr>
<td>cluster</td>
<td></td>
</tr>
<tr>
<td>skew</td>
<td></td>
</tr>
</tbody>
</table>

*(table continues)*
not that they can say exactly what the definition is, but when they don’t say exactly what the definition is then that’s better! I mean it’s good to have the first step. ...I think when they’ve connected to something else is important... when they’ve explained it in a different way. That... that will help me understand.

In the following excerpt, Maggie delved into the meaning of the word “mean.” Note that inaudible text is recorded for the non-participating student (NPS).

What Does the Mean Mean? Lesson 4

045 Maggie: What’s the mean again?
046 Dylan: Seventy-nine and a half.
047 Maggie: No. Correct. The value of the mean, but what does mean mean? What does the mean mean?
048 NPS: (inaudible)
049 Maggie: It does mean average. Right. Hey, what are the other things that could give us average besides mean?
050 Dylan: Don’t you add them all up and divide by the number of numbers there is?
051 Maggie: That is how you get the mean.
055 Maggie: But there are two other ways to find an average.
056 Chloe: Really?
057 Maggie: Yeah. OK, so the word “average” has three different... there are three different things that actually could be sort of the average of anything else. Mean... what’s another one?
Maggie also expected students to become more specific and precise in the words they spoke and in the descriptions they wrote.

_Be More Specific, Lessons 5, 7, and 13_

204 Maggie: Be more specific. (Lesson 5)

340 Maggie: There are other words, but I want to make sure you’re clear about your descriptions. I just don’t want you to say, “It’s spread out.” OK. That’s not a good description. We’re probably going to include other words but I wanted you to think about that. OK. And then that’s important. This is an important learning point. (Lesson 7)

082 Maggie: I want to make sure when you answer, “Explain how you will make sure your sample... that your samples are chosen randomly” that you are really specific. Don’t give me a vague calculator answer. I want you to be really specific. Tell me exactly what you did. (Lesson 13)

Maggie used the inductive method when introducing new vocabulary. The following excerpt provides an example as she developed the meaning of “variability” with Ava, Diana, and Evie during their Small-Group Discussion.

_What Do You Think “Variability” Means? Lesson 8, Small-Group Discussion_

271 as Maggie: You can start describing the variability of the numbers.
271 at Ava: What is that variability mean?
271 au Maggie: Oh, yeah. What do you think variability means? Let’s go...
271 av Ava: Like the... I don’t understand. But they’re pretty much the same.
271 aw Maggie: Diana, do you have an idea? What do you think variability means?
271 ax Diana: Um, how it is or something?
271 ay Maggie: Yeah. When you say variability... that’s actually, that’s the first thing that would probably come out of my mouth. How it varies. So what does that... what if something looked like if it didn’t vary a lot? If people’s movies.
271 az Ava: It’d be the same.
It would be like really random and there’d be no order, whereas these ones are like... a lot of them are really close to being the same.

So, does it have any variance? Does it vary at all?

Yeah, because like she had less numbers than me or had numbers where I didn’t.

How about if you just see the whole thing at once. Let’s just focus on one of these, OK, for a second. Does that have variability?

Yeah.

How so? Tell me about it.

Like it doesn’t... it doesn’t keep like a constant. Like it jumps up and down.

Um... hm. So can you tell me some things you could describe? Could you tell me what the range is?

If something had no variance at all, what would the range be?

There wouldn’t be one.

Or zero, right? If it had no variance at all, the range would be zero.

So the range would be like zero to 17?

Um... hm. Um... hm. So I want you to describe the best you possibly can, of the spread of the data. OK? The variability.

Maggie also asked students to self-evaluate their use of mathematical talk. Her goal was to raise their awareness of its use.

Focused On Mathematical Talk, Lesson 10

Hold for a second. That you did not talk about other things. You were focused on mathematical talk for that time. That would be a five. A one would be I basically... I didn’t do anything I was supposed to do.

Maggie knew that being able to use the vocabulary appropriately was a step along her students’ journey to becoming full participants in the mathematical Discourse. This ability didn’t necessarily mean they had developed a deep conceptual understanding of what they were saying. Understanding the nuances, subtleties, and characteristics of a
word's role in the language can only be understood through repeated exposures to the word in a variety of contexts over time (Beck et al., 2002). In the following excerpt, Maggie shared her thoughts about the role language played in reflecting her students' understanding of mathematics.

Does the Use of Mathematical Language Reflect Understanding? Interview 3

007 Maggie: Yeah, that’s a sometimes answer, which is not usually helpful. But, uh, sometimes, if the kid is using really precise mathematical language, or has acquired that, then it’s a sure... it’s a good sign they’ve really started to internalize it, really started conceptual understanding....

And I see that sometimes with my students. Like they can give you the answers... They can do everything, but they really can’t quite put the language to it. And so I do believe that there is a certain level of conceptual understanding, even when students can’t have that appropriate lexicon. But I’m sure there’s a difference to the depth of which their understanding is if they can... if they can write it.

There was some work done on like mean when somebody would ask students about what is the mean of something. That the stuff that was actually in their head was very different from what they had learned to write or say about what the mean is. And so students would still default on something that was more procedural even what’s ... if people could really get inside there, take time to really talk to them and probe that, they weren’t really using that at all. But they were using some other concept of whether it would be averaging or something else. Even though the language didn’t match up with what they were doing. So, I think sometimes that when we are pushing language, which I think is good, but sometimes that then students will default to saying, “OK, this is what you want me to say.” But it is really... But this is not really how I’m conceptualizing.
Maggie also shared her thoughts about whether conversing fluently with mathematical language makes a difference in her students' journey to becoming mathematicians. She had interesting observations about the use of language for both her eighth graders and her university students.

*On a Mathematical Journey, Interview 3*

007 Maggie: I think it's an interesting question because I think about lots of pre-service teachers I work with that can say a lot of words. But they still don't have a lot of understanding. And so, um, this is one of those things I could debate both sides of. That sometimes the words... being able to say the word and... is different than truly being able to understand it, even though that usually that's what follows. You know, when you understand, you can explain it and everything. But they can't necessarily explain it, but they can say it... say it. Does that make sense?

And so, um, I think it's definitely been helpful to them, because they're obviously going into a field, which uses mathematics. They could... they could use that precise language, but sometimes I think some of my eighth graders could explain the heck out of some of them. Compared to some. Not all of them, obviously, but um, so...

Uh, yes. Typically my answer would be a strong yes. But I think that there are counterexamples to that. When somebody can just use the correct terminology, but they really don't have the depth behind it. Is that? Yeah. But it makes them feel much more confident. Makes them feel like they're making progress, for sure. And they are. They are. They can own those... those words.

**Summary**

This chapter focused on Maggie as the designer and facilitator of the reform-based mathematics classroom environment. In order to understand this environment, it was important to get to know Maggie as a teacher, understand what she valued, how she
organized her lessons, and what she said to ensure all her students were valued members of the learning community.

We met Maggie, the teacher participant in the study, who was a master teacher of mathematics and highly regarded across the state as a leader among mathematics educators. She taught 14 eighth graders, up to four pre-service secondary mathematics education majors, and a mathematics education graduate student during the course of the 14-lesson unit on data and statistics.

Maggie believed that all her students were capable of learning, though some were challenged due to institutional expectations for instructional pacing and the different connections they made. In order to support her students, Maggie was efficient in her use of instructional time, maximized her availability, set high expectations, and provided encouragement to them.

Maggie held three beliefs about mathematics education. Students should actually do mathematics and talk about mathematics. She also believed, however difficult to enact, that the Standards for Mathematical Practice (CCSSI, 2010) outweighed the content standards.

To this end, Maggie set three goals for her students. She wanted her students to learn how to learn math, not be afraid of math, and see the beauty and structure of math as represented in the interconnectedness of it in their world and the world around them.

Maggie also set short-term goals for the data and statistics unit, the focus of the study. First and foremost, she wanted her students to be wise creators and wise consumers of data. Specific skills that enabled them to achieve this overarching goal was
to be able to represent and interpret single variable data through histograms, box-and-whiskers plots, and line plots; represent and interpret relationships between bi-variate data by using a scatterplot; understand convenience, voluntary response, systematic, and random sampling methods; and how to choose an appropriate sample size.

Maggie was able to enact these beliefs and goals in her classroom with ease due to the depth of her own content knowledge and sophistication of her content knowledge for teaching. Maggie was a continuous learner who engaged in multiple professional development opportunities that stretched her thinking and extended her own understanding of the content.

Maggie organized her lessons according to three talk formats: Whole-Class Discussion, Partner Talk, and Small-Group Discussion (Chapin et al., 2009). Within that framework, she skillfully wove these talk formats with the lesson models of Direct Instruction, Guided Discovery, and Open-Ended Exploration (Rubenstein et al., 2004), variations of the typical reform-based three-phase lesson (Sherin, 2002; Stein et al., 2008). She supplemented her lessons with Meaningful Distributed Instruction (Rathmell, 2010) when appropriate. A special note was made of Maggie’s infrequent use of independent work as a contrast to the prominence it plays in the traditional classroom lesson (Huang et al., 2005).

Maggie conveyed her beliefs, goals, and the mathematical content by what she said to her students. Maggie’s classroom was inclusive. Everybody was expected to participate by attending and contributing to the discussion. She probed students’ thinking and insisted that they support their claims with mathematical reasons. She achieved this
through her use of teacher talk moves (Chapin et al., 2009), by posing higher-order questions (L. W. Anderson & Krathwohl, 2001) for students to discuss, and engaging them in higher-level tasks (Stein & Smith, 1998) in the context of a language-rich environment.

Her classroom was best described by paraphrasing NCTM's (2000) vision for school mathematics:

Maggie's classroom was a place where all students had access to high-quality, engaging mathematics instruction. There were ambitious expectations for all, with accommodations for those who needed it. Maggie was a knowledgeable teacher, who had adequate resources to support her work and was continually growing as a professional. The curriculum was mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology was an essential component of the environment. Students confidently engaged in complex mathematical tasks chosen carefully by their teacher. They drew on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they found methods that enabled them to make progress. Maggie helped her students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning to confirm or disprove those conjectures. Students were flexible and resourceful problem solvers. Alone or in groups and with access to technology, they worked productively and reflectively, with the skilled guidance of their teacher. Orally and in writing, students communicated their ideas and results effectively. They valued mathematics and engaged actively in learning it. (adapted from p. 3)
CHAPTER V

THE QUANTITY AND QUALITY OF STUDENTS’ MATHEMATICAL LANGUAGE

I turn now to the students, where the environment Maggie designed and facilitated—her beliefs, goals, knowledge, lesson organization, talk formats, lesson types, and language—impacted how they engaged in a unit of study on the topic of data and statistics. While the purpose of this study was to explore how they and their teacher used language to advance their conceptual understanding of mathematics in a reform-based mathematics classroom, I specifically sought to describe how the reform-based mathematics classroom environment impacted the quantity and quality of their mathematical language. Since the students were at the heart of the study, this chapter focuses on the quantity and quality of mathematical language they expressed.

The methods I used for observing the classroom discourse and taking field notes were completed as described by Bogdan and Biklen (2003). What follows was synthesized from 14 hours of video-recordings from each of 10 Flip Video™ camcorders located strategically around the classroom and 14 hours of audio-recordings from each of seven digital voice recorders placed between every two students that captured all classroom talk that occurred during classroom observations over fourteen 52-minute lessons from March 21, 2011 to April 8, 2011. When transcribed, the recordings yielded over 600 pages of data. Field notes supplemented the video- and audio-recordings with additional information pertaining to the students’ utterance patterns during mathematics instruction. I also considered relevant information regarding observations of non-speaking students and additional comments I wanted to remember. Prior to this study, I
had completed a pilot study during March 2010 with one classroom in which I became familiar with the electronic equipment, began exploring possible categories of student talk, and developed questions for further study.

I want to note that the limited number of observations in this single classroom might not represent the full range of utterances that each student expressed. Therefore, patterns of students' talk found in this study should not be generalized to all eighth-grade mathematics students in reform-based mathematics classrooms.

In order to analyze the quantity and quality of students' mathematical language as completely and accurately as possible, I listened carefully to students' voices as they were invited to join the Discourse. All mathematical talk was transcribed for Whole-Class Discussion, Partner Talk, and Small-Group Discussions and was included in the analysis. Examples excerpted from the transcriptions include line numbers that refer to speech turns from the original transcripts. Citations refer to the sequence of the lesson from which the talk was excerpted. All names are pseudonyms to protect the identity of the teacher and her students.

**Quantity of Student Talk**

Throughout the unit on data and statistics, students were expected to talk. Maggie believed that communication in the classroom was important in order to achieve her goal for students to be able to talk about mathematics. This belief aligned with NCTM's (2000) communication standard, which at grades six through eight asks teachers to use oral and written communication in mathematics to give students opportunities to—think through problems; formulate explanations; try out new vocabulary or
notation; experiment with forms of argumentation; justify conjectures; critique justifications; [and] reflect on their own understanding and on the ideas of others. (p. 272)

The research on classroom discourse would suggest, however, that school is dominated by teacher talk. In traditional classrooms, nearly 70% of classroom time is consumed by teacher talk (Flanders, 1970). This figure is raised to well over 80% if only lesson time is observed (McHoul, 1978). Simple subtraction would imply that students in traditional classrooms talk 20% of the time. Students in Maggie’s reform-based classroom talked nearly twice as much as the literature would suggest. Table 10 describes the total quantity of student talk compared to teacher talk during Maggie’s mathematics instruction.

Table 10.

<table>
<thead>
<tr>
<th>Initiator of Talk</th>
<th>Words</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Talk</td>
<td>47,081</td>
<td>37%</td>
</tr>
<tr>
<td>Teacher Talk</td>
<td>81,870</td>
<td>63%</td>
</tr>
<tr>
<td>Total</td>
<td>128,951</td>
<td>100%</td>
</tr>
</tbody>
</table>

Note: * words spoken by non-participating student were not included in this total

It would appear that Maggie still talked over 25% more than her students. Table 10 does not tell the complete story, however. To better understand the proportion of student to teacher talk, we need to look at the various talk formats that were used in Maggie’s classroom. Maggie divided her mathematics instruction between Whole-Class Discussions, Partner Talk, and Small-Group Discussion talk formats (Chapin et al.,
Since the talk formats were implemented for different amounts of time, a comparison between student and teacher talk was conducted for each. These various talk formats yielded considerable differences in the proportion of student talk in her classroom. Table 11 describes the quantity of student talk compared to teacher talk during the various talk formats. As expected, the percent of student talk surged during Partner Talk and Small-Group Discussion when there was less teacher involvement and students were able to all talk at once.

Table 11.

<table>
<thead>
<tr>
<th>Eighth-Grade Mathematics Instruction: Quantity of Student Talk Compared to Teacher Talk While Engaged in Three Talk Formats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Talk Format</td>
</tr>
<tr>
<td>Whole-Class Discussion Lessons 1-14</td>
</tr>
<tr>
<td>Partner Talk Lessons 1-6; 8-11; 14</td>
</tr>
<tr>
<td>Small-Group Discussion Lessons 7-8; 10; 12</td>
</tr>
</tbody>
</table>

Note: * words spoken by non-participating student were not included in this total

Maggie also used various lesson models during Whole-Class Discussion. Each of these lesson models, *The Guided Discovery Lesson, The Open-Ended Exploration Lesson*, and the *Integrating Direct Instruction with Guided Discovery Lesson* (Rubenstein et al., 2004) were observed and were variations of the three-phase launch, explore, summarize lesson format. Because the *Integrating Direct Instruction with Guided*
Discovery combined two lesson models, these were separated for analysis purposes.

Since the talk formats were implemented for different amounts of time, a comparison between student and teacher talk was conducted for each.

Maggie was the facilitator of the Whole-Class Discussion; therefore, it was not surprising that her talk dominated the discourse during this talk format. What is interesting to note is the variance in the proportion of student talk to teacher talk according to the lesson model. Table 12 describes this difference.

Table 12.

<table>
<thead>
<tr>
<th>Eighth-Grade Mathematics Instruction: Quantity of Student Talk Compared to Teacher Talk While Engaged in Three Whole-Class Discussion Lesson Models</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Whole-Class Discussion Lesson Model</strong></td>
</tr>
<tr>
<td>Students</td>
</tr>
<tr>
<td><strong>Words</strong></td>
</tr>
<tr>
<td>Direct Instruction Lessons 3-14</td>
</tr>
<tr>
<td>Guided Discovery Lessons 1-3; 5-11; 14</td>
</tr>
<tr>
<td>Open-Ended Exploration Lesson 6</td>
</tr>
</tbody>
</table>

Note: * words spoken by non-participating student were not included in this total

It is important to remember that students led the 17-minute Open-Ended Exploration Whole-Class Discussion. Maggie intended to step out of the discussion by removing herself to the back of the room. Nevertheless, she questioned students and clarified comments so that the total percentage of her talk exceeded half the total words spoken.
Given that each lesson model and talk format consumed differing amounts of time (see Tables 3 and 4), the actual number of words students spoke during each needed to be compared to realize the proportion of time students expressed themselves. Cobb (in Sfard et al., 1998) noted that a crude measure of the amount of talk occurring in classrooms is to determine the number of words spoken per minute. Table 13 describes the density of students' talk when compared across lesson models and talk formats using this measure.

Table 13.

**Eighth-Grade Mathematics Instruction: Density of Student Talk While Engaged in Three Whole-Class Discussion Lesson Models, Partner Talk, and Small-Group Discussion**

<table>
<thead>
<tr>
<th>Model</th>
<th>Minutes</th>
<th>Words</th>
<th>Words per Minute</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct Instruction (Lessons 3-14)</td>
<td>260</td>
<td>4,514*</td>
<td>17.4</td>
</tr>
<tr>
<td>Guided Discovery (Lessons 1-3; 5-11; 14)</td>
<td>243</td>
<td>8,464*</td>
<td>34.8</td>
</tr>
<tr>
<td>Open-Ended Exploration (Lesson 6)</td>
<td>17</td>
<td>1,157*</td>
<td>68.1</td>
</tr>
<tr>
<td>Partner Talk (Lessons 1-6; 8-11; 14)</td>
<td>104</td>
<td>18,175*</td>
<td>174.8</td>
</tr>
<tr>
<td>Small-Group Discussion (Lessons 7-8; 10; 12)</td>
<td>77</td>
<td>14,744*</td>
<td>191.5</td>
</tr>
</tbody>
</table>

Note: *words spoken by non-participating student were not included in this total

It is interesting to note that while half of the Whole-Class Discussion instructional minutes were devoted to the Direct Instruction lesson model, students talked nearly half as much, when actual word counts were compared, as they did when the Guided Discovery Lesson model was employed. When word counts were divided by instructional minutes, student talk averaged just over 17 words per minute when Maggie
taught using the Direct Instruction lesson model. Given that 47% of the Whole-Class Discussion instructional minutes were allocated to the Guided Discovery lesson model, this proved to be a much more language dense experience for students. Despite the fact that Maggie talked just over half the time during the Open-Ended Exploration lesson model, this lesson model provided the richest Whole-Class Discussion language experience for students, averaging 68 words per minute. When students were able to engage in Partner Talk or Small-Group Discussion it provided many more opportunities for them to share their thinking.

When analyzing students’ mathematical talk across Whole-Class Lesson Models and talk formats represented in Tables 11, 12, and 13, it became apparent that student talk increased, from 14% to 63% of total talk, the less structured the lesson became. Student talk became more than 11-fold denser when rates of student talk were compared during Direct Instruction to Small-Group Discussion, when multiple students talked at once.

Was this pattern true for all students? Or were some students monopolizing the discussion and skewing the results? Maggie indicated during the initial interview that some students were more verbal than others. In fact, the total words spoken by individual students varied widely. Tyler, the quietest student spoke a total of 1,046 words during the unit, compared to Evie, the most verbal, who spoke 7,265 words. Maggie felt this imbalance in the quantity of talk had mostly to do with peer pressure.

The “Coolness” Factor, Interview 1

008 Maggie: This is a great group, so it’s not about “this is not a good class” or anything else. But there’s a lot of quieter kids by nature in there, and there’s a little bit of the whole “coolness” – “not coolness” thing going on in there, more
than in second hour. So that has a little bit of a fear factor involved, and we try... I think we overcome that. Some kids don’t care about that, and they’re the ones most likely to talk. But, this group is harder to do that with than others. Or some of it... I have a couple of really verbal kids that you’ll see, that they have to... like they really have to talk to be able to understand it. And by that, they start monopolizing the time in which other people could possibly talk.

There was an additional variable, besides the “coolness” factor that impacted the total amount some students spoke. Tony was absent four class periods, and Dan, Diana, Evie, and Tyler were each absent once.

I analyzed the total talk for each individual student to determine if the lesson model or talk format impacted his or her rate of participation, assuming that the pattern of talk would be consistent for a student on any given day. Since the lesson models and talk formats were implemented for different amounts of time, the percent of talk made comparison possible. Table 14 describes the quantity of individual students’ talk when compared across lesson models and talk formats.

Profiles for individual students indicated a different distribution of talk than the aggregated totals did. Every student talked more during Guided Discovery Lessons compared to Direct Instruction. It was interesting to note, however, that while the overall percentage of student talk was comparable to teacher talk during Open-Ended Exploration lessons and the density of student talk increased during this Whole-Class lesson model, this was not where students spent most of their time talking. Even Chloe, who was the recorder and the manager of turn taking, only spent 5% of her total talk in this lesson format.
Table 14.

<table>
<thead>
<tr>
<th>Student</th>
<th>Direct Instruction</th>
<th>Guided Discovery</th>
<th>Open-Ended Exploration</th>
<th>Partner Talk</th>
<th>Small-Group Instruction</th>
<th>Total Talk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No.</td>
<td>%</td>
<td>No.</td>
<td>%</td>
<td>No.</td>
<td>%</td>
</tr>
<tr>
<td>Landon</td>
<td>115</td>
<td>4%</td>
<td>268</td>
<td>10%</td>
<td>24</td>
<td>1%</td>
</tr>
<tr>
<td>Diana</td>
<td>67</td>
<td>2%</td>
<td>604</td>
<td>14%</td>
<td>118</td>
<td>3%</td>
</tr>
<tr>
<td>Andrew</td>
<td>178</td>
<td>10%</td>
<td>298</td>
<td>17%</td>
<td>29</td>
<td>2%</td>
</tr>
<tr>
<td>Evie</td>
<td>914</td>
<td>13%</td>
<td>1735</td>
<td>24%</td>
<td>317</td>
<td>4%</td>
</tr>
<tr>
<td>Dylan</td>
<td>576</td>
<td>10%</td>
<td>1114</td>
<td>20%</td>
<td>34</td>
<td>1%</td>
</tr>
<tr>
<td>Connor</td>
<td>403</td>
<td>9%</td>
<td>579</td>
<td>14%</td>
<td>229</td>
<td>5%</td>
</tr>
<tr>
<td>Chloe</td>
<td>921</td>
<td>14%</td>
<td>1615</td>
<td>24%</td>
<td>337</td>
<td>5%</td>
</tr>
<tr>
<td>Tony</td>
<td>858</td>
<td>20%</td>
<td>947</td>
<td>23%</td>
<td>-*</td>
<td>-*</td>
</tr>
<tr>
<td>Ava</td>
<td>291</td>
<td>7%</td>
<td>573</td>
<td>15%</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Matt</td>
<td>19</td>
<td>1%</td>
<td>183</td>
<td>9%</td>
<td>27</td>
<td>1%</td>
</tr>
<tr>
<td>Tyler</td>
<td>47</td>
<td>4%</td>
<td>127</td>
<td>12%</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Gavin</td>
<td>89</td>
<td>6%</td>
<td>223</td>
<td>15%</td>
<td>13</td>
<td>1%</td>
</tr>
<tr>
<td>Dan</td>
<td>36</td>
<td>2%</td>
<td>198</td>
<td>10%</td>
<td>29</td>
<td>2%</td>
</tr>
</tbody>
</table>

Note: * Student was absent for this lesson model.

Evie and Chloe were the two most verbal students in the class, and did most of their talking during Whole-Class Discussion. There was not a direct correlation for all students, however, between the distribution of their total talk and talk format. Tony, Chloe’s partner, also did most of his talking during Whole-Class Discussion, despite the fact that he was absent for the Open-Ended Exploration, one of the Whole-Class Discussion lesson models. Several other students who were not very verbal during Whole-Class Discussion actually said as many words as some of their peers who appeared more talkative. They simply did not contribute much during the large group setting. Nevertheless, students talked the most when they were permitted to talk with partners and in small groups. The quantity of talk ranged from 57% to 89% for individual students during this time. The data was clear. Students were talking more in
Maggie’s reform-based classroom than in the typical traditional classroom. It was important to find out what they were saying.

**Quality of Student Talk**

What students actually say is more important than how many times they said certain sorts of things (Gee, 2005). The important questions I needed to be concerned with was what was the nature of all these conversations, and did they provide productive conditions for mathematical learning (Sfard et al., 1998)? It was important to Maggie that her students talked in her classroom, but she wanted that talk to be about mathematics (008, Interview 1). In the next section, I will describe what Maggie’s students were actually saying while they engaged in the unit on data and statistics. I illustrate the qualitative difference that could be heard in their talk as it shifted over time, as they transitioned from the use of vague language, to the spontaneous use of talk moves, critique, and eventual self-correction. This description is further supported by an analysis of the change in students’ talk as measured by the lexical density (Vee], 1999) of their utterances during Partner Talk and Small-Group Discussion. In addition, I also analyzed two mathematical Discourses (Moschkovich, 2002; Setati, 2005) that emerged from transcripts of these talk formats: Procedural Discourse (A. G. Thompson, Philipp, R. A. Thompson, & Boyd, 1994) and Conceptual Discourse (Sfard et al., 1998). Fine-grained analysis of students’ Conceptual Discourse revealed 10 categories of mathematical reasoning, which are highlighted and explained.
Starting At the Beginning

When Maggie began the unit on data and statistics, she found that her students knew very little about the topic; even less than she had anticipated when planning for it. Maggie remembered that the seventh grade math teacher had skipped the data and statistics unit the previous year due to lack of time. Perhaps other teachers had done the same?

I Was Surprised, Interview 2

001 Maggie: I know they haven't had a lot of stats, but I thought they would probably come with more experiences, just by living. You know, just by being in the world today. And maybe I just haven't given them the chance to share those, and so maybe they do have more of that. But I guess that I was surprised that they really didn't know what a histogram was. And they really, it seemed like they had more limited understanding than I thought they would have come with.

In order to find out exactly what her students did know, Maggie spent the first lesson of the unit pre-assessing them. She asked students to discuss with their partners, “What do you know about data and statistics?” The following is the extent of their responses. The transcripts of partnerships are grouped together. These discussions occurred simultaneously.

What Do You Know About Data and Statistics? Lesson 1, Partner Talk

Ava and Diana

042a Ava: It sounds like a mouthful of words.
042d Diana: Yeah. I like statistics. The other day I looked to see how many steps we walk... average people walk. Guess what? Amish people walk a lot and do you want to know why? They don't have cars. They walk like 16,000 steps.
Evie and Dan

042a Evie: I think of plots. Plotting.
042b Dan: And tables.
042c Evie: And data stuff. Numbers.
042d Dan: Yes. And lines.
042e Evie: And equations. And... two numbers to make a statistic.

Dylan and Landon

042h Dylan: Sometimes it's like graphs, charts, and information.
042i Landon: Well, yeah, like information and other stuff.

Connor and Matt

044a Connor: That is where you get information.
044c Connor: I don't know.
044d Matt: It shows the data of two or more things.

Andrew, Tyler, Gavin and Chloe had no response. Tony was absent.

Next, Maggie asked her students to talk to a partner, “Tell the person next to you all the kinds of plots you know about. What kind of graphs or plots?” Students responded to this question similarly to the previous one.

What Kind of Graphs or Plots? Lesson 1, Partner Talk

Ava and Diana

056a Ava: Plots and graphs, like you could plot the mean plot? Like graphs?
056b Diana: Exponential, linear, quadratic...

Evie and Dan

056a Evie: Dot things that are in graphs.
056b Dan: Oh, that stuff. I don't know.

Dylan and Landon

060a Dylan: What's a plot?
Landon: So, a long and a regular. I don’t really know different kinds of plots.

Connor and Andrew

Connor: Food plot. Is that a real thing, food plot?
Andrew: Plot a graph.

Gavin and Tyler

Gavin: What’s a plot?

Chloe and Matt had no response. Tony was absent.

Maggie continued to probe her students’ prior knowledge with additional questions before polling her students on their knowledge of various types of graphs and charts. She asked them to vote with their thumbs to indicate their response. A thumbs up signaled that the students thought they were experts on a particular graph. A side-ways thumb indicated they could create the graphical representation. A downward thumb indicated they had heard of the term. On occasion, Andrew’s thumb was out of site of the camera and his response could not be seen. Table 15 represents the students’ responses. Eighth graders felt confident about their knowledge of pie charts and bar graphs, were somewhat familiar with the dot plots, had barely heard of histograms or box-and-whisker plots, and most gave no response when asked about relative frequency plots.

A cumulative effect of little to no instruction was devastating, though not as unusual as it might seem. For Maggie, the topic of statistics and probability was incredibly important, yet it was the most likely mathematical area to be left out when teachers ran out of time to address everything in the curriculum.
Table 15.

Eighth Grade Mathematics Instruction: Percent of Students Familiar With 5 Common Graphical Representations of Data

<table>
<thead>
<tr>
<th>Type of Graph or Plot</th>
<th>I’m totally an expert (thumb up)</th>
<th>I think I can create it (thumb sideways)</th>
<th>I’ve heard of it (thumb down)</th>
<th>No response</th>
<th>Unable to determine response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pie Chart</td>
<td>67%</td>
<td>25%</td>
<td>0%</td>
<td>0%</td>
<td>8%</td>
</tr>
<tr>
<td>Bar Graph</td>
<td>67%</td>
<td>17%</td>
<td>0%</td>
<td>8%</td>
<td>8%</td>
</tr>
<tr>
<td>Dot Plot</td>
<td>42%</td>
<td>42%</td>
<td>8%</td>
<td>0%</td>
<td>8%</td>
</tr>
<tr>
<td>Box-and-Whiskers Plot</td>
<td>0%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Relative Frequency Plot</td>
<td>0%</td>
<td>8%</td>
<td>25%</td>
<td>67%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Note: N=12. One student was absent during the pre-assessment.

_After That Comes the Statistical Knowledge, Interview 1_

004 Maggie: They don’t skip the arithmetic. You know, you’re going to learn how to add, subtract, multiply, and divide all different kinds of numbers. They don’t skip the algebraic ideas, and then after that comes the geometry stuff, if we have time, and then after that comes the statistical knowledge. And that’s hard, that’s a hard shift to make as to what teachers select as what is important and not important, even when you have a curriculum that’s supposed to be aligned all the way through. And for me, I think the statistics is hugely important, and it’s still hard for me, even though I believe statistics is probably one of the most important areas that most students need to have access to. It’s even hard for me because I know that it’s not on the tests and it’s not on the other things, so you just always have that interplay between what you actually think is important in mathematics compared to what somebody else thinks is really important in mathematics.

It was evident that students knew very little about data and statistics when they began the unit. Nevertheless, Maggie encouraged her students by looking positively at the situation.
Lot To Learn! Lesson 1

139  Maggie:  This is great! Because you have a lot to learn, then.

Maggie didn’t waste a moment’s time before launching the lesson. The students began by thinking about their spring break data.

Mathematical Vagueness

At the beginning of the unit, students used vague language (Rowland, 2000) to avoid asserting any conviction in their response. Students frequently changed their mind or hedged when they talked in class. In the first excerpt, students predicted how the interval size would change the height of the bars in a histogram. They had difficulty committing to a response.

Yes, No, Maybe, Lesson 3

086  Maggie:  Will the heights of the bars change at all?
087  Tony:  No.
088  Andrew:  Maybe.
089  Tony:  No. Yes. They may. No they won’t.
090  Maggie:  Yes? No?

After Maggie changed the interval size, Chloe made the following observation. Her reasoning was hard to follow.

102  Chloe:  Well, it’s better in a way that it’s easier to read the data, but it’s not better in a way because we lost a bar. Well we didn’t lose a bar. No we didn’t. Never mind. We gained a bar.

Initially, students lacked the content knowledge to make informed contributions. As the lesson progressed, they sometimes had difficulty articulating their ideas and provided vague answers because they lacked confidence in their thinking. In the following excerpt, Tony revealed how he was nervous to respond because he was afraid
Tony continued to struggle to articulate his thinking four lessons later, when he attempted to explain how he knew that 1.5 divided by 10 was 0.15.

*Fifteen Hundredths, Lesson 9*

055 Tony: Because, I know that with hundredths, it's not going to be like hundredths with ten. Wait. No, I'm not going to say that. With like... I don’t know.

In the following excerpt, Matt claimed he “forgot” in order not to respond. Then he used a variety of hedging moves including the use of a vague pronoun referent “he” and multiple restarts. Despite the fact that the non-participating student tried to intervene, Maggie’s guiding questions, use of talk moves, and gentle persistence helped Matt eventually contribute to the discussion as he shared why he chose Player B to be on...
his basketball team for the activity, *Who Would You Choose?* Note that inaudible text is recorded for the non-participating student (NPS).

*Um, Lesson 9*

<table>
<thead>
<tr>
<th></th>
<th>Maggie:</th>
<th>Matt:</th>
<th>NPS:</th>
<th>Maggie:</th>
<th>Matt:</th>
<th>NPS:</th>
<th>Maggie:</th>
<th>Matt:</th>
<th>NPS:</th>
<th>Maggie:</th>
</tr>
</thead>
<tbody>
<tr>
<td>224</td>
<td>OK, Matt.</td>
<td>Um...</td>
<td>(inaudible)</td>
<td>OK. We’ll let Matt go.</td>
<td>I forgot it already.</td>
<td>(inaudible)</td>
<td>No, go. Go.</td>
<td>I forgot it.</td>
<td>You did?</td>
<td>Yeah, I did. I forgot it.</td>
</tr>
<tr>
<td>225</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Give me a little nugget. Can you remember any part of it?</td>
</tr>
<tr>
<td>226</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>So I might be able to help you.</td>
</tr>
<tr>
<td>227</td>
<td></td>
<td></td>
<td>(inaudible)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>228</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>229</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>230</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>231</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>232</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>233</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>234</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>235</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>236</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>237</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>238</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>239</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>240</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>241</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>242</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>243</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>244</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Maggie wanted students to commit to their claims and expected them to respond with mathematical reasoning. On occasion, she called them on their hedging behavior. In the following excerpt, students were to decide if a sample size of 20 would be substantially different than a sample size of 30.

*Do You Think It Will Make a Difference? Lesson 9*

<table>
<thead>
<tr>
<th></th>
<th>Maggie:</th>
<th>Evie:</th>
</tr>
</thead>
<tbody>
<tr>
<td>385</td>
<td>Do you think it will make a difference of 20 versus 30?</td>
<td>Si'.</td>
</tr>
</tbody>
</table>
Maggie: You do?
Evie: No.
Maggie: No, because I said, “You do?”
Evie: No. Yes.
Maggie: Just checking.

Students’ use of vague language and hedging behaviors gradually began to subside as Maggie challenged their thinking and asked them to become more specific in their responses. Maggie also asked her students to engage in the collective construction of knowledge by using talk moves (Chapin et al., 2009) as part of her repertoire to support mathematical thinking and learning.

Taking Ownership of the Talk Moves

Students heard Maggie incorporate the five teacher talk moves on a daily basis during Whole-Class Discussion. The only time these moves were not modeled was during Lesson 12, when the students were engaged in the Archeology Mystery Small-Group Discussion PBIT, and during Lesson 13, when they were assessed. These talk moves became part of the language of the class. Before long, students incorporated the talk moves into their own speech patterns and used them spontaneously. Most of the following examples occurred during Whole-Class Discussion, except for the Revoicing excerpt, which occurred during Partner Talk.

Revoicing. Students knew that clarity of expression was important when communicating their ideas. They did their best to help each other and used the revoicing move if they thought it would help. In the following excerpt, Dan attempted to bring clarity to Evie’s ideas as she shared her interpretation of a graph about people’s sleeping habits.
The More You Work, The More You Sleep Possibly, Lesson 1, Partner Talk

294c Evie: And it’s showing like the increase that’s different, like, the percentages at different times that people... lengths of peoples’... the length that people...

294d Dan: Like the more you work, the more you sleep possibly. (Revoicing)

295e Evie: Yeah.

Repeating. Repeating was modeled only once during Whole-Class Discussion.

Student did not use this talk move spontaneously.

Reasoning. Some students began offering unprompted reasoning beginning in Lesson 2. In the following excerpt, Tyler explained why he thought the example Maggie showed the class was a bar graph.

I Think It’s Like a Bar Graph, Lesson 2

099 Tyler: Uh, I think it’s like a bar graph, because the categories are specific. (Reasoning)

As the unit progressed, students understood that mathematical reasoning was an expectation of them. In the following excerpt, Maggie asked the class if anyone could predict the time period of the two arrowhead sites for the Archeology Mystery PBIT in Lesson 12. Chloe wanted to wait to make a claim until she had sufficient evidence to support her reasoning.

Have You Decided Where Site One and Site Two Are From? Lesson 12

084 Maggie: Have you... don’t say it out loud, but have you decided where site one and site two are from?
085 Chloe: We have an idea.
086 Maggie: You have an idea?
087 Chloe: We just need stuff to back it up. (Reasoning)
Adding On. Students knew that contributions were encouraged. As time went on, students were eager to offer ideas. In the following excerpt, Evie wanted to share credit with her partner, Dan, for an idea about how to determine slope. He was not as confident as she was that the idea would work, but Evie offered it anyway. She asked to maintain the floor to explain it fully.

I Have an Idea, Lesson 10

155 Evie: I have an idea. (Adding on)
156 Maggie: Yeah.
157 Evie: Well, me and Dan have an idea. (Adding on)
158 Dan: (shakes head)
159 Maggie: OK.
160 Evie: We were thinking...OK. So two times five is ten. And four times five is twenty. So it could be five. But, let me finish.

Waiting. Students also began to spontaneously request wait time in order to think. In the following excerpt, Maggie asked Landon a question about whether four and a half over thirty was equivalent to one and a half over ten. She wrote $\frac{4.5}{30} = \frac{1.5}{10}$ on the whiteboard as a visual cue. Landon responded by asking for her to wait before he answered her. Later, he used the word “well” to buy himself more time.

Um, Wait, Lesson 9

074 Maggie: Landon, what do you think?
075 Landon: Um, wait... (Waiting)
076 Maggie: Yeah, I'm good with thinking about it. Does it look like these are equivalent?
077 Landon: Yeah.
078 Maggie: And why does it look like it... they are equivalent to you?
079 Landon: Well, I'm not sure if it is, but if you take three times one point five, that equals four point five.
080 Maggie: Yes. Yes. And what else?
081 Landon: Well... (Waiting)
It's not enough just to do three times this, right? What else do you need to do?

Well, you need to take three times ten, also.

Spontaneously Checking and Self-Correcting

Though not identified as a talk move by Chapin et al. (2009), Maggie often asked students to verify their answers with other sources and check their response. "How do you know?" and "How could you check?" were typical questions Maggie asked students following a response, prompting them to explain their reasoning and provide evidence for their logic (Johnston, 2004). As with the talk moves, students started to spontaneously provide reasoning and self-check their answers. Students who understood a method or procedure were able to continually monitor its use to determine if they were accurate. They asked themselves if their work made sense and self-corrected if necessary.

In the following excerpt, Maggie asked the class to discuss with their partners, "Here are two points on a line: (2, 10) and (4, 20). What is the slope?" After providing almost a minute and a half of Partner Talk time, she brought the class back to Whole-Class Discussion. She asked individual students to share their thinking in order to contribute to the collective understanding of the group. Evie was the first to share what she and her partner, Dan, had talked about. Her answer for slope was accurate, but her reasoning for how she visualized the two points on a graph was not. Later, she noticed her error and self-corrected. That exchange occurred right after Chloe self-corrected an error in speaking, and Tony shared how he discovered an error in his original thinking.

Cause I Just Realized It, Lesson 10

We were thinking... OK. So two times five is 10. And four times five is 20. So it could be five. But, let me finish.
OK. He thought more of what the answer was, and I thought more of like what it would look like on a graph, like if I made it into a graph. So I was thinking about like if it went from two to ten, and you times it by two, it would be four and twenty. I was thinking it would be exponential, or whatever.

Or something. Because I was thinking how it would look good for an answer and he was thinking of the answer, so we worked together to find it.

Awesome!

Yes!

Great!

I could get mixed up.

OK. So, Evie, you gave us a couple different things to think about, so that’s good. And one thing that I liked here is you thought, “Hey, this has got to be a common thing that when you multiply across this way.” (makes an xy chart on the whiteboard and under xy writes 2 x 5 10 and below that writes 4 x 5 20)

Right? Two times five is 10. Four times five is 20.

Awesome. Ok, Chloe. Give your attention and respect here please. So I want you to be listening.

Rather than expanding on Evie’s contribution, Chloe and Tony shared their thinking as to how they solved the problem. Though not a self-correction in her thinking, Chloe did self-correct when she mis-spoke to make sure she communicated her ideas accurately.

Tony shared how his original thinking was in error and how he realized his mistake.

So we did the change in y divided by the change in x. We just did 10 divided by two and four divided by... er, 20 divided by four. (Self-correction)

Yeah, so I tried to eyeball it and I knew what I was talking about, but I absolutely messed up the equation. I said like two x plus five. I don’t know why I said that. Then I realized it was five. (Self-correction)

And (NPS), I think you did something similar to this, right? The change in y and the change in x? And then you
At this point Evie, realized her error. She signaled by raising her hand that she noticed something. Maggie acknowledged her with a non-verbal signal, but waited to call on her until she finished probing the class for additional ways to find slope using the talk move: adding on.

174 Maggie: Anybody do it differently? Anybody else do it differently than either of those? Those are a lot of good ways. Did anybody think about the point in between (2, 10) and (4, 20) and would go, “Oh, well three... would have to been 15?” Anybody think about that? No? Ok, Evie?

175 Evie: *I just realized it wouldn't be exponential, I don't think.* (Cross-checked and noticed error)

176 Maggie: I was going to get back to it.

177 Evie: *I think it would be linear.* (Self-correction)

178 Maggie: It has... Yeah, it's going to be linear, and I told you....

179 Evie: *'Cause I just realized it.* ( Noticed by cross-checking)

180 Maggie: And what's... What's a glaring “It’s got to be linear”? Yeah?

181 Evie: The line you made. (Verified with source)

182 Maggie: The... And... yeah... “...on the line.” (points to words in the question.)

183 Tony: Oh.

184 Maggie: On a line. Yep.

185 Evie: Yeah. (Confirmed)

Thus, student talk during Whole-Class Discussion began to sound different.

But it is one thing to use mathematical vocabulary and mathematical reasoning when they heard it modeled by their teacher and practiced under her guidance. Were they able to transfer it to their independent use?

Students did most of their talking during Partner Talk and Small-Group Discussion. As noted in Chapin et al. (2009), Maggie circulated as partners and groups discussed, but she didn’t control the discussions. Sometimes, she interjected when...
appropriate, but her role was diminished in these talk formats. Therefore Maggie could not guarantee that the talk would be productive. It is in this context that I explored the quality of student talk during these talk formats.

**Partner Talk and Small-Group Discussion**

To review, Partner Talk was the second most-used talk format in Maggie's classroom. She incorporated this talk format in Lessons 1 through 6, 8 through 11, and 14. Maggie did not use Partner Talk in Lessons 7 and 12, as students were engaged in Small-Group Discussion, or in Lesson 13, when students were formally assessed. Students were engaged in Partner Talk 104 out of 701 minutes (15%) of the time Maggie devoted to mathematics instruction. Maggie injected the use of Partner Talk into Whole-Class Discussion by pausing the discussion for short periods of time while she posed a question for students to discuss. She also used Partner Talk as an opportunity for students to solve mathematics problems during her MDI mini-lessons on slope, work with partners to create graphical representations, engage in short mathematical activities, or discuss, complete, or start homework during class.

Maggie also engaged her students in Small-Group Discussion. This was the least-used talk format in Maggie's classroom. She incorporated Small-Group Discussion in Lessons 7, 8, 10, and 12. Students were engaged in Small-Group Discussion 77 out of 701 minutes (11%) of the time Maggie devoted to mathematics instruction. Maggie engaged her students in two PBITS during this time.
Students uttered 2,299 responses, or individual turns, during Partner Talk and another 2,028 responses, or individual turns, during Small-Group Discussion. In all, 4,327 responses were analyzed.

Lexical Density

Earlier in this chapter, I described how students’ talk began to sound different as they became more precise in what they contributed to the discussion. What started as hesitant contributions full of vague language and hedging became more refined as students began talking more knowledgably. When students began spontaneously integrating the talk moves into their own discourse patterns, the talk became more focused. They also, upon Maggie’s urging, began to use mathematical language. The following excerpts provide examples of students reminding each other to use mathematical words.

Use Mathematical Words, Lessons 10 and 12, Small-Group Discussion

219v Chloe: We have to... we have to write... use mathematical words. (Lesson 10)

083aj Diana: Oh, we gotta think of reasons using the words. (Lesson 12)

089bq Chloe: So use median, and quartile, and all those big words? (Lesson 12)

Other times, students expressed that they thought they had fulfilled some imagined quota of math words. Chloe self-evaluated her writing and decided her use of mathematical vocabulary was sufficient.

That’s Enough Mathematical Words, Lesson 12, Small-Group Discussion

219ax Chloe: OK. I think that’s enough mathematical words.
Nevertheless, students’ use of language became richer and they were better able to articulate their understanding as the lessons progressed. During Lesson 12, students worked in small groups to estimate the time periods during which two archeological sites were inhabited based on the arrowheads found there. They were provided two sample sets of data based on the length, width, and neck width of the arrows, which they compared to arrowhead data sets from two known sites. In the following excerpt, Tyler and Diana compared their reasoning as they worked to write the summary of their results. Although they used implicit language, which is often done to “achieve solidarity” (Gee, 2005, p. 42) as they looked over their shared box plots, they included the mathematical vocabulary of statisticians in their descriptions.

*Talking Knowledgably*, Lesson 12, Small-Group Discussion

089c Diana: I said that the maximums for all those up here were a lot bigger than anywhere else.

089d Tyler: OK. I said the minimums were almost the same. And the maximums were... Like the IQR. (points to box plot) It overlaps that... almost it overlaps and the median here is in between those. And so is the third quartile.

089e Diana: I also said that the minimums were really close to this end. And all of these overlap.

089f Diana: (14 seconds later) So is there anything else we should say?

Working on the same assignment as Tyler and Diana above, Evie, Matt, and Gavin had collaborated to determine the time period for each archeological site. They had decided to portion out the work of writing the answers to the assigned questions. In the following excerpt, Evie, who had recorded her reasoning, read to the others what she had documented so far.
**Our Conclusion, Lesson 12, Small-Group Discussion**

089ay Evie: Four thousand B.C. to 500 A.D. We know this because when we made the box plots, we compared which one, one or two, was closest to which other site data. Then decided off of that. Two. 500 to 1,600 A.D. We know this because once we finished making the box plot, we looked which box plot was the most... was most, er, was most similar to the one we made to determine which time period it came from and from that data it came from two. OK. And we said yes. Because the width from site 1 matched up with Laddie Creek and Dead Indian Creek matches up with Big Goose Creek and with...

As the lesson progressed, the group made some revisions to what she had written.

Nevertheless, I only needed to reflect back to Lesson 2 to recognize the progress she had made in her ability to articulate her reasoning and in her use of mathematical language.

In Lesson 2, students explored a computer program that enabled them to create histograms from prepared data sets. Guiding questions focused their discovery of how the choice of interval size affected readers' perception of data. In the following excerpt, Evie recorded her answer to the first question, “How do you decide the interval widths?” Her response illustrates how she expressed her thinking 10 lessons prior to the archeology lesson, and how much she had grown over the course of the unit.

*How do you decide the interval widths?* Lesson 2, Partner Talk

257k Evie: ... the... range... of... of... One second. We decide the width by the range of... We decided by like by how wide it was but it’s a percent... it’s like a... it’s like a range, too. Inside there are numbers.

Evie was not the only student who demonstrated growth in the use of mathematical language over the course of the unit. There was a significant difference in students’ use of mathematical language over time for the entire class.
The excerpts above are anecdotal examples of change over time. I wanted to know if calculating the lexical density of student talk would support these observations. In order to find out, I compared in-depth analyses of two lessons that included a substantial portion of student talk. Lesson 2 was a logical lesson to analyze from the beginning of the unit. Much of Lesson 1 was pre-assessment and students were often merely repeating what they heard Maggie say. By the middle of Lesson 2, they were engaged in their first math activity, “Creating Histograms,” described above, which lasted just over 21 minutes. During this Partner Talk activity, students spoke a total of 891 clauses, all of which were included in the analysis. I chose Lesson 12 for the end-of-unit lesson, since Lesson 13 was an assessment day in which no talking took place, and Lesson 14 was a bridge lesson between data and statistics unit and the upcoming algebra unit. In Lesson 12, students were engaged in a culminating activity during Small-Group Discussion referred to as the “Archeology Mystery” PBIT for nearly 37 minutes. In this lesson, students uttered 1,799 clauses. Between both lessons, a total of 2,690 clauses were analyzed.

There are several accepted ways to calculate lexical density (R. Veel, personal communication, June 30, 2011). I chose to calculate the number of mathematical terms per clause. Mathematical terms included specialized vocabulary (e.g., interval, median, histogram) and words that can be used in ordinary English, but have mathematical meaning when used in context (e.g., range, scale, table). I also included terms that were necessary for students to use to discuss the data, such as the names of the data sets for Lesson 2 and the names of the archeological sites for Lesson 12. A dependent two-tailed
A t test was calculated using Microsoft® Excel to compare the mean Lexical Density of Lesson 2 with the mean Lexical Density of Lesson 12. The mean on Lesson 2 was 0.25 \((sd = 0.19)\), and the mean on Lesson 12 was 0.88 \((sd = 0.46)\). A significant increase from Lesson 2 to Lesson 12 was found \((p < 0.001)\). Students were talking more mathematically. Table 16 illustrates the increase in the lexical density of students’ speech over time when Lesson 2 and Lesson 12 were compared.

Table 16.

**Eighth Grade Mathematics Instruction: Increase in the Lexical Density of Student Talk From Lesson 2 to Lesson 12**

<table>
<thead>
<tr>
<th>Student</th>
<th>Clauses</th>
<th>Lexical Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Landon</td>
<td>74</td>
<td>0.16</td>
</tr>
<tr>
<td>Diana</td>
<td>14</td>
<td>0.57</td>
</tr>
<tr>
<td>Andrew</td>
<td>37</td>
<td>0.05</td>
</tr>
<tr>
<td>Evie</td>
<td>179</td>
<td>0.46</td>
</tr>
<tr>
<td>Dylan</td>
<td>75</td>
<td>0.09</td>
</tr>
<tr>
<td>Connor</td>
<td>97</td>
<td>0.08</td>
</tr>
<tr>
<td>Chloe</td>
<td>141</td>
<td>0.52</td>
</tr>
<tr>
<td>Tony</td>
<td>116</td>
<td>0.20</td>
</tr>
<tr>
<td>Ava</td>
<td>18</td>
<td>0.50</td>
</tr>
<tr>
<td>Matt</td>
<td>50</td>
<td>0.04</td>
</tr>
<tr>
<td>Tyler</td>
<td>31</td>
<td>0.16</td>
</tr>
<tr>
<td>Gavin</td>
<td>30</td>
<td>0.03</td>
</tr>
<tr>
<td>Dan</td>
<td>29</td>
<td>0.31</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Lesson 2</th>
<th></th>
<th>Lesson 12</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Partner Talk</td>
<td>Clause</td>
<td>Lexical Density</td>
<td>Clause</td>
<td>Lexical Density</td>
</tr>
<tr>
<td>Mathematical Words</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>0.16</td>
<td>36</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.57</td>
<td>103</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.05</td>
<td>31</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>83</td>
<td>0.46</td>
<td>153</td>
<td>212</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.09</td>
<td>181</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.08</td>
<td>260</td>
<td>164</td>
</tr>
<tr>
<td></td>
<td>74</td>
<td>0.52</td>
<td>229</td>
<td>383</td>
</tr>
<tr>
<td></td>
<td>23</td>
<td>0.20</td>
<td>190</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.50</td>
<td>215</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.04</td>
<td>154</td>
<td>175</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.16</td>
<td>53</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.03</td>
<td>113</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.31</td>
<td>81</td>
<td>40</td>
</tr>
</tbody>
</table>

**Note:** * \( p < 0.001 \), two-tailed

Students took pride in their progress. After finishing the group’s final conclusions for the Archeology Mystery PBIT, Matt exclaimed how pleased he was with what he had contributed to the group.
I Feel Good, Lesson 12, Small-Group Discussion

Matt: I feel so good about myself.

When students achieve feelings of accomplishment for effort toward a job well done, it fosters an internal motivation to the activity. The more often students notice and rehearse the connection between their effort and feelings of self-worth and independence, it strengthens their intrinsic motivation to do so again (Johnston, 2004). As students talked more mathematically, their feelings of self-efficacy increased. Not only were Maggie’s students speaking, listening, reading, writing, and thinking mathematics, they were also feeling, believing, and valuing mathematics. They were becoming members of the Mathematics Discourse (Gee, 2008).

Procedural and Conceptual Discourses

Moschkovich (2002) noted, “There is no one mathematical Discourse” (p. 199). Due to the purposes of Partner Talk and Small-Group Discussion activities, two types of mathematical student talk emerged. Similar to the South African students in Setati’s study (2005), Maggie’s students engaged in Procedural and Conceptual Discourses.

Procedural Discourse is consistent with what A. G. Thompson et al. (1994) described as a “computational orientation” (p. 86), where mathematics is viewed as computational procedures, and doing mathematics is computing for no other reason than for having been asked to do so. It is not unusual to find students talking about mathematical procedures in mathematics classrooms. In this study, mathematical procedural talk included responses in which students questioned the steps of procedures, sought reassurance that they were completing a procedure accurately, provided rote
mathematical answers, or reported their actions, which also included procedural actions taken when accessing computer programs to solve mathematical tasks. Reading aloud from textbooks, handouts, the whiteboard, items projected on the document camera or from the teacher’s computer often served the function of signaling the beginning of a new problem. Students also used these resources as a reference for the next step of a procedure. Students frequently read aloud answers they wrote for procedural questions as a way of bringing closure to a problem.

There was another type of student talk that occurred in Maggie’s classroom during Partner Talk and Small-Group Discussions. Conceptual Discourse emerged as a focus of student talk when students were thinking about mathematical concepts while doing mathematics. Conceptual Discourse refers to discussions “in which the reasons for calculating in particular ways can also become explicit topics of conversation” (Sfard et al., 1998, p. 46). “In conceptual Discourses, learners articulate, share, discuss, reflect upon, and refine their understanding of the mathematics that is the focus of the interaction” (Setati, 2005, p. 249). In Maggie’s classroom, this type of talk was referred to as mathematical reasoning.

Determining whether student talk was classified as procedural talk or mathematical reasoning was dependent on the context in which it was expressed. The function of each utterance was carefully considered within the sequence in which it occurred in the natural flow of discussion. Table 17 describes the proportion of student talk focused on mathematical procedures and mathematical reasoning during Partner Talk, a talk format utilized in all lessons except Lessons 7 and 12 when students were
engaged in Small-Group Discussion, and Lesson 13, when they were assessed. A total of 2,299 responses were analyzed. Since many of the questions or tasks during the time allocated to Partner Talk challenged students to engage in higher-order thinking about mathematical concepts (see Table 8), 60% of their talk focused on mathematical reasoning. The other 40% of talk focused on mathematical procedures around calculating slope or the mechanics of constructing graphical representations. Note that 6% of student responses during Partner Talk were inaudible due to background noise produced by all of the students talking at once.

Table 17.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Mathematical Procedures</th>
<th>Mathematical Reasoning</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No.</td>
<td>%</td>
<td>No.</td>
</tr>
<tr>
<td>Lesson 1</td>
<td>46</td>
<td>17%</td>
<td>225</td>
</tr>
<tr>
<td>Lesson 2</td>
<td>255</td>
<td>51%</td>
<td>248</td>
</tr>
<tr>
<td>Lesson 3</td>
<td>0</td>
<td>0%</td>
<td>67</td>
</tr>
<tr>
<td>Lesson 4</td>
<td>46</td>
<td>66%</td>
<td>24</td>
</tr>
<tr>
<td>Lesson 5</td>
<td>62</td>
<td>27%</td>
<td>165</td>
</tr>
<tr>
<td>Lesson 6</td>
<td>142</td>
<td>39%</td>
<td>221</td>
</tr>
<tr>
<td>Lesson 8</td>
<td>138</td>
<td>61%</td>
<td>87</td>
</tr>
<tr>
<td>Lesson 9</td>
<td>98</td>
<td>57%</td>
<td>73</td>
</tr>
<tr>
<td>Lesson 10</td>
<td>14</td>
<td>33%</td>
<td>28</td>
</tr>
<tr>
<td>Lesson 11</td>
<td>51</td>
<td>34%</td>
<td>99</td>
</tr>
<tr>
<td>Lesson 14</td>
<td>65</td>
<td>31%</td>
<td>145</td>
</tr>
<tr>
<td>Total</td>
<td>917</td>
<td>40%</td>
<td>1,382</td>
</tr>
</tbody>
</table>

Table 18 describes the proportion of student talk focused on mathematical procedures and mathematical reasoning during Small-Group Discussion, which occurred
in Lessons 7, 8, 10, and 12. Students engaged in two PBIT activities during Small-Group Discussion.

Table 18.

*Eighth Grade Mathematics Instruction: Comparison of Student Responses Focused on Mathematical Procedures and Mathematical Reasoning While Engaged in Small-Group Discussion*

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Mathematical Procedures</th>
<th>Mathematical Reasoning</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No.</td>
<td>%</td>
<td>No.</td>
</tr>
<tr>
<td>Lesson 7</td>
<td>282</td>
<td>86%</td>
<td>45</td>
</tr>
<tr>
<td>Lesson 8</td>
<td>259</td>
<td>92%</td>
<td>23</td>
</tr>
<tr>
<td>Lesson 10</td>
<td>233</td>
<td>60%</td>
<td>155</td>
</tr>
<tr>
<td>Lesson 12</td>
<td>699</td>
<td>68%</td>
<td>332</td>
</tr>
<tr>
<td>Total</td>
<td>1,473</td>
<td>73%</td>
<td>555</td>
</tr>
</tbody>
</table>

In the first PBIT, students organized into groups of three with each individual group member generating a random sample of 30 from a 100-member data set regarding individuals’ sleep and movie-watching habits. Their talk focused on mathematical procedures in Lessons 7 and 8 while they constructed line plots and box plots for their individual random sample, and gathered information about the plots and graphs their group members had constructed. As students finished their graphical representations during Lesson 10, their language shifted to include more mathematical reasoning as they began discussing the variability of the data between the three random samples.

In the second PBIT, completed during Lesson 12, students determined the time period and location of two archeological sites based on the length, width, and neck-width of a data set of arrowheads found there and compared this information to the time period of known sites. Students were more proficient in their ability to construct box plots and
engaged in less talk about the procedures necessary to create the graphical representations during this second PBIT. More of their talk focused on the mathematical reasons for determining the location of the sites. A total of 2,028 responses were analyzed. Due to the nature of the tasks, 73% of the overall talk spoken during this talk format focused on the mathematical procedures of constructing graphical representations and 27% was focused on mathematical reasoning. Note that 1% of student responses during Small-Group Discussion were inaudible due to background noise produced by all of the students talking at once. (See Table 8 for an analysis of the cognitive complexity of these tasks.)

Mathematical Reasoning

Maggie encouraged her students to think mathematically and to provide mathematical reasons to support their conjectures. As part of her beliefs about mathematics, Maggie wanted to develop in her students the Standards for Mathematical Practice. In the unit on data and statistics, one of the standards on which she focused was Standard 3: Construct viable arguments and critique the reasoning of others. Since Maggie enacted her beliefs on a daily basis, students began to internalize these beliefs and demonstrated them through their own actions. In the following excerpt, Chloe and Tony critiqued each other’s reasoning during Partner Talk as they discussed reasons why they would choose either Player A or Player B to be on their basketball team, given a summary of the players’ individual performance statistics. In the end, Tony decided Chloe provided the strongest argument by providing the most convincing mathematical reasons.
185m Chloe: OK. So I think we should choose Player B because he has a bigger mean, he has a larger median, and a larger minimum. So even though he doesn’t score the most points, he's consistent. Eh?

185n Tony: I think we should score the player with the most points.

185o Chloe: Why? But his one game... a couple of games he had single digits. A lot of games. One, two, three, four, five, six...

185p Tony: I say we should click person B.

185q Chloe: All of his... All of his are double-digit games.

185r Tony: That’s so... but that’s very bad (inaudible).

185s Chloe: Um...hm. Because Player B has a bigger mean, a bigger median, and a bigger minimum.

185t Tony: Wow. Chloe is so cool.

185u Chloe: And a bigger first quartile.

185v Tony: I give. OK.

185w Chloe: It just has a smaller...

185x Tony: You’re right. Yeah, you’re right. I give.

Due to the emphasis placed on mathematical reasoning in Maggie’s classroom, this type of student talk became a focus of interest. As I listened to students provide reasons for their responses, I noticed a qualitative difference between the ideas. I used transcripts of student’s mathematical talk expressed during Partner Talk and Small-Group Discussion for the analysis as it more likely reflected independent continuous student talk with less involvement from Maggie or the university participants. These responses were analyzed through the grounded theory approach.

As no preconceived classification system was derived before the study, a constant comparative analysis was used to develop the categories. Initially, I compiled a sample of student talk by combing the transcripts of Partner Talk and Small-Group Discussion to obtain the most sophisticated utterance spoken by each participating student in response to every question or task Maggie posed. These responses were arranged by question or
task for each lesson. Recognizing that students' utterances were shaped according to the
question or task Maggie posed to them, I determined that the sample set of responses
were suitable given the range of thinking required. See Table 8 for a complete analysis
of the cognitive complexity of Maggie's questions and tasks.

Through open coding, 11 categories of mathematical reasoning emerged from the
data. I reviewed these categories with three others familiar with teacher talk moves and
mathematical discourse. One reviewer was a university instructor and author on the topic
of teacher talk moves, one was a university instructor who worked closely with
practitioners regarding teacher discourse, and the third was a university researcher who
specialized in mathematical discourse. The purpose of this review process was to
determine if the categories made sense according to the sample set of data.

I applied these categories to the entire set of student utterances transcribed from
Partner Talk and Small-Group Discussions. Upon initial analysis of the data, I
determined that further refinement was necessary. The categories were revisited to
tighten the descriptors and ensure that they were discrete. Further refinement resulted in
10 core categories, which were again applied to the entire set of student utterances
transcribed from Partner Talk and Small-Group Discussions.

When students' responses of Mathematical Reasoning for Partner Talk and Small-
Group Discussion were combined, 1,937 responses were analyzed and classified
according to the 10 categories of mathematical reasoning that emerged from the data, as
follows:
Category 1: I don’t know.

Category 2: Agrees or disagrees with partner without evidence of independent reasoning.

Category 3: Personalizes the data.

Category 4: Responds without evidence of reasoning.

Category 5: Indicates thinking but provides no explanation.

Category 6: Asks a question to seek understanding.

Category 7: Responds with partial or incomplete reasoning.

Category 8: Responds with simple reasoning.

Category 9: Responds with complex reasoning.

Category 10: Self-checking response.

These categories of students’ mathematical reasoning are reported in this section.

Excerpts from transcripts provide examples of each and graphical representations are provided to illustrate the frequency of students’ use as it was distributed across the 14 lessons.

Category 1: I don’t know. When students expressed confusion regarding a concept being discussed or lack of understanding to a question posed, a typical answer was “I don’t know.” This was different than when students questioned how to “do” a math problem, which was categorized as talk about mathematical procedures. There was often a considerable amount of vague language and hedging that accompanied this category. In the first excerpt, students were asked to talk about the kinds of graphs or plots they knew about.
What Kind of Graphs or Plots Do You Know? Lesson 1, Partner Talk

062a Landon: So, a long and a regular. I don't really know different kinds of plots. (Category 1)
62b Dylan: I think that plots (inaudible).
62c Landon: So, um, really, so um, I don't really know. (Category 1)

In the second excerpt, students were provided four questions to guide their Partner Talk around the discussion of choosing the interval widths for a histogram. Evie and Dan discussed the first question, “How do you decide the interval widths?” Students used a computer program, Creating Histograms, to complete this activity.

How Do You Decide the Interval Widths? Lesson 2, Partner Talk

244a Evie: OK. Do you understand what she’s wanting?
244b Dan: You mean that question?
244c Evie: I don’t. (Category 1)
244d Dan: I don’t understand that question. That is... (inaudible). (Category 1)
244e Evie: OK. [reading] How do you decide the interval widths?
244f Dan: Oh, my gosh! I don’t know. (Category 1)

In the third example, Landon, Dylan, and Dan attempted to generate a random sample of 30 out of a 100 sets of data about people’s sleep and movie-watching habits using a random number generator. The process and the terminology proved to be confusing.

What Do You Mean, Data? Lesson 7, Small-Group Discussion

368ad Dylan: What do you mean, data? What the heck?
368ae Dan: I don’t know. (Category 1)

The graphs shown in Figure 6 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 1 across the lessons.
Figure 6. Students' Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 1: I Don't Know.
Category 2: Agrees or disagrees with partner without evidence of independent reasoning. As students worked with partners, they often expressed agreement or sometimes disagreement with their partner without any evidence of independent reasoning. In the first excerpt, Diana and Ava discussed whether they could make a pie graph out of a particular set of information. Students agreed by saying, “Yeah,” or some other form of affirmation. Likewise, they expressed disagreement by saying, “No” without reasoning. Note that this is different from the use of “Yes” or “Yeah” in lines 177c and 177d, when Ava and Diana provided reasons for their opinion in those responses. In this example, 177c was classified as Category 8: Responds with simple reasoning, and 177d was classified as Category 7: Responds with partial reasoning.

Could You Make a Pie Chart? Lesson 1, Partner Talk

177a Ava: You could make a pie graph. Well, yeah, you could 'cause you could make percents by figuring out how many people did this.
177b Diana: No. (Category 2)
177c Ava: Yes, you could. You could get a percent like so many people like out of whole. Yeah. How many people out of... so it would be 15 people in a class, right? How many people total... like percentage play played 120 minutes or whatever.
177d Diana: Yeah. But that really wouldn’t make sense.
177e Ava: You could do it on how many people played.
177f Diana: Yeah. (Category 2)
177g Ava: For longer than that or whatever.
177h Diana: Or just played, right?
177i Ava: Yeah. (Category 2)

In the second excerpt, Ava agreed with Connor’s prediction about the location of archeology sites based on the arrowhead data. She provided no additional reasoning.
What's the Time Period for This? Lesson 12, Small-Group Discussion

Connor: I still think that these two go to site one.
Ava: Yeah. Me, too. (Category 2)

In the third excerpt, Diana and Ava disagreed on whether they would rather have intervals of 15 or 25 for the histogram Maggie displayed on the projected computer display using the Creating Histograms program. Note that the first time they offer an answer it is classified as Category 4: Responds without reasoning. It is only after they begin repeating their answers and the responses become a disagreement that they are classified as Category 2. Neither student provided reasoning for their response during this entire 22-second segment until Diana finally offered one in line 180j, to which Ava laughed it off.

Would You Rather Have Intervals of 15 or 25? Lesson 3, Partner Talk

180a Ava: 15.
180b Diana: 25.
180c Ava: 15. (Category 2)
180d Diana: 25. (Category 2)
180e Ava: 15. (Category 2)
180f Diana: 25. (Category 2)
180g Ava: 15. (Category 2)
180h Diana: 25. (Category 2)
180i Ava: No. 15. (Category 2)
180j Diana: 15 is too small. People who can’t see can’t see (inaudible).
180k Ava: [chuckles to Chloe and Tony] She's like... Diana's like it can't be 15 because some people can't see... can't see it. (Category 2)

Sometimes students conveyed a response of agreement that appeared to express cooperation or support for the thinking of their partner. In the fourth excerpt, Diana and Evie discussed similarities and differences in the amount of sleep people got on Saturday and Sunday nights.
What Are the Similarities and Differences, Lesson 5, Partner Talk

210j Diana: But overall, most people got more sleep than they did.
210k Evie: Yeah. (Category 2)

The graphs shown in Figure 7 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 2 across the lessons.

**Category 3: Personalizes the data.** At times, students were tempted to respond to the data by personalizing it. Students shared their opinions or talked about how they would respond to the particular survey question rather than analyzing the information represented by the data. Category 3: Personalizes the Data is not off-topic talk as the students attended to the information presented. This response was heard during the early lessons, but faded as the students become more proficient with the analysis process. In the first excerpt, Tony and Chloe were supposed to be responding to questions for the Creating Histograms assignment when a particular graph about NBA players’ salaries caught Tony’s attention. Instead of discussing interval widths, Tony personalized the information presented in the graph by sharing his opinion of the data.

Benchwarmer, Lesson 2, Partner Talk

242ai Tony: *Benchwarmer. Really though? People are sitting on the bench? And you get paid (inaudible). I’ll keep warming the bench if I can make the bench for that much. Some of those players don’t even play. They just come out and like practice.* (Category 3)
242aj Chloe: *Just get a draft. They draw you in, OK.* (Category 3)
242ak Tony: *They still live paycheck to paycheck. Which is stupid.* (Category 3)
242al Chloe: *There you go. (inaudible) for a billion dollars.* (Category 3)
242am Tony: *Football equipment would cost a lot.* (Category 3)
242an Chloe: *Not millions and millions.* (Category 3)
Figure 7. Students' Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 2: Agrees or Disagrees Without Evidence of Independent Reasoning.
242ao Tony: No. (inaudible) for a thousand dollars a year. There’s not a lot of hope. (Category 3)

In the second excerpt, Dylan responded to information presented on a handout regarding information about populations and samples. Two particular statements caught his attention: The average child eats 1,500 peanut butter and jelly sandwiches before graduating from high school; and The average American child watches 30,000 commercials each year. Instead of analyzing how this information was obtained, he related it to his own personal commercial viewing and eating habits.

_I Don’t Eat Peanut Butter and Jelly Sandwiches, Lesson 5, Partner Talk_

366a Dylan: The average child watches... commercials every year. Geez! That’s a lot of commercials. Lot of peanut butter and jelly sandwiches. I don’t eat peanut butter and jelly sandwiches very much. (Category 3)

In the third excerpt, Diana, Ava, and Evie compared their random samples for the movies watched. They discussed whether the data represented 20 movies watched or 20 hours of movies watched. Evie personalized the data by commenting on her own viewing habits.

_Movies Watched, Lesson 10, Small-Group Discussion_

219bd Ava: There’s multiple numbers for 20 hours or go to movies for 20 hours.
219be Diana: It’s not... it’s the number of movies watched not the movies.
219bf Evie: So, I could easily watch 20 movies. (Category 3)

The graphs shown in Figure 11 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 3 across the lessons.
Figure 8. Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 3: Personalizes the Data.
Category 4: Responds without evidence of reasoning. Responses were classified as Category 4 when students did not reveal a rationale for their answer. This category does not imply that students weren’t thinking, but rather they did not make their reasoning explicit. On average, 22% of the student responses were of this category type.

In the first example, Maggie’s prompted Diana to describe the variability of her sleep/movie data plotted from her random sample of 30 pieces of data. Diana responded but did not elaborate with any reasoning.

What’s One Thing You Can Say, Lesson 8, Small-Group Discussion

271cw Maggie: Would you write what you’d do for your own? Is this yours, Diana? How would you describe your own? Give me a start. Why don’t you start? What’s one thing you can say about that?

271cw Diana: *That it’s close together.* (Category 4)

Andrew responded, albeit inaccurately, to a question on his handout that asked which type of sampling method was used to survey every fourth person in the cafeteria line. He provided no support for his response.

Random, Lesson 6, Partner Talk

090cl Andrew: *I would say this one’s random.* (Category 4)

A mathematical response could also be classified as Category 4 if it required the student to think, rather than provide a rote answer such as counting or recording numbers produced by the random number generator, or a procedural one when announcing the steps in an applied algorithm. In the following excerpt, Austin determined the slope from the equation described by the least squares regression line. He didn’t, however, explain how he knew.
Which Number Is Slope? Lesson 14, Partner Talk

144a  Connor:  

Slope is point four nine six.  (Category 4)

The response “Yeah” could be classified as Category 4 if it was in response to a question and not used as a statement of agreement regarding a partner’s response. In the following example, one of the university participants asked Connor about the attributes that influenced his choice of basketball players for the activity Who Would You Choose?

You’re Looking For a Player You Can Depend On, Lesson 9, Partner Talk

187l  Mr. Dunn:  

So you’re looking more for the player that you can depend on?

187m  Connor:  

Yeah.  (Category 4)

187n  Mr. Dunn:  

To win the game?

187o  Connor:  

Yeah.  (Category 4)

The graphs shown in Figure 9 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 4 across the lessons.

Category 5: Indicates thinking but provides no explanation. When students responded with phrases such as “I think...,” used the conditional tense “it would,” or responded with a process, it indicated they were thinking about their response. If no further explanation accompanied the utterance, then it was classified as Category 5. In the first excerpt, Chloe and Tony discussed how to calculate the slope given two points on the line. Both shared how they arrived at an answer. Through discussion, Tony indicated he had thought inaccurately about the process. Later he acknowledged the magnitude of his error, but elaborated no further.
Students' Use of Mathematical Reasoning During Partner Talk
Category 4: Responds Without Evidence of Reasoning

Students' Use of Mathematical Reasoning During Small-Group Discussion
Category 4: Responds Without Evidence of Reasoning

Figure 9. Students' Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 4: Responds Without Evidence of Reasoning.
In the second excerpt, Maggie asked Landon to identify the population he was planning to describe through the survey questions he was writing. Landon indicated that he needed to think more about that. Again, he did not explain further.

\textit{Who's Your Population? Lesson 14, Partner Talk}

\begin{quote}
Who's your population? \\
Um, I haven't... I need to figure that out. (Category 5)
\end{quote}

In the third excerpt, Matt expressed that he had another idea about how the group could predict the length of an arrowhead if given the width. This was an indication that he was thinking, but no explanation was provided during this turn.

\textit{I've Got Another Idea, Lesson 12, Small-Group Discussion}

\begin{quote}
I want you to give a good estimate of what you think the length is going to be. \\
All right. \\
OK? \\
Well, I got another idea for that. (Category 5)
\end{quote}
The graphs shown in Figure 10 illustrate students' use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 5 across the lessons.

**Category 6: Asks a question to seek understanding.** Students wanted to make sense of the mathematics they were learning. When they were confused or wanted more information, they often asked each other or their teacher in order to better understand.

In the first excerpt, Tony and Chloe tried to clarify what the intervals were when working on the *Creating Histograms* questions. What follows was their line of questioning to each other as they attempted to construct a common understanding of the graph.

**These Are Each Interval, Right? Lesson 2, Partner Talk**

257ak Chloe: OK. Now go down here. Go to histogram. Now we have to write how we think we have decided the interval width.

257al Tony: *These are each interval, right?* (Category 6)

257am Chloe: Hm?

257an Tony: *These are each interval, right?* (Category 6)

257ao Chloe: Um...

257ap Tony: The widths are all the same.

257aq Chloe: *Is it?* (Category 6)

257ar Tony: Yes. *Why are there all those little spaces?* (Category 6)

In the second excerpt, Dylan was trying to understand what data the group was supposed to collect from the population of 100 individuals' sleep/movie habits. He asked questions to seek understanding from his group members.

**You're Supposed to Have Different Data? Lesson 7, Small-Group Discussion**

268ag Dylan: *You're supposed to have different data or what?* (Category 6)

268ah Dan: Yeah. Yeah.

268ai Dylan: *How?* (Category 6)

268aj Dan: Just randomly choose 30 people...
Students' Use of Mathematical Reasoning During Partner Talk
Category 5: Indicates Thinking But Provides No Explanation

Students' Use of Mathematical Reasoning During Small-Group Discussion
Category 5: Indicates Thinking But Provides No Explanation

Figure 10. Students' Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 5: Indicates Thinking But Provides No Explanation.
In Lesson 14, students were expected to write a second question that correlated with an initial survey question they had written several days before. In the second excerpt, Ava wanted to clarify the task and asked a university participant for help.

*Do You See How Those Might Be Related? Lesson 14, Partner Talk*

283ab Ava: *So I’d have to find data for both of those things from each person?* (Category 6)
283ac Mr. Dunn: And you’re going to plot it. And this one might be harder and you’ll have to change it a little bit.

The graphs shown in Figure 11 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 6 across the lessons.

*Category 7: Responds with partial or incomplete reasoning.* When students provided a response with partial or incomplete reasoning, it was classified as Category 7.

In the first example, Evie tried to explain why she thought the plot displayed by the computer projector was a bar graph. She had difficulty articulating her reasoning, leaving it incomplete.

*What Type of Plot Is This? What Does It Tell You? Lesson 1, Partner Talk*

295a Evie: I think it’s out of 100 percent. And it’s like a bar graph or something.
295b Dan: Something like that.
295c Evie: *And it’s showing like the increase that’s different, like, the percentages at different times that people... lengths of peoples’... the length that people...* (Category 7)
295d Dan: Like the more you work, the more you sleep possibly.

In the second example, Evie attempted to explain to Dan why the missing score of a quiz had to be less than then mean of 6 scores. She provided reasoning, but it was not mathematically sound. This discussion was prompted by Maggie’s question, “Why do we know the score should be less than 79 and a half?”
Students' Use of Mathematical Reasoning During Partner Talk
Category 6: Asks a Question To Seek Understanding

Students' Use of Mathematical Reasoning During Small-Group Discussion
Category 6: Asks a Question To Seek Understanding

Figure 11. Students' Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 6: Asks a Question To Seek Understanding.
Why Should the Score Be Less Than 79.5? Lesson 4, Partner Talk

174a Dan: (inaudible) and why? Hm...
174b Evie: Because almost everything is above 79 and a half. (Category 7)
174c Dan: Yeah, so the lowest is 71.
174d Evie: Right. And if that the lower it would be like median, it has to be equivalent on each side of what it is, so it would be like 91 or above it. (Category 7)

In the third example, Ava, Evie, and Diana compared their random samples of individuals’ sleep/movie watching habits. Diana provided reasoning for why the data appeared similar, but her reasoning was faulty. The sample was similar because it was sampled randomly, not because everyone was identical.

Why Would It Be Similar? Lesson 10, Small-Group Discussion

279c Ava: Oh, yeah. It’s going to be pretty close.
279d Evie: Similar. Similar.
279e Diana: Why would it be similar? Because the data stays the same and nobody is unique and different. That’s why I think. (Category 7)

The graphs shown in Figure 12 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 7 across the lessons.

Category 8: Responds with simple reasoning. When students provided one reason or an explanation for their thinking, their response was classified as Category 8. These responses contained less hedging that resulted in a more confident tone.

In the first example, though not spontaneous, Gavin provided a single reason for why he chose Player B to be on his basketball team for the activity Who Would You Choose? When Maggie pressed him for a second reason, he was unable to provide one. Note that Gavin’s initial response given in line 185d was classified as Category 4 because
Figure 12. Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 7: Responds With Partial or Incomplete Reasoning.
no reasoning was provided. Only after he explained why he chose Player B in line 185f, was his response classified as Category 8.

*I Would Choose B, Lesson 9, Partner Talk*

185c Maggie: So what do you guys think? What’s your first reaction, Gavin? Who would you choose? A or B?
185d Gavin: I would choose B.
185e Maggie: And why would you choose B?
185f Gavin: *Because of the higher average.* (Category 8)
185g Maggie: Because of the higher average. Does he have a higher average?
185h Gavin: (inaudible)
185i Maggie: Is that enough? Is that enough reason? Is there any other reason why you’d choose B?
185j Gavin: I don’t know.
185k Maggie: But that’s good. I would tell you that most people’s first initial thing is let’s look at their average and choose that.

In the second excerpt, Matt determined the location of site 2 of the Archeology Mystery PBIT based on the arrowhead data. It took him several turns to express why he chose the location he did because his group members kept interrupting him as they recorded the response. Eventually, he was able to support his claim with a mathematical reason. Most of the sequence was classified as Category 8 because it was expressed as a singular sentence.

*Due To Our Data, Lesson 12, Small-Group Discussion*

089dh Matt: *Due to our data, we found that site two...* (Category 8)
089di Gavin: Site...
089dj Matt: *...two is like Big Goose Creek and Wortham Shelter because...* (Category 8)
089dk Gavin: To which ones?
089dl Matt: *...Big Goose Creek and Wortham Shelter....* (Category 8)
089dm Gavin: I’m pretty sure that’s how you spell goose.
089dn Matt: There’s two o’s.
089do Gavin: Yeah, I know that.
089dp Matt: *...Goose Creek and Wortham Shelter... OK.* (Category 8)
I made a heart straw for a cherry.

All right. You got that?

Um...hm.

...Wortham Shelter because it is bunched together, er, it has a smaller range... just cause they all have a smaller range.

(Category 8)

In the third excerpt, Diana and Ava determined slope from the equation describing the least squares regression line for the Archeology Mystery PBIT and predicted what information the slope conveyed. Even though Diana elaborated on her response in line 144e, she reiterated the same explanation. Both responses were classified as Category 8.

Which Number is Slope and What Can You Predict? Lesson 14, Partner Talk

144a Diana: OK. Slope is the one before “w.”
144b Ava: Point four nine six.
144c Diana: Yeah. And it tells you that the width is longer than the neck width. (Category 8)
144d Ava: What?
144e Diana: It tells you that... look it... see that's width. No it's bigger so that means the width is always bigger than the neck width. Which makes sense. We could figure that out before. (Category 8)

The graphs shown in Figure 13 illustrate students' use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 8 across the lessons.

Category 9: Responds with complex reasoning. At times, students responded with more complex reasoning by providing two, three, or more reasons to support their thinking. These responses were usually well thought out and explained clearly. Sometimes students shared multiple reasons over time. If the second reason followed along the same line of thinking as the original response, but provided additional
Figure 13. Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 8: Responds With Simple Reasoning.
reasoning than expressed earlier, then the second response was classified as Category 9.

At other times, students shared multiple reasons during a single turn.

In the first excerpt, Diana determined disadvantages of systematic sampling by surveying every fourth person in the cafeteria line. In the first response, she provided one disadvantage: that surveying only those in the cafeteria line would exclude people who brought their lunch. Her initial response was classified as Category 8. Then she provided two more reasons in the second response. She thought this method wouldn’t provide much information and it would include only those who wanted to take the survey. Since she provided two additional responses, the second response was classified as Category 9.

Disadvantages, Lesson 6, Partner Talk

090aw Diana: Disadvantage, people who don’t eat so it excludes people who eat... who bring their own lunch. It excludes people who eat... who bring their own lunch. So it excludes people who bring their own lunch. So that’s the population you would survey. (writes) Excludes the people...

090ax Ava: It’s random. (inaudible due to background noise)

090ay Diana: So there’s not a lot of information. It’s going to be people who want to take it. Only people who want to take it so not enough information. (Category 9)

In the second excerpt, Chloe provided multiple reasons in a single response for choosing Player B and multiple reasons in a single response for not choosing Player A for her basketball team for the activity Who Would You Choose? Both responses were scored as Category 9.

We Needed Player B, Lesson 9, Partner Talk

187c Chloe: OK, so we thought we needed Player B because he has a larger mean, a larger median, and first quartile. And he
may not have scored the most points, but he is consistent because he has all double-digit games. (Category 9)

187d Mr. Dunn: Good. So why did you not choose Player A even though he scores...

187e Chloe: Player A. Even though he scored 28 points a game, six or seven games of his, he only he only had single-digit games and so that really brought down his average. And so maybe he had some good games here and there, but do you want the ones that you don’t know what they’re going to do or do you want the ones that are consistent? (Category 9)

In the third excerpt, Tyler and Diana compared arrowhead data for the Archeology Mystery PBIT. Diana provided a single reason for selecting one of the archeology sites in her first response, which was classified as Category 8. Tyler provided multiple reasons for selecting a second site, classified as Category 9. Diana followed up by providing additional support for her data in line 089e. Her second response was also classified as Category 9.

Archeology Sites, Lesson 12, Small-Group Discussion

089c Diana: I said that the maximums for all those up here were a lot bigger than anywhere else.

089d Tyler: OK. I said the minimums were almost the same. And the maximums were... Like the IQR. (points to box plot) It overlaps that... almost it overlaps and the median here is in between those. And so is the third quartile. (Category 9)

089e Diana: I also said that the minimums were really close to this end. And all of these overlap. (Category 9)

The graphs shown in Figure 14 illustrate students’ use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 9 across the lessons.

Category 10: Self-Checking Response. As the unit progressed, some students began to cross-check their response against sources to determine whether they responded accurately. Other students self-corrected errors in thinking. Responses in which there
Figure 14. Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 9: Responds With Complex Reasoning.
was evidence of self-checking or self-correcting were classified as Category 10. No response was identified in this category until Lesson 5.

In the first example, Maggie provided students time to complete homework so the lesson could move forward. While working on her assignment, Evie noticed additional information that would help in constructing her box plot for the movie/sleep data. While many of her responses in this sequence were classified as mathematical procedures, the self-checking response was recorded as mathematical reasoning Category 10 because the new information caused her to rethink her original ideas and make a self-correction. In this excerpt, Evie noticed the additional information and announced her observation to the university participant. When Maggie stopped by the partnership to check in with Dan, Evie’s partner, Evie announced the self-correcting behavior to her teacher, too.

*Oh, I Didn’t See This*, Lesson 8, Partner Talk

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2ao</td>
<td>Mr. Dunn:</td>
<td>There shouldn’t be any decimals for your movies watched.</td>
</tr>
<tr>
<td>2ap</td>
<td>Evie:</td>
<td>It’s not. These are sleep. <em>Oh, I didn’t see this.</em> (Category 10)</td>
</tr>
<tr>
<td>2aq</td>
<td>Mr. Dunn:</td>
<td>You’re fine.</td>
</tr>
<tr>
<td>2ar</td>
<td>Maggie:</td>
<td>(whispers) Dan, do you know what you’re doing right now?</td>
</tr>
<tr>
<td>2as</td>
<td>Evie:</td>
<td><em>I actually made a mistake.</em> (Category 10)</td>
</tr>
<tr>
<td>2at</td>
<td>Maggie:</td>
<td>OK.</td>
</tr>
<tr>
<td>2au</td>
<td>Evie:</td>
<td><em>I’m fixing it.</em> (Category 10)</td>
</tr>
</tbody>
</table>

In the second example, group members were charged with gathering sleep/movie data from each other. Diana looked over her work and realized her data didn’t make sense. She had misread a comma as a decimal point when she had determined the median hours of sleep. In the following excerpt, Diana cross-checked information and self-corrected her error.
It's a Comma, Lesson 8, Small-Group Discussion

271aa Diana:  

*What's the middle between seven point eight and...? Eight point five. Wait. Why is there eight point...? Never mind. It's not eight point eight, it's eight comma eight.* (Category 10)

In the third example, Diana predicted a bowler's shoe size by correlating this data with his height of 74 inches using a least squares regression line placed on a scatterplot.

Then, Diana self-checked her prediction by recalculating the bowler's shoe size using the linear equation. She acknowledged that she would multiply in line 238c, prior to Maggie's suggestion to use a calculator. Diana didn't complete the self-check process until line 240f.

*Based Off the Line, What Would You Predict and Why?* Lesson 14, Partner Talk

238a Diana:  

Eleven and a half.

238b Ava:  

(chuckles)

238c Diana:  

You multiply it later.

238d Ava:  

Thanks, Diana.

238e Diana:  

That's OK. (inaudible) one and two, you just add.

238f Diana:  

Wait. What did you say? Eleven and a half?

238g Diana:  

Yeah. I just used the line.

239 Maggie:  

You may use a calculator. I might even want a calculator on this one.

240a Ava:  

What?

240b Diana:  

I just used the line.

240c Ava:  

(inaudible) really bad.

240d Diana:  

I'm not sure (inaudible)

240e Ava:  

What?

240f Diana:  

*Seventy-four times four equal to...* (checks answer with multiplication; Category 10)

The graphs shown in Figure 15 illustrate students' use of mathematical reasoning during Partner Talk and Small-Group Discussion for Category 10 across the lessons.
Figure 15. Students’ Use of Mathematical Reasoning During Partner Talk and Small-Group Discussion Category 10: Self-Checking Response.
It wasn’t enough for Maggie’s students to simply talk during her math class. She expected them to use mathematical words and expected them to share the mathematical reasons for their claims. In other words, she expected them to talk and reason like mathematicians within the conceptual Discourse. As students became more fluent in this Discourse, it appeared as if their reasoning become more refined and complex. Though results were tentative, the data suggested that, over time, students decreased their use of “I don’t know” statements and reduced the amount of time that they personalized the data. This occurred while students increased their content knowledge and data analysis skills as they engaged in the unit. Students sought understanding from others, expressed more complex reasoning, and initiated self-checking behavior as the unit progressed. They were reasoning more mathematically. These findings were consistent with Cobb’s research (see Sfard et al., 1998) in which he noted, “We have found that, within a few weeks, most students routinely give conceptual explanations as the need arises and that they ask other clarifying questions that bear directly on their underlying task interpretations” (p. 47).

**Off-topic talk.** It is a rare adolescent who is able to stay focused on an academic topic 100% of the time, especially when left to work independently with peers. Chapin et al. (2009) observed that, “students can spend time on off-task talk” (p. 21) during Small-Group Discussion and Maggie acknowledged it happened as well. While she believed that students should *talk* about mathematics and she provided substantial amounts of time for them to do so, she had no guarantee that they actually did. When I asked her about her expectations for student talk in her classroom, Maggie replied,
They Might Come In and Out of It, Interview 1

008 Maggie: One, that they actually do talk about math. They will want to veer from that because I do allow them lots of time to talk. And I think like us, as adults, you come into the concentrated talk and then you go back out. So know that there’s some allowance that I give for that. And, because I’ve been thinking about this a lot, ’cause I’m pretty quick to like... “No! You’re not talking about math! You’re not talking about math! Get talking about math!” And then I was thinking about how much time that I really have to diverge my thinking for a little bit from talking about whatever I’m talking about and then I come back to it. And so I’ve been trying to relax my response to that, because I think that... that response is... that’s my first response... “Stay focused!” But it’s not the most productive response. So, student talk... I want them to be talking about math, but they might come in and out of it.

Students did, on occasion, come in and out of talking about math, but this off-topic talk served three purposes for students in Maggie’s classroom: as a filler if they finished a task before the allotted time was up, as small talk while working on mathematical procedures, and as a diversion if they didn’t know what they were supposed to do or didn’t know how to do it.

Students entered into off-topic talk when they had finished their discussion of the question posed or task assigned during Partner Talk or Small-Group Discussion and chatted about other things. In the following excerpt, Chloe and Tony discussed the similarities and differences between two box plots that represented the number of hours students slept on Sunday night versus Saturday night with Miss Miller, a university participant, for 10 consecutive turns. After they brought that discussion to a close, Chloe and Tony proceeded to talk about Miss Miller’s last day, Chloe’s relative’s wedding plans, Chloe’s mother’s wedding, which trailed off to stories about her mother’s
engagement, and accounts of other engagement narratives. These topics are noted below with the initial turn indicating a new topic. The entire off-topic conversation was not transcribed, as off-topic talk was not the focus of the study. Maggie allowed three and a half minutes for students to compare the box plots.

*We Finished Early, Lesson 5, Partner Talk*

210a Chloe: You know that’s funny between the days because there’s a huge difference between the minimum and the maximum but the medium... the median is still a quarter of an hour off. That is weird!

210b Miss Miller: Why do you think that?

210c Tony: ’Cause the data was like really...

210d Chloe: ’Cause people don’t want to disrupt their sleep schedule as much.

210e Miss Miller: Yeah. Most people want to get kind of the same amount of sleep.

210f Tony: Yeah. I’m not one of those people.

210g Miss Miller: Yeah, and then there are some people who...

210h Chloe: Stay up?

210i Miss Miller: Or sleep on the weekends. And then there’s some people that don’t get any sleep on the weekends.

210j Chloe: And there’s people who stay with their same schedule pretty much.

210k Tony: You know it’s her last day, right?

210r Chloe: She’s [Chloe’s relative] all ready checking it [reserving church for the wedding].

210ab Chloe: My mom got married in that church.

201ak Chloe: I remember when she got engaged at [church], too. It was so cute.

210ao Chloe and Tony: (continued to talk about engagement stories)

Students also participated in off-topic conversation was when it served as small talk while completing mathematical procedures. In the following excerpt, Diana, Ava,
and Evie worked as a small group to construct their box plots for the sleep/movie data, when Evie interjected a question about a shopping trip. Evie continued counting to determine the size of her data set in order to locate the median even while chatting about the mall.

*Did You Go To the Mall?* Lesson 10, Small-Group Discussion

219ag Evie: One, two, three, four, five. One, two, three, four... So did you go to the mall?
219ah Ava: Just write the thing down!
219ai Evie: Did you go to the mall?
219aj Ava: Yeah.
219ak Evie: One, two, three, four, five six. I tried to wave, but you didn’t see me.
219al Ava: Was I with my cousin and my other cousin?
219am Evie: Four, five, six. Yes. That’s so ironic. On number three, there’s three.

The third reason that students engaged in off-topic talk was when they didn’t understand the question or know how to do a task. This off-topic talk was often related to Category 1: I don’t know, and was most concerning for Maggie. In the following excerpt, she asked students, “Could you make a type of graph that represented what was your favorite activity?” Andrew, Connor, and Matt didn’t know of any graphs, so they talked about the spring season sports in which they planned to participate. Miss Miller, a university participant, helped them get back on topic.

*I Don’t Know How I Would Do It,* Lesson 1, Partner Talk

216d Connor: Really? Are you doing track?
216e Matt: No, I’m doing golf.
216f Connor: Oh.
216g Miss Miller: Do you guys know what you’re supposed to be doing?
216h Andrew: No, I don’t.
216i Matt: Well, I do but I don’t know how I would do it. So...
216j Miss Miller: What kind of graph did you make?
I asked Maggie how she knew students understood the concepts or tasks she presented. Some students shared their thinking so explicitly that they left no doubt if they had questions.

_I Have a Couple Barometers, Interview 1_

009 Maggie: I have a couple of barometers of that, a couple students that are really good at letting me know and one is, I think I mentioned Ava .... Ava’s frustration... She’ll be like, “I don’t have a clue of what we’re supposed to be doing, Mrs. Baldwin! I don’t know.”

Most of the time, however, Maggie employed her keen listening and observational skills to determine if students needed additional guidance to get them going.

_I Listen, Interview 1_

009 Maggie: I just stop and I listen and see if they’re talking about what I want them to be talking about. [Then], there are a couple others also that are good about being on-task on the wrong thing for a while, for an extended period of time. Dan can be on-task on the wrong thing for a short period of time and Dylan can do that, too. But they don’t mean that and usually it’s a misunderstanding.... So there are a couple, but most of them will ask for clarity. Mm...hm. Most of them will. There are a couple that will just sit. I mean they won’t be doing anything! They’ll just sit and wait for it.... wait for it.... wait for it... see if she’ll come to me. Well, it’s not about the learning for them.

Students were not engaged in off-topic talk for any significant periods of time or for any other reasons during Partner Talk or Small-Group Discussion. Because it was not the focus of the study, not all of the off-topic talk was transcribed. However, notations were made when off-topic talk occurred and the nature of this talk in which students periodically engaged.
Reflection

As a result of the data and statistics unit, Maggie’s students became better consumers and producers of information. They read and interpreted graphical representations and developed the skills to create dot plots, line plots, histograms, and box-and-whisker plots. They acquired specific vocabulary to talk about the data and various statistics, including a five-number summary, and were able to use that vocabulary to share their thinking and reasoning with others. Considering that Maggie’s students knew very little when the unit started, they achieved quite a bit in 14 days.

They Will Get There, Interview 3

So do I think progress there was made? I’m just saying to the degree of which you want them... where you want them to be. They’re not there yet. So I do think that there was progress made? I just don’t know if it would be on the completely, “Hey, you’ve got this. I’m not worried about it any longer” stage.... Chloe moved greatly. Do I think Connor moved? He moved. But he didn’t move as much as I hoped. I think it was relative. They did move. They’re not... they’re not to where I want them to be yet. They will get there.

Summary

In order for students to be full participants in Mathematics Discourses, Maggie expected her students to be able to talk about mathematics, and talk and reason mathematically. I wanted to find out how the environment she created, as described in Chapter IV, impacted how much her students said and what they talked about. Therefore, the quantity and quality of students’ mathematical language was reported in this chapter.

Students talked nearly twice as much in Maggie’s classroom as in typical classrooms. Comparisons were made between student talk and teacher talk for three
lesson models and three talk formats. Student talk increased as lessons became less structured. Although Maggie spent most of her instructional time leading Whole-Class Discussion, the highest percentage of student talk occurred during Partner Talk and Small-Group Discussion. It didn’t matter whether I compared student talk with the amount of Maggie’s talk, calculated the density of talk in the classroom, or compared the percent of an individual student’s talk to different lesson models and talk formats over the course of the unit. Students were talking a lot during this less-structured time. Since Maggie played a diminished role during these talk formats, it wasn’t easy for her to ensure that the talk was mathematically productive. For these reasons, an in-depth analysis was conducted of student talk during Partner Talk and Small-Group Discussion to determine what students were talking about when they were working independently.

The qualitative substance of students’ talk changed over the course of the unit. Initially, students incorporated vague language and hedging (Rowland, 2000) as novice learners. As the unit progressed, they adopted the talk moves (Chapin et al., 2009) that Maggie modeled as their own, and began to critique the reasoning of others. Students also incorporated more mathematical vocabulary into their utterances as they demonstrated a greater understanding of the content. An in-depth analysis showed a statistically significant increase in students’ lexical density from the beginning to the end of the unit. They were talking more mathematically.

Further analysis of transcripts of Partner Talk and Small-Group Discussion revealed that students were engaged in two mathematical Discourses: Procedural Discourse and Conceptual Discourse (Setati, 2005; Sfard et al., 1998; A. G. Thompson et
al., 1994). It is common for students to talk about mathematical procedures in mathematics classrooms. Conceptual Discourse emerged as a focus of student talk when students were thinking about mathematical concepts while doing mathematics.

Given the focus of this study on how language was used to advance students’ conceptual understanding of mathematics in a reform-based mathematics classroom, and Maggie’s emphasis on mathematical reasoning, this type of talk became a focus of analysis. Ten categories of mathematical reasoning emerged from the data. Though results were tentative, the data suggested that, over time, students decreased their use of “I don’t know” statements and reduced the amount of time that they personalized the data. This occurred while students increased their content knowledge and data analysis skills as they engaged in the unit. Students sought to gain understanding from others, expressed more complex reasoning, and initiated self-checking behavior as the unit progressed. These results were consistent with prior research (Cobb in Sfard et al., 1998). Students were reasoning more mathematically.
Shorter days, cooler nights, and crimson sumac betrayed the final days of summer. Maggie’s students started their high school adventures last week as Maggie welcomed a new class of eighth grade students to talk about mathematics and to talk and think mathematically. I reflected on Maggie’s teaching and on last year’s eighth grade class, the students of the study. What did I learn from Maggie? What did I learn from her students? To fully appreciate these questions, they must be placed in context of the Mathematics Discourse; the circumstances that precede and follow an event that clarifies its meaning according to ways of “behaving, interacting, valuing, thinking, believing, speaking, ... reading and writing” (Gee, 2008, p. 3).

What Did I Learn From Maggie and Her Students?

I wanted to explore how Maggie and her students used language to advance her students' conceptual understanding of mathematics in a reformed based mathematics classroom. Specifically I sought to describe how Maggie’s reform-based mathematics classroom environment impacted the quantity and quality of mathematical language expressed by her students.

This newest reform effort, Maggie’s reform effort, raised questions about the values of a traditional mathematics education. Following a series of reformers, and those that influenced reform, (Eliot, Harris, Committee of Ten, Hall, E. L. Thorndike, Rice, Ward, Dewey, Kilpatrick, Snedden, Bobbitt, Charters, Prosser, Bestor, Hutchins, Conant,
NSF, Piaget, Vygotsky, NCEE, and NRC) that spanned more than a century, this latest movement, motivated by the National Council of Teachers of Mathematics (2000), redefined what constituted mathematics and advocated for new pedagogical practices (Klein, 2003). And unlike reform movements of the past, Maggie’s reform welcomed everyone into the student-centered Discourse where they talked and reasoned mathematically.

Johnston (2004) wrote that the language teachers (and students) use in their classroom is a “big deal” (p. 10). It is guided by their goals, beliefs, and knowledge (Ball, 1996; Brendefur & Frykholm, 2000; Schoenfeld, 1998) and it determines to a large extent what students learn (Fennema & Frank, 1992). Throughout the study we heard Maggie’s voice as she described her vision for mathematics education. We heard her enact this vision in her classroom, though not without its challenges. Time was Maggie’s worst enemy.

Maggie often wrestled with how she spent her precious instructional time. She had only 52 minutes each day to meet the different learning needs of her students while negotiating the expectations of a mathematics department, the testing culture fostered by NCLB (2002), and a content that may not be as relevant to students in the twenty-first century as she’d like. Maggie knew that a “whole lot of people ... are not going to use the equation of a circle again” (004, Interview 2) and if they did, a Google search would yield 15.8 million hits on how to calculate it in only a tenth of a second. Instead, Maggie felt it was a higher priority to help her students be critical mathematical thinkers. After all, her students were the ones who would have to make sense of the 1.8 zettabytes of
information (Gantz & Rainsel, 2011) zipping along the global networks they navigated daily on their iPads and cell phones. Over the next two years the amount of information was expected to double, with only exponential growth expected into the future. She wanted her students to be able to be critical consumers of this data into adulthood.

Despite these parameters, Maggie believed that all her students were capable of learning mathematics and she did everything she could to be accessible for them, whether in person or virtually. She believed her students should actually do mathematics (Stein & Smith, 1998) and talk about mathematics (NCTM, 2000) as she sought to develop their expertise in the Standards for Mathematical Practice (CCSSI, 2010) in her classroom. To this end, Maggie set three goals for her students. She wanted her students to learn how to learn math, not be afraid of math, and see the beauty and structure of math as represented in the interconnectedness of it in their world and the world around them. She was able to enact these beliefs and goals in her classroom with ease due to her depth of her own content knowledge and sophistication of her content knowledge for teaching (Ball et al., 2008; H. Hill & Ball, 2009). Maggie was a continuous learner who engaged in multiple professional development opportunities that stretched her thinking and extended her own understanding of the content. She created an inter-generational learning community of eighth graders, undergraduate, and graduate students. In this context she taught a 14-lesson unit on data and statistics for which she wanted her students to be critical consumers and producers of information.

Maggie organized her lessons according to three talk formats: Whole-Class Discussion, Partner Talk, and Small-Group Discussion (Chapin et al., 2009). Within that
framework, she skillfully wove these talk formats with the lesson models of Direct Instruction, Guided Discovery, and Open-Ended Exploration (Rubenstein et al., 2004), variations of the typical reform-based three-phase lesson (Sherin, 2002; Stein et al., 2008). She supplemented her lessons with Meaningful Distributed Instruction (Rathmell, 2010) when appropriate. A special note was made of Maggie’s infrequent use of independent work as a contrast to the prominence it plays in the traditional classroom lesson (Huang et al., 2005).

Maggie’s classroom was inclusive. Everybody was expected to participate by attending and contributing to the discussion. She probed students’ thinking, and insisted that they support their claims with mathematical reasons. She achieved this through her use of teacher talk moves (Chapin et al., 2009), by posing higher-order questions (L. W. Anderson & Krathwohl, 2001) for students to discuss, and engaging her students in higher-level tasks (Stein & Smith, 1998). Maggie provided a language-rich environment for students to learn new mathematical vocabulary and engage in mathematical Discourses (Moschkovich, 2002; Setati, 2005).

As a result, students talked nearly twice as much in Maggie’s classroom (37%) as in typical classrooms (20%; McHoul, 1978). I compared the amount of talk for Maggie and her students for the three lesson models and three talk formats. Students increased their talk as lessons became less structured. Although Maggie spent most of her instructional time leading Whole-Class Discussion (74%), the highest percentage of student talk occurred during Partner Talk and Small-Group Discussion. It didn’t matter whether I compared student talk (60% Partner Talk; 63% Small-Group Discussion) with
the amount of Maggie’s talk (40% Partner Talk; 37% Small-Group Discussion), calculated the density of talk in the classroom (174.8 w.p.m. Partner Talk; 191.5 w.p.m. Small-Group Discussion), or compared the percent of an individual student’s talk to different lesson models and talk formats over the course of the unit (range 57%-89% for Partner Talk and Small-Group Discussion combined): Students talked a lot during this less-structured time. Since Maggie played a diminished role during these talk formats (Chapin et al., 2009), it wasn’t easy for her to ensure that the talk was mathematically productive (Cobb in Sfard et al., 1998). For these reasons, an in-depth analysis was conducted of student talk during Partner Talk and Small-Group Discussion to determine what students were talking about when they worked independently.

Upon the onset of the study, I was interested in whether it was possible to document shifts in students’ thinking and reasoning by listening to their talk as an indication that they were developing a conceptual understanding of the mathematics Maggie was teaching. In fact, the qualitative substance of students’ talk did change over the course of the unit. Initially, students incorporated vague language and hedging (Rowland, 2000) as novice learners. As the unit progressed, they adopted the talk moves (Chapin et al., 2009) that Maggie modeled as their own, they self corrected their misunderstandings, and they began to critique the reasoning of others (CCSSI, 2010). Students also incorporated more mathematical vocabulary into their utterances as they developed a greater understanding of the content. An in-depth analysis showed a statistically significant (p = 0.00042593) increase in students’ lexical density from the beginning to the end of the unit.
Further analysis of transcripts of Partner Talk and Small-Group Discussion revealed that students were engaged in two mathematical Discourses: Procedural Discourse and Conceptual Discourse (Setati, 2005; Sfard et al., 1998; A. G. Thompson et al., 1994). It is common for students to talk about mathematical procedures in mathematics classrooms (A. G. Thompson et al., 1994). Conceptual Discourse emerged as a focus of student talk when students were doing mathematics or talking about mathematical concepts (Setati, 2005), a fulfillment of Maggie’s beliefs. Since the focus of this study was on how language was used to advance students’ conceptual understanding of mathematics in a reform-based mathematics classroom, which aligned with Maggie’s emphasis placed on mathematical reasoning, this type of talk became a focus of analysis. Ten categories of mathematical reasoning emerged from the data. Though results were tentative, the data suggested that, over time, students decreased their use of “I don’t know” statements and reduced the amount of time that they personalized the data. This occurred while students increased their content knowledge and data analysis skills as they engaged in the unit. Students sought to gain understanding from others, expressed more complex reasoning, and initiated self-checking behavior as the unit progressed. Results regarding students’ clarifying questions and conceptual explanations were consistent with prior findings (Cobb in Sfard et al., 1998).

This study provided insight into reform-based mathematics education. Maggie was a teacher who walked the talk. In other words, her actions were consistent with her expressed beliefs and goals. Maggie’s belief system was grounded in the philosophical theory of social constructivism. She wanted her students to construct their own
understanding by *doing* mathematics, but she also wanted her students *talking* about their understanding of mathematics with others as they developed the Standards for Mathematical Practice as junior mathematicians. Her goals, instructional practices, curriculum decisions, and assessment methods were consistent with mathematics education research and in alignment with the NCTM (2000) *Principles and Standards for School Mathematics* and the *Common Core* (CCSSI, 2010). Her breadth and depth of content knowledge and content knowledge for teaching were obtained through a graduate degree in secondary mathematics and continually refined through on-going professional development.

Her interactions with students were purposeful as she engaged all of them in a community of inter-generational learners. During the unit on data and statistics, her students gained procedural skills in constructing graphical representations of information, but they also developed a conceptual understanding of what the graphs represented. Students incorporated mathematical vocabulary in their talk to communicate their ideas more precisely. They developed reasoning to explain their thinking. As a result, Maggie’s students expressed pride in their accomplishments as learners. In summary, Maggie provided a model of a reform-based mathematics classroom by successfully enacting NCTM’s (2000) vision for school mathematics.

**Why Does It Matter?**

This study contributes to basic research in the areas of educational leadership, literacy education, mathematics education, and teacher education by providing a description of how a teacher and her students used language to advance students’
conceptual understanding of mathematics in a reform-based mathematics classroom and specifically how that reform-based mathematics classroom environment impacted the quantity and quality of students' language. Several implications for practitioners can be derived from this study.

Many theoretical and philosophical articles concerning mathematics reform were found in professional journals, however, only a handful of actual studies were located that focused on the dialogic discourse advocated by reformists. These focused on teacher talk and the complexities involved in implementing and orchestrating classroom talk in a reform-based mathematics classroom (Berry & Kim, 2008; Chazan & Ball, 1995; Hamm & Perry, 2002; Nathan & Knuth, 2003; Sherin 2002; Wallach & Even, 2005). Only the university researchers were successful in doing so (Chazen & Ball, 1995; Leinhardt & Steele, 2005; Schoenfeld, 1998).

Two studies specific to the impact of the reform-based mathematics environment on the quantity of student talk were located (Baxter et al., 2001; Nathan & Knuth, 2003). No studies specific to the quality of student talk were found.

Research described the strong connection between vocabulary knowledge and comprehension (R. C. Anderson & Freebody, 1981; Carver, 2003; Davis, 1942, 1944, 1968, 1972; R. L. Thorndike, 1974; Whipple 1925) and general knowledge (Gee, 2008; Monroe & Panchyshyn, 1995; Stahl & Nagy, 2006), concluding that increased vocabulary enables increased thought. Yet, no research was located that studied students' vocabulary acquisition over time, as measured by lexical density or any other means, as an indicator of their increased gains in content knowledge and fluency with academic
discourse. Other studies of lexical density focused on comparisons of student talk with teacher talk, textbooks, and tests (Veel, 1999). One study was located that indicated procedural and conceptual discourses emerged from South African students' language as observed in one classroom (Setati, 2005), but no studies that further described student thinking or reasoning were found.

**Educational Leadership**

This study contributes to the Educational Leadership field in this continued era of accountability and calls for reform. This study provides a rich description of what a reform-based mathematics classroom looks like and sounds like for principals who are considering implementing mathematics reform in their school. When facilitated by a master teacher such as Maggie, this study provides school leaders with reasonable expectations for student engagement in academic discourse and conceptual understanding of mathematics content.

When learning leaders understand the reform-based mathematics they are asked to support, they are better able to engage students and teachers in their school in productive mathematical discussions about thinking and reasoning. Knowledgeable learning leaders are better able to make important hiring decisions regarding teachers grounded in mathematical theory and research, to guide and facilitate student learning, provide for enriching professional development experiences, and engage teachers in more productive evaluation conferences. Leaders are better able respond to community members who question why students' mathematics assignments look different than the ones they
remember. They are better able to understand and allocate resources and protect the precious instructional time Maggie relished.

**Literacy Education**

This study contributes to the field of Literacy Education in at least three ways. It distinguishes content area literacy from the new Literacy Studies, operationalizing theory into practice; informs practice by sharing results of inductive vocabulary instruction; and provides a way for those interested in language as thinking to categorize and identify shifts in student reasoning over time.

First, content area literacy has long been an interest of literacy educators, yet little research has been recently conducted regarding the application of literacy ideas in the mathematics content area. Most suggestions from literacy educators are not well received by mathematics educators (Siebert & Draper, 2008) due to messages that neglect, deemphasize, or misrepresent mathematics. Shanahan and Shanahan (2008) attempted to address this void, but maintained a traditional view of literacy when they conducted research in the areas of reading (proofs, problems, and textbooks) and writing (classroom notes) mathematics.

Framed in the field of new Literacy Studies, this current study provides a rich description of what it means to become a member and participant in the mathematics Discourse, expanding the traditional view of literacy as a cognitive process to that of a socio-cultural approach (Gee, 2008). Maggie’s reform-based mathematics environment was framed in a “cognitive, social, interactional, cultural, political, institutional, economic, moral, and historic context” (p. 2). Each of these socio-cultural contexts
impacted how much Maggie and her students talked and made a qualitative difference in what they had to say.

Second, Maggie's use of the inductive model of teaching to introduce new vocabulary provides literacy educators with an alternative way to introduce the meanings of unfamiliar words. Through inductive reasoning and subsequent language immersion, Maggie introduced 86 new vocabulary words into students' lexicon in 14 days. Students were able to incorporate this vocabulary into their own language as shown by the statistically significant increase in their lexical density from the beginning to the end of the unit. This increased vocabulary allowed them to be more precise when contributing their ideas and communicating their reasoning.

Third, this study documents a qualitative shift in Maggie's students' ability to think and reason over time as revealed by the language they used, both at the word and grammar level. Literacy educators have long known that "reading is thinking" (F. Smith, 2004, p. 191), but conceptually oriented mathematics educators, such as Maggie, want their students to think, too (A. G. Thompson et al., 1994). The hallmark of mathematical discourse is when students are able to justify and explain why they performed mathematical procedures or conceptualized a mathematical reason (Siebert & Draper, 2008). Initial student talk revealed uncertainty through the use of vague language, inclusion of hedging, imprecise vocabulary, and "I don't know" statements. Students agreed with whatever ideas their peers shared and responded without evidence of independent thought. Over time, as students gained content knowledge and skills and acquired an academic vocabulary to name their ideas, their word-choice became more
precise, and their sentences were more complete. Students began to question to seek additional understanding and offered conjectures of plausible thinking. As the unit progressed, students demonstrated a shift from partial to complete reasoning, moving from simple to more complex explanations. Ultimately, students verified and cross-checked information, resulting in self-correcting behavior. This study provides educators a way to categorize the progression of students’ reasoning and note change in their ability to express ideas over time.

**Mathematics Education**

This study contributes to the field of mathematics education, to practicing mathematics educators, and to pre-service mathematics education students entering the profession by providing a rich description of how Maggie and her students used language to develop her students’ conceptual understanding of mathematics and the impact a reform-based mathematics classroom has on the *quantity* and *quality* of students’ talk in that endeavor.

Mathematics reformists have tried for several decades to implement reform practices into mathematics classrooms (NCTM, 1989) with limited success. Theoretical articles presented a vision to mathematics teachers, but with little guidance on how to pull it off (Ball, 1996). Based on research of classroom practices, though worthy of its goals, many classroom teachers have found it difficult to do. Maggie is a leading exemplar of a classroom teacher who successfully implements reform-based mathematics and skillfully orchestrates dialogic discourse.
An important point to note regarding the implications of focusing on the quantity of student talk in this study, is that the goal of reform-based mathematics classrooms, and in Maggie’s classroom in particular, is that classroom discourse is not an end in itself, as Cobb cautions (in Sfard et al., 1998). Rather, the attention to the quantity of discourse is directly related to attending to the changing quality of students’ participation, and thus to their mathematical learning. The quantity of student talk means little if the quality of that talk is ignored.

It is also important to note to the degree to which Maggie’s philosophy was steeped in a theoretical framework, and the degree to which the depth and breadth of her own understanding of the context of mathematics education and the depth and breadth of her conceptual understanding of mathematical content knowledge grounded her. These played an instrumental role in her ability to articulate her beliefs and goals. In addition, her capacity to draw on various lesson models and lesson types, and incorporate talk moves that facilitated dialogic discourses allowed her to implement her unit with ease. Mathematics educators may want to self-evaluate their practice based on these criteria if they plan to successfully implement reform teaching in their classroom. If teachers find an area lacking in their own repertoire, they may consider additional professional development to enhance or refine their competencies.

It is obvious that the mathematics education students that participated in Maggie’s classroom did not have the expertise or experience that she did. Though they were not the focus of the study, the university mathematics education students who participated in Maggie’s classroom were similar to the novice teachers reported in Schoenfeld’s (1998)
and Frykholm's (1999, as cited in Brendefur & Frykholm, 2000) studies exhibiting more of a unidirectional communication pattern and a limited understanding of content knowledge for teaching. On more than one occasion, Maggie needed to interject into their lessons when they weren't sure how to provide a relevant example, unsuccessfully anticipated students' misconceptions, or were moving forward with their predetermined lesson plans without stopping to listen to students' questions or challenges. Mathematics education programs may find it helpful to provide undergraduate students with more opportunities to deepen their content knowledge for teaching (H. Hill & Ball, 2009) and practice teacher talk moves as a way to foster more dialogic communication patterns.

Teacher Education

This study contributes to the general field of teacher education by providing a rich description of the dialogic discourse patterns observed between Maggie and her students. Uni-directional/univocal discourse is common among many teachers (J. P. Smith, 1996) and is heard in up to 85% of beginning teachers (Frykholm, 1999, as cited in Brendefur & Frykholm, 2000). It is noted for its high quantity of teacher talk and initiation-response-evaluation (IRE) exchange patterns that accounts for 70% to 80% of teacher-student interactions (McHoul, 1978; Mehan, 1979; Nassagi & Wells, 2000; Sinclair & Coulthard, 1975; Wells, 1999).

In order to increase student participation, and consequently increase student learning (Sfard et al., 1998), implementing a dialogic teacher-student interaction pattern might be helpful in other subject areas, in addition to mathematics, especially where the goal is an inquiry-based classroom and where students are encouraged to contribute to
class discussion. Maggie regularly used teacher talk moves in order to support productive talk. This study provides examples of how Maggie implemented these talk moves and how students spontaneously adopted them as their own as they interacted with others.

Where Do We Go From Here?

Because few research studies have been conducted regarding the impact of the reform-based mathematics environment on the quantity and quality of student talk, there are many possibilities for further study. Three specific areas for consideration are presented.

First, this study was conducted in a single classroom for 14 days. It would be important to replicate it in different classrooms across different grade levels, mathematical areas, and time periods to determine if similar teacher-student interactions are established and whether the results are consistent. It would also be important in these subsequent studies to establish inter-rater reliability by having a second rater score the transcripts in order to validate the results.

Second, it would be of value to conduct a comparative study in a traditional mathematics classroom. Prior research would suggest that students do not talk as much in traditional classrooms, but no research could be found that studied the quality of student talk in these classrooms. It would be helpful in drawing conclusions from the current study to be able to compare the quality of student talk in a reform-based classroom with the quality of student talk found in a traditional classroom.
Third, the transcripts of the study provide rich data to be reexamined for different purposes. It was established in this study that students’ lexical density increased over time and that their reasoning could be categorized. A future study that would be useful is to examine whether there is a correlation between these two findings. Also, it would be useful to reanalyze the teacher-student interactions that precipitated these changes to determine what interactions may have had the greatest effect on students’ learning. Possible variables to explore include the frequency of student participation, differentiated teacher attention, cognitive complexity of the task or question posed, or specific teacher-to-student or student-to-student talk moves.
REFERENCES


*Amendment to Education of the Handicapped Act, PL 94-192 (S6) (1975).*


instruction in textbooks as part of the equation. *Learning Disability Quarterly, 31*(1), 21-35.


Joyce, B., & Showers, B. (2002). *Student achievement through staff development* (3rd ed.). Alexandria, VA: ASCD.


National Reading Panel. (2000). *Teaching children to read: An evidence-based assessment of the scientific research literature on reading and its implications for


