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# Some convergence properties of Minkowski functionals given by polytopes

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SOME CONVERGENCE PROPERTIES OF MINKOWSKI FUNCTIONALS  
GIVEN BY POLYTOPES

An Abstract of a Thesis  
Submitted  
in Partial Fulfillment  
of the Requirement for the Degree  
Master of Arts

Jesse Moeller  
University of Northern Iowa  
May 2016

## ABSTRACT

In this work we investigate the behavior of the Minkowski Functionals admitted by a sequence of sets which converge to the unit ball ‘from the inside’. We begin in  $\mathbb{R}^2$  and use this example to build intuition as we extend to the more general  $\mathbb{R}^n$  case. We prove, in the penultimate chapter, that convergence ‘from the inside’ in this setting is equivalent to two other characterizations of the convergence: a geometric characterization which has to do with the sizes of the faces of each polytope in the sequence converging to zero, and the convergence of the Minkowski functionals defined on the approximating sets to the Euclidean Norm. In the last chapter we explore how we can extend our results to infinite dimensional vector spaces by changing our definition of polytope in that setting, the outlook is bleak.

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This Study by: Jesse Moeller

Entitled: SOME CONVERGENCE PROPERTIES OF MINKOWSKI FUNCTIONALS  
GIVEN BY POLYTOPES

Has been approved as meeting the thesis requirement for the  
Degree of Master of Arts.

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Date

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Dr. Douglas Mupasiri, Chair, Thesis Committee

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Date

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Dr. Olena Ostapyuk, Thesis Committee Member

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Dr. Theron Hitchman, Thesis Committee Member

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Date

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Dr. Kavita Dhanwada, Dean, Graduate College

*I dedicate this work to my parents and to my grandparents, whose love and support have been both sufficient and necessary.*

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## CHAPTER 1

### INTRODUCTION

In a geometry course, one might spend some time looking at the shapes unit balls produced by various norm functions and metrics. It is also possible, however, to go the other direction. That is, to take a shape (if it is nice enough) and define a norm with it. It is quite simple, really: set a point  $x$  somewhere in your space and pick a nice shape to sit at the origin. Now, scale the shape with positive numbers until the shape just barely touches  $x$ . Once this is achieved, record the final scalar that is used. This scalar is taken to be the ‘norm’ of  $x$  with respect to the shape that is used. What we have just described is how the Minkowski functional works. Quite naively, the question we explored in this paper is this: If we have some sets which approximate the unit ball, do the Minkowski functionals admitted by these sets approximate the Minkowski functional admitted by the unit ball (the euclidean norm)?

The main theorem of this paper does not connect to the popular literature in a way that would be typically worth mentioning in the introduction. The question that is asked above is first pursued in Chapter 3 in  $\mathbb{R}^2$  where the approximating sets are regular polygons. It was immediately obvious when I moved into  $\mathbb{R}^3$  that there wasn’t a good generalization of ‘regular’ polytopes in higher dimensions, thus the generalization that is made in higher dimensions abandons this notion of regularity. The initial  $\mathbb{R}^2$  results proved in Chapter 3 are successfully extended to  $\mathbb{R}^n$  in Chapter 5. Chapter 2 is dedicated to defining the traditional concepts related to topological vector spaces and it is also dedicated to proving some simple, but necessary, theorems about seminorms and Minkowski functionals. We spend Chapter 4 proving properties of convex sets which were taken for granted in Chapter 3.

Chapter 4 is a showcase of some useful properties of convex sets in a general, not necessarily finite dimensional, setting. In particular, the main theorem decomposes the boundary of a convex set into the union of the faces. The main theorem also decomposes

the vector space into the ‘facial cones’ which are made by the convex set. And lastly, the main theorem in Chapter 4 also decomposes the convex set itself into a union of convex hulls of the faces joined with an interior point.

The final theorem in Chapter 5 is a three way equivalence which characterizes the convergence of Minkowski functionals defined on polytopes to the norm function in finite dimensional spaces. It relies on a lemma which states that between any two concentric balls there lies a convex polytope, with respect to set containment. All of these statements together make for a colorful mix of standard analysis techniques which are accessible to any graduate student who has taken a real analysis course.

Chapter 6 is dedicated to exploring the future work related to this problem. An obvious area of exploration would be to extend results to infinite dimensional spaces. We review the work we have done and identify which properties restrict us to  $\mathbb{R}^n$ . Additionally, we explore how to generalize the notion of a convex polytope in infinite dimensional spaces.

CHAPTER 2  
SEMINORMS AND THE MINKOWSKI FUNCTIONAL

In functional analysis the earliest encountered structure is the topological vector space. A topological vector space is a vector space which has been endowed with a topology that respects the operations of vector addition and scalar multiplication. Requiring the topology to satisfy the  $T_1$  separation axiom (or, equivalently, requiring every singleton to be closed), forces the topology to be  $T_2$ . The original definitions and arguments that we present in this chapter, besides Lemma 2.8 and Lemma 2.9, can be found in [5]. However, I have filled in many of the dismissed details.

**Definition 2.1** (Topological Vector Space). *If  $X$  is a vector space over the scalar field  $\Lambda$ , usually  $\mathbb{R}$  or  $\mathbb{C}$ , and  $\tau$  is a topology on  $X$  then the pair  $(X, \tau)$  is a topological vector space if*

- i)  $x \in X \Rightarrow \{x\}$  is closed in  $\tau$ .*
- ii) The mapping  $+$  :  $X \times X \rightarrow X$  is continuous.*
- iii) The mapping  $\cdot$  :  $X \times \Lambda \rightarrow X$  is continuous.*

We say that a topological vector space with a **norm** is a normed vector space, or a normed linear space.

**Definition 2.2** (Norm). *A norm on a topological vector space  $X$  is a nonnegative function such that*

- i)  $p(\lambda x) = |\lambda|p(x)$*
- ii)  $p(x + y) \leq p(x) + p(y)$*
- iii)  $p(x) = 0 \Rightarrow x = 0$*

There are topological vector spaces which are not normable. However, on any vector space it is possible to define the concept of a seminorm, which is weaker than that of a norm. A **seminorm**  $p$  is a real valued function which only satisfies i) and ii) from Definition 2.2. Before we continue it is necessary to establish some terminology.

**Definition 2.3** (Absorbing, Balanced, Convex). *Let  $X$  be a vector space with field  $\Lambda$ .*

- i)  $A \subset X$  is absorbing if for every  $x \in X$  there is a nonnegative number  $t$  so that  $x \in tA$ .*
- ii)  $A \subset X$  is balanced if for every  $\lambda \in \Lambda$  with  $|\lambda| \leq 1$  we have  $\lambda A \subset A$ .*
- iii)  $A \subset X$  is convex if it is closed under convex combinations. That is, for every pair  $x, y \in A$ , the set of all convex combinations of  $x$  and  $y$ , denoted  $[x, y]$  and given by  $\{tx + (1 - t)y : t \in [0, 1]\}$ , is a subset of  $A$ .*

Note that any vector space  $X$  contains a trivially absorbing set, all of  $X$  itself. So on any vector space we may define the following functional, which is the subject of our investigation in this paper:

**Definition 2.4** (Minkowski Functional). *Let  $A \subset X$  be an absorbing set. Then the Minkowski Functional of  $A$ , denoted  $\mu_A$ , is given by*

$$\mu_A(x) = \inf\{t > 0 : x \in tA\}$$

The Minkowski functionals are important because they connect seminorms to their unit balls in the following sense: take a seminorm and consider the set of points where that seminorm has value less than 1 (i.e. generate the unit ball). Now, view this unit ball as an absorbing set. It turns out that the Minkowski functional determined by this absorbing unit ball is actually the seminorm that we started with. This is the content of the next theorem:

**Theorem 2.5.** *Let  $X$  be a vector space and let  $p$  be a seminorm on  $X$ . Then  $B = \{x : p(x) < 1\}$  is an absorbing, balanced, and convex set. Moreover,  $p(x) = \mu_B(x)$ .*

Proof. Let  $x \in X$ . Let  $s \in \Lambda$  so that  $p(x) < s$ . Then  $p(s^{-1}x) = s^{-1}p(x) < 1$ . This implies that  $s^{-1}x \in B$ , so we get  $x \in sB$ , thus  $B$  is absorbing. Let  $\lambda \in \Lambda$  with  $|\lambda| \leq 1$ . Then for any  $x \in B$  it follows that  $p(\lambda x) = |\lambda|p(x) \leq p(x) < 1$ , which means that  $B$  is balanced. Let  $x, y \in B$  and let  $t \in [0, 1]$ . Then

$$\begin{aligned} p(tx + (1-t)y) &\leq p(tx) + p((1-t)y) \\ &= tp(x) + (1-t)p(y) \\ &< t + 1 - t = 1 \end{aligned}$$

and thus  $B$  is convex. Now we need to show that  $p(x) = \mu_B(x)$ . Suppose that  $p(x) < \mu_B(x)$ . Then let  $s \in \mathbb{R}$  so that  $p(x) < s < \mu_B(x)$ . From the left inequality we get that

$$\begin{aligned} p(x) &< s \\ s^{-1}p(x) &< 1 \\ p(s^{-1}x) &< 1 \end{aligned}$$

which means that  $s^{-1}x \in B$ . From this we get  $x \in sB$ . Since  $\mu_B(x)$  is the infimum of such numbers,  $\mu_B(x) \leq s$ . But this contradicts  $s < \mu_B(x)$ . Hence it must be true that  $p(x) \geq \mu_B(x)$ . Now suppose that  $\mu_B(x) < p(x)$ . Since  $\mu_B(x)$  is defined as the greatest lower bound of  $\{t > 0 : x \in tB\}$ ,  $p(x)$  cannot be a lower bound for this set. Hence let  $t \in \mathbb{R}$  with  $\mu_B(x) \leq t < p(x)$  so that  $x \in tB$ . From this inequality we get that

$$\begin{aligned} t &< p(x) \\ 1 &< t^{-1}p(x) \\ 1 &< p(t^{-1}x) \end{aligned}$$

which means that  $t^{-1}x \notin B$ , thus  $x \notin tB$ . This is a contradiction, hence  $\mu_B(x) \geq p(x)$  and  $\mu_B(x) = p(x)$ .  $\square$

**Example 2.6** (Taxicab Norm). *If  $z \in \mathbb{R}^2$  and  $z = (x, y)$  then the Taxicab norm of  $z$  is given by  $\|z\|_T = |x| + |y|$ . The previous theorem tells us that the unit ball should be absorbing, balanced, and convex. We can compute the unit ball directly:*

$$\begin{aligned} \|z\|_T &< 1 \\ |x| + |y| &< 1 \\ |y| &< 1 - |x| \\ |x| - 1 &< y < 1 - |x| \end{aligned}$$

This last inequality yields the diamond shape in Figure 2.1 and is the “unit ball” for the Taxicab Norm. But, suppose that we did not start out with a seminorm as Theorem 2.5 requires. Suppose that we simply started just started out with an absorbing, balanced, and convex set. Using the Minkowski functional, we obtain a seminorm.

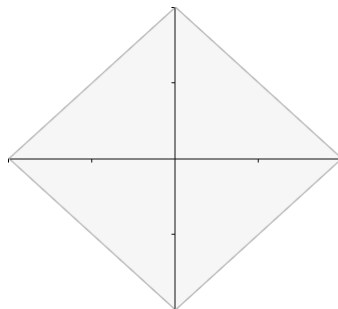


Figure 2.1: The Taxicab Unit Ball in  $\mathbb{R}^2$ .

**Theorem 2.7.** *Let  $A$  be an absorbing, balanced, and convex set. Then  $\mu_A(x)$  is a seminorm.*

*Proof.* We will begin by proving property i) from Definition 2.2 but we first prove a weaker statement. Let  $A$  be an absorbing, balanced, and convex set.



**Claim.** For  $\lambda \geq 0$ ,  $\mu_A(\lambda x) = \lambda \mu_A(x)$

Proof of Claim. Let  $x \in X$ . Suppose that  $\lambda = 0$ . Since  $A$  is absorbing we know that  $\mu_A(0) = 0$ . Then

$$\mu_A(\lambda x) = \mu_A(0x) = \mu_A(0) = 0 = 0\mu_A(x) = |\lambda|\mu_A(x)$$

as desired. Now suppose that  $\lambda > 0$ . We want to show that the following equality holds:  $\{t > 0 : \lambda x \in tA\} = \lambda\{t > 0 : x \in tA\}$ . Let  $t_0 > 0$  such that  $\lambda x \in t_0A$ . By the continuity of scalar multiplication,  $x \in \frac{t_0}{\lambda}A$ . Since both  $t_0$  and  $\lambda$  are positive numbers,  $\frac{t_0}{\lambda}$  is a positive number. Thus  $\frac{t_0}{\lambda}$  meets the criterion for belonging to  $\{t > 0 : x \in tA\}$ . Since  $\frac{t_0}{\lambda} \in \{t > 0 : x \in tA\}$ ,  $t_0 \in \lambda\{t > 0 : x \in tA\}$ . This shows the first inclusion, that  $\{t > 0 : \lambda x \in tA\} \subset \lambda\{t > 0 : x \in tA\}$ . Now let  $t_0 > 0$  such that  $x \in t_0A$ . Again since  $\lambda$  and  $t_0$  are positive,  $\lambda t_0$  is positive. Additionally,  $x \in t_0A \iff \lambda x \in \lambda t_0A$ . Note now that  $\lambda t_0$  meets the criterion for belonging to  $\{t > 0 : \lambda x \in tA\}$ . The fact that  $\lambda t_0$  originated from  $\lambda\{t > 0 : x \in tA\}$  gives us our second inclusion, hence  $\{t > 0 : \lambda x \in tA\} = \lambda\{t > 0 : x \in tA\}$ . Recall that the infimum is positive homogeneous:

$$\begin{aligned} \mu_A(\lambda x) &= \inf\{t > 0 : \lambda x \in tA\} \\ &= \inf \lambda\{t > 0 : x \in tA\} \\ &= \lambda \inf\{t > 0 : x \in tA\} \\ &= \lambda \mu_A(x) \end{aligned}$$

This ends the claim.

So now suppose that  $|\lambda| > 0$ . The balanced property of  $A$  tells us that for any  $t > 0$ ,  $\lambda x \in tA \iff |\lambda|x \in tA$ . Therefore we get that

$\{t > 0 : \lambda x \in tA\} = \{t > 0 : |\lambda|x \in tA\}$  and hence  $\mu_A(\lambda x) = \mu_A(|\lambda|x)$  ( $\star$ ). We already

showed, however, that property i) holds for positive scalars in the Claim. Hence

$\mu_A(|\lambda|x) = |\lambda|\mu_A(x)$  ( $\star\star$ ). Combining ( $\star$ ) and ( $\star\star$ ) we get that

$\mu_A(\lambda x) = \mu_A(|\lambda|x) = |\lambda|\mu_A(x)$ . This completes the proof of property i). To show property ii), select  $t_0$  from  $\{t > 0 : x \in tA\}$  and  $s_0$  from  $\{s > 0 : y \in sA\}$  so that  $\mu_A(x) < t_0$  and

that  $\mu_A(y) < s_0$ . Since  $x \in t_0A$  and  $y \in s_0A$ ,  $t_0^{-1}x$  and  $s_0^{-1}y$  are both in  $A$ . We may write

$$\frac{x+y}{t_0+s_0} = \frac{x}{t_0+s_0} + \frac{y}{t_0+s_0} = \left(\frac{t_0}{t_0+s_0}\right)t_0^{-1}x + \left(\frac{s_0}{t_0+s_0}\right)s_0^{-1}y.$$

Thus  $\frac{x+y}{t_0+s_0}$  is a convex combination of  $t_0^{-1}x$  and  $s_0^{-1}y$ , which are both in  $A$ . Since  $A$  is convex,  $\frac{x+y}{t_0+s_0}$  is also in  $A$ . Thus  $x+y \in (t_0+s_0)A$ . Then we obtain that

$\mu_A(x+y) \leq t_0+s_0$ . Since we may repeat this process for  $t_0$  and  $s_0$  chosen to be arbitrarily close to  $\mu_A(x)$  and  $\mu_A(y)$  respectively, by the definition of these objects as an infimum, the desired result follows:  $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$ .  $\square$

Here are some practical properties of the Minkowski functional which we will use often

**Lemma 2.8.** *If  $A \subset B$  and both are absorbing, then  $\mu_B(x) \leq \mu_A(x)$ .*

Proof. Let  $x \in X$ . Then for any  $t \geq 0$  ( $A \subset B \Rightarrow tA \subset tB$ ). Thus since  $\mu_B(x)$  is a lower bound for the collection  $\{t > 0 : x \in tB\}$ , it is also a lower bound for the collection  $\{s > 0 : x \in sA\}$ . Since  $\mu_A(x)$  is the greatest lower bound of  $\{s > 0 : x \in sA\}$ , it follows that  $\mu_B(x) \leq \mu_A(x)$ , as desired.  $\square$

**Lemma 2.9.** *If  $p$  is a norm and  $B_r$  denotes the  $p$ -ball of radius  $r$  for  $r \in \mathbb{R}^+$ , then*

$$\mu_{B_r}(x) = \frac{p(x)}{r}$$

Proof. By direct manipulation:

$$\begin{aligned} \mu_{B_r}(x) &= \inf\{t > 0 : x \in tB_r\} \\ &= \inf\{t > 0 : \frac{x}{t} \in B_r\} \\ &= \inf\{t > 0 : p\left(\frac{x}{t}\right) < r\} \\ &= \inf\{t > 0 : p\left(\frac{x}{tr}\right) < 1\} \\ &= \inf\{t > 0 : \frac{x}{tr} \in B_1\} \end{aligned}$$

$$\begin{aligned} &= \inf\{t > 0 : \frac{x}{r} \in tB_1\} \\ &= \mu_{B_1}\left(\frac{x}{r}\right) = p\left(\frac{x}{r}\right) = \frac{p(x)}{r} \end{aligned}$$

□

We are curious: If two absorbing sets  $A$  and  $B$  are ‘close’ in some sense, does  $\mu_A \approx \mu_B$ ? In the next chapter we will approach this question through a particular example which is recognizable: we will approximate the unit ball in  $\mathbb{R}^2$  with regular inscribed polygons and we will observe that the Minkowski functionals associated with these inscribed polygons converge pointwise to the euclidean norm on  $\mathbb{R}^2$ .

CHAPTER 3  
SIMPLE CASE: REGULAR POLYGONS IN  $\mathbb{R}^2$

In this Chapter we will prove a unit ball approximation theorem in a familiar setting. Just as Archimedes used many-sided regular polygons to approximate the value of  $\pi$ , we will use many-sided inscribed regular polygons, denoted  $P_n$ , to approximate the  $\mathbb{R}^2$  euclidean norm  $\|x\|$  using  $\mu_{P_n}(x)$ . We begin in this setting so that as we move to prove the case in  $\mathbb{R}^n$  the  $n$ -dimensional polytopes will be easier to visualize. This first lemma illustrates an important strategy that we will return to: polygons (and polytopes) which have vertices in the boundary of the unit ball contain, as a subset, a ball of smaller radius. Once we have some way of determining this radius, we can develop a criterion which will yield a sequence of polygons which converge to the unit ball in a sense that we will define later. For the rest of this paper we will use the term polygon and polytope to mean convex polygon and convex polytope. The following lemma is obvious upon observing Figure 3.1, but we prove the lemma using the techniques that are used higher dimensions as to better prepare our intuition.

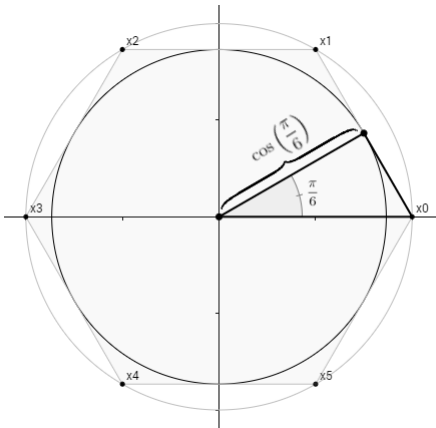


Figure 3.1: Regular 6-gon Inscribed in the Unit Circle  $\mathbb{R}^2$ .

**Lemma 3.1.** *In  $\mathbb{R}^2$ , let  $P_n$  denote the regular  $n$ -gon inscribed in the unit circle. Also, let  $B_r$  denote the set  $\{x \in \mathbb{R}^2 : \|x\| < r\}$ . Then  $B_{\cos(\pi/n)} \subset P_n$ .*

Proof. Let  $P_n$  be a regular polygon inscribed in  $B_1$  with vertices  $x_0 = \langle 1, 0 \rangle, \dots, x_{n-1} = \left\langle \cos\left(\frac{(n-1)2\pi}{n}\right), \sin\left(\frac{(n-1)2\pi}{n}\right) \right\rangle$  and let  $x \in B_{\cos(\pi/n)}$ . If  $x = 0$ , then it is clear that  $0 \in P_n$  for any  $n$ . Suppose instead that  $x \in B_{\cos(\pi/n)}$  and that  $x \neq 0$ . Observe that the collection  $\text{Arg}(x_0), \dots, \text{Arg}(x_{n-1})$  forms a partition of  $S^1$ . It follows that there is an  $i \leq n-1$  so that  $\text{Arg}(x_i) \leq \text{Arg}(x) < \text{Arg}(x_{i+1})$ , with the convention that  $x_{n-1+1} = x_0$ . Hence  $x$  is in the conical hull of  $x_i$  and  $x_{i+1}$ . That is, there exist nonnegative constants  $\alpha \geq 0$  and  $\beta \geq 0$  so that  $x = \alpha x_i + \beta x_{i+1}$ . By the continuity of scalar multiplication and addition, the function  $c : [0, 1] \rightarrow \mathbb{R}^2$  given by  $c(t) = tx_i + (1-t)x_{i+1}$  is a continuous function and hence  $\|c(t)\|$  is continuous. Let  $\theta_1 = \frac{i2\pi}{n}$  and  $\theta_2 = \frac{(i+1)2\pi}{n}$  so that  $x_i = \langle \cos(\theta_1), \sin(\theta_1) \rangle$  and  $x_{i+1} = \langle \cos(\theta_2), \sin(\theta_2) \rangle$ . Then observe that

$$\begin{aligned}
0 &= \frac{d}{dt} (\|c(t)\|) \\
&= \frac{d}{dt} (\|t\langle \cos(\theta_1), \sin(\theta_1) \rangle + (1-t)\langle \cos(\theta_2), \sin(\theta_2) \rangle\|) \\
&= \frac{d}{dt} \sqrt{(t \cos(\theta_1) + (1-t) \cos(\theta_2))^2 + (t \sin(\theta_1) + (1-t) \sin(\theta_2))^2} \\
&= \frac{d}{dt} \sqrt{t^2 + (1-t)^2 + 2t(1-t) \cos(\theta_2 - \theta_1)} \\
&= \frac{4t - 2 + 2(1-2t) \cos(\theta_2 - \theta_1)}{\sqrt{t^2 + (1-t)^2 + t(1-t) \cos(\theta_2 - \theta_1)}} \\
&= (2t - 1) - (2t - 1) \cos(\theta_2 - \theta_1)
\end{aligned}$$

Since  $\theta_2 - \theta_1 \neq 0$ , it follows that  $\frac{d}{dt} (\|c(t)\|) = 0 \iff t = 1/2$ . Since  $\|c(t)\|$  is continuous, the extreme value theorem tells us that the minimum value of  $\|c(t)\|$  on  $[0, 1]$  is  $\min\{\|c(0)\|, \|c(1/2)\|, \|c(1)\|\}$ . Plugging in these values we obtain that  $1 = \|c(0)\| = \|c(1)\|$ . Also, since  $\theta_2 - \theta_1 = \frac{2\pi}{n}$ , using a half-angle identity we get that

$$\|c(1/2)\| = \sqrt{1/2 + 1/2 \cos\left(\frac{2\pi}{n}\right)} = \cos\left(\frac{\pi}{n}\right) < 1$$

Hence any convex combination of  $x_i$  and  $x_{i+1}$  will have norm at least  $\cos\left(\frac{\pi}{n}\right)$ . But the norm of  $x$ , since  $x \in B_{\cos(\pi/n)}$ , is strictly less than  $\cos\left(\frac{\pi}{n}\right)$ . Therefore we have that  $\|x\| < \|tx_i + (1-t)x_{i+1}\|$  for all  $t \in [0, 1]$ . From this we determine that  $\alpha + \beta < 1$ . Otherwise, if  $\alpha + \beta \geq 1$ , we would have

$$\|x\| = (\alpha + \beta) \left\| \frac{\alpha}{\alpha + \beta} x_i + \frac{\beta}{\alpha + \beta} x_{i+1} \right\| \geq \left\| \frac{\alpha}{\alpha + \beta} x_i + \frac{\beta}{\alpha + \beta} x_{i+1} \right\| > \|x\|$$

which is clearly false. Thus  $x$  may be written as a convex combination of  $0$ ,  $x_i$  and  $x_{i+1}$ .

This is precisely what it means to belong to a sector of  $P_n$ . If  $x$  is in a sector of  $P_n$ , then  $x$  belongs to  $P_n$ .  $\square$

**Theorem 3.2.** *The functionals  $\{\mu_{P_n}\}$  converge to  $\|\cdot\|$  pointwise on  $\mathbb{R}^2$ .*

*Proof.* Let  $x \in \mathbb{R}^2$  and let  $\epsilon > 0$ . Since  $\{1 - \cos(\pi/n)\}_{n=3}^{\infty} \rightarrow 0$ , choose  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $1 - \cos(\pi/n) < \frac{\epsilon \cos(\pi/3)}{\|x\|}$ .

$$|\mu_{P_n}(x) - \|x\|| = \mu_{P_n}(x) - \|x\| \tag{3.1}$$

$$\leq \mu_{B_{\cos(\pi/n)}}(x) - \|x\| \tag{3.2}$$

$$= \frac{\|x\|}{\cos(\pi/n)} - \|x\| \tag{3.3}$$

$$= \frac{\|x\|}{\cos(\pi/n)} (1 - \cos(\pi/n)) \tag{3.4}$$

$$\leq \frac{\|x\|}{\cos(\pi/3)} (1 - \cos(\pi/n)) \tag{3.5}$$

$$< \frac{\|x\|}{\cos(\pi/3)} \frac{\epsilon \cos(\pi/3)}{\|x\|} = \epsilon \tag{3.6}$$

Since  $B_{\cos(\pi/n)} \subset P_n$  by Lemma 3.1, Lemma 2.8 tells us that  $\mu_{P_n}(x) \leq \mu_{B_{\cos(\pi/n)}}(x)$ , which justifies lines 3.1 to 3.2. Lines 3.2 to 3.3 are justified by Lemma 2.9. For 3.4-3.5, Since cosine is decreasing on  $[0, \pi]$ , we get that  $\frac{1}{\cos(\pi/n)} \leq \frac{1}{\cos(\pi/3)}$  for any  $n \geq 3$ .  $\square$

**Remark.** *The 3 in the previous argument is really just there to emphasize there are no regular 2-gons.*

It is easy to visualize what is happening in Lemma 3.1 because we are familiar with the picture and the setting. In addition, it doesn't take much justification to believe that the arguments of the vertices of each  $P_n$  form a partition of  $S^1$ . However, in higher dimensions the corresponding statement is not obvious at all. Moreover, it is not clear what a higher dimensional analogue would look like. In higher dimensions we will show that if  $P$  is a convex set which contains the origin of a normed linear space, then there is a homeomorphism between the boundary of  $P$  and the boundary of the unit ball in that space. Also, we will show that a convex set is the union of its 'sectors'.

CHAPTER 4  
 PROPERTIES OF POLYTOPES IN LOCALLY CONVEX SPACES

The main result of this chapter is Theorem 4.8, which matches the observations made about the regular polygons in Lemma 3.1. This theorem will justify extending the methods used in  $\mathbb{R}^2$  to  $\mathbb{R}^n$ . And, possibly, since the theorems in this chapter are only dependent on the topological vector space being normed, we may be able to recover some results in infinite dimensional vector spaces. In order to get things done in this chapter we will need the Hyperplane Separation Theorem, which is a famous corollary of the Hahn-Banach theorem.

**Theorem 4.1** (The Hyperplane Separation Theorem). *Let  $P$  be a convex set in a locally convex topological vector space  $X$  and let  $x$  be disjoint from  $P$ . Then there exists a nonzero continuous linear functional  $f$  so that  $f(x) = \alpha$  and so that for all  $y \in P$  we have  $f(y) \geq \alpha$ . Alternatively, if  $x \in \partial P$  then there exists a nonzero continuous linear functional  $f$  so that  $f(x) = \alpha$  and for all  $y \in \bar{P}$  we have  $f(y) \geq \alpha$ . A proof of this can be found in [4].*

The following is an interesting result which is not true for most sets. An obvious example of a set for which the following lemma would fail are the rationals. Clearly,  $\bar{\mathbb{Q}} = \mathbb{R}$ , and  $\mathbb{R}$  is open, so  $\bar{\mathbb{Q}}^\circ = \mathbb{R}$ . However,  $\mathbb{Q}^\circ = \emptyset$ . While the result is probably known folklore, I could not find it in the literature and so I have proved it myself.

**Lemma 4.2.** *If  $P$  is convex, then  $\bar{P}^\circ = P^\circ$*

Proof. We will prove this by double inclusion. Let  $x \in P^\circ$ . Then let  $V_x \subset P^\circ$  be an open neighborhood containing  $x$ . But  $V_x \subset P^\circ \subset P \subset \bar{P}$  means that  $x$  is an interior point of  $\bar{P}$ , hence  $P^\circ \subset \bar{P}^\circ$ . Now let  $x \in \bar{P}^\circ$  and by way of contradiction suppose that  $x \notin P^\circ$ . Then there is a ball around  $x$  completely contained in  $\bar{P}$ , call it  $V_x \subset \bar{P}$ . Since  $x \notin P^\circ$ , it follows that  $x \in \partial P$ . Hence there is a point  $z \in V_x$  which is not in  $P$ . Since  $P$  is convex and  $z$  is not in  $P$ , by Theorem 4.1 let  $f$  be a nonzero continuous linear functional such



that  $f(z) = \alpha$  with  $f(y) \geq \alpha$  for all  $y \in P$ . Since  $V_x$  is open, let  $V_z \subset V_x$  be a symmetric neighborhood of  $z$  and let  $D = \{y \in V_z : f(y) < \alpha\}$ . Then  $D$  is open,  $D \subset \overline{P}$  and  $D \cap P = \emptyset$ . Therefore  $D \subset \partial P$ . But  $D$  is open, and boundaries have empty interior. Thus  $D$  is empty. Now consider the functional  $g = f - \alpha$ . Since  $f$  is not identically zero,  $g$  is not identically zero. Hence, since  $g(z) = 0$ , let  $w \in V_z$  so that  $g(w) \neq 0$  and let  $w' = -w$ . Then by linearity of  $g$ ,  $g(w) = g(-w') = -g(w')$ . Without loss of generality suppose that  $g(w) > 0$  and  $g(w') < 0$ . Then in fact  $f(w) > \alpha$  and  $f(w') < \alpha$ . Since  $V_z$  is symmetric and  $w \in V_z$ , we get that  $w' \in V_z$ . These last two statements guarantee that  $w' \in D$ . This is a contradiction.  $\square$

**Lemma 4.3.** *If  $P$  is a convex set containing a neighborhood of 0 in a linear space  $X$ , then  $x \in \partial P \iff (\{tx : 0 \leq t < 1\} \subset P^\circ) \wedge (x \notin P^\circ)$ .*

Proof. ( $\Rightarrow$ ) Let  $x \in \partial P$ . Since  $P$  contains a neighborhood of 0, let  $V \subset P$  be that neighborhood. Now consider the set  $C_{x,V} = \bigcup_{0 \leq t < 1} (tx + (1-t)V)$ . By the convexity of  $\overline{P}$ ,  $C_{x,V} \subset \overline{P}$ . By the continuity of addition and multiplication,  $C_{x,V}$  is open. Hence  $C_{x,V} \subset P^\circ$ . By Lemma 4.2,  $C_{x,V} \subset P^\circ$ . Since  $\{tx : 0 \leq t < 1\} \subset C_{x,V}$  we are done.

( $\Leftarrow$ ) Suppose that  $(\{tx : 0 \leq t < 1\} \subset P^\circ) \wedge (x \notin P^\circ)$ . Let  $t_n < 1$  be a sequence of real numbers converging to 1. Then  $t_n x$  converges to  $x$ . Since  $\{t_n x\} \subset P^\circ$ , that makes  $x \in \overline{P}$ . By supposition  $x \notin P^\circ$ . Hence  $x \in \partial P$ .  $\square$

**Lemma 4.4.** *Let  $X$  be a normed linear space and let  $B_r$  denote the ball of radius  $r$  with respect to  $\|\cdot\|$ . If  $P$  is a bounded convex set containing a neighborhood of 0, then  $f : \partial P \rightarrow \partial B_1$  given by  $x/\|x\|$  is a bijection.*

Proof. Let  $P$  be convex and let 0 be an interior point of  $P$ . Suppose that  $f(x) = f(y)$ . If  $\|x\| = \|y\|$  then we are done. Hence suppose that  $\|x\| < \|y\|$ . Then we may write  $x = \frac{\|x\|}{\|y\|}y$ . Since  $0 \in P^\circ$ ,  $\frac{\|x\|}{\|y\|} < 1$  and  $y \in \partial P$ , by Lemma 4.3 we get that that  $x \in P^\circ$ , but  $x \in \partial P$ . This is a contradiction hence  $f$  is one-to-one. Now we wish to show the onto property. Let  $x \in \partial B_1$ . Since 0 is an interior point, let  $B_\epsilon$  be given such that  $\overline{B_\epsilon} \subset P^\circ$ . Then  $\|\epsilon x\| = \epsilon$  and thus  $\epsilon x$  is in  $P$ . Now, since  $P$  is bounded there is  $M > 0$  so

that  $\|y\| < M$  for all  $y \in P$ . In particular,  $L = \{s \geq 0 : s\epsilon x \in P^\circ\}$  is a bounded set of real numbers. By convexity of  $P$ ,  $L$  is actually an interval. Let  $s' = \sup L$  and observe that  $s'\epsilon x \notin P^\circ$ . Then  $L = [0, s')$  and therefore  $\frac{1}{s'}L = [0, 1)$ . Hence  $s'\epsilon x \frac{1}{s'}L = \{t(s'\epsilon x) : 0 \leq t < 1\}$ . But also we know that  $s'\epsilon x \frac{1}{s'}L = \epsilon x L \subset P^\circ$ . So by Lemma 4.4 we get that  $s'\epsilon x \in \partial P$ .  $\square$

Instead of trying to parametrize angles in  $\mathbb{R}^n$  or higher dimensions as we did in  $\mathbb{R}^2$  in Chapter 4, we will think of a particular heading or direction as a point in the unit ball. Since every point in our space admits a particular heading, which is a point in the unit ball, by our previous theorem every point in our space can be projected onto the boundary of a convex set sitting at the origin. Once we define what a face is, a definition borrowed from [1], we will be able to describe our space as a union of ‘facial sectors’. Part i) of Theorem 4.8 is known and can be found in [1] as well, but the proof was again too brief for my level and without detail. Parts ii) and iii) of Theorem 4.8 are natural corollaries of i) which I proved out of necessity and found more useful.

**Definition 4.5.** *If  $P$  is convex, then a convex set  $F \subsetneq \overline{P}$  is said to be a face of  $P$  provided that for any pair  $x, y \in \overline{P}$  and for any  $t \in (0, 1)$ , if  $tx + (1 - t)y \in F$ , then  $x, y \in F$ . The idea is that if  $F$  contains the interior point of some segment (a segment is all convex combinations of two points), then it contains the whole segment.*

**Definition 4.6.** *The cone of a set  $S$ , given by  $\text{cone}(S)$ , is the set containing all linear combinations of elements in  $S$  where the scalars are chosen from  $[0, \infty)$ . That is,*

$$\text{cone}(S) = \{\alpha x + \beta y : x, y \in S \wedge \alpha, \beta \geq 0\}$$

**Definition 4.7.** *A convex hull of a set  $S$ , given by  $\text{hull}(S)$ , is the set containing all convex combinations of elements in  $S$ .*

$$\text{hull}(S) = \{tx + (1 - t)y : x, y \in S\}$$

**Theorem 4.8** (Face-Hull Decomposition Theorem). *Let  $X$  be a normed topological vector space and let  $P$  be a bounded and convex set with  $0 \in P^\circ$  and let  $\mathcal{F}$  denote the collection of faces of  $P$ . Then*

$$i) \partial P = \bigcup_{F \in \mathcal{F}} F$$

$$ii) X = \bigcup_{F \in \mathcal{F}} \text{cone}(F)$$

$$iii) \bar{P} = \bigcup_{F \in \mathcal{F}} \text{hull}(F \cup \{0\})$$

Proof. To begin i) we will first show that if  $F$  is a face of  $P$ , then  $F \subset \partial P$ . So let  $F$  be a face of  $P$  and let  $x \in F$ . Since  $F \neq \bar{P}$ , let  $y \in \bar{P} \setminus F$ . Suppose that for some  $t > 1$  we have  $z = tx + (1 - t)y \in \bar{P}$ . Then

$$\frac{1}{t}z + \left(1 - \frac{1}{t}\right)y = \frac{1}{t}(tx + (1 - t)y) + \left(1 - \frac{1}{t}\right)y = x$$

Since  $F$  is a face, it means that  $z$  and  $y$  should also be in  $F$ . This is not possible. Hence, if  $t > 1$  then  $z = tx + (1 - t)y \notin \bar{P}$ . Then let  $\{t_n\}$  be a sequence of real numbers which converge to 1 from above. We have that  $\{t_n x + (1 - t_n)y\}$  is a sequence which is disjoint from  $\bar{P}$  but converges to  $x$ , hence  $x \in \partial P$ . For the other direction suppose that  $x \in \partial P$ . By Theorem 4.1, there is a continuous linear functional  $f$  so that  $f(x) = \alpha$  and for all  $y \in \bar{P}$  we have  $f(y) \geq \alpha$ . Let  $F = \{y \in \bar{P} : f(y) = \alpha\}$ . We want to show that  $F$  is in fact in  $\mathcal{F}$ . To that end, let  $y, z \in \bar{P}$  and suppose that for some  $t \in (0, 1)$  we have  $ty + (1 - t)z \in F$ . By way of contradiction, suppose that one of these elements is not in  $F$ . Without loss of generality, suppose that  $f(z) > \alpha$ . Then by linearity of  $f$ ,

$$f(ty + (1 - t)z) = tf(y) + (1 - t)f(z) > tf(y) + (1 - t)\alpha \geq t\alpha + (1 - t)\alpha = \alpha$$

This is contrary to the fact that  $f(ty + (1 - t)z) = \alpha$ . Hence, since  $y$  and  $z$  are in  $\bar{P}$ , the only option is that  $f(y) = f(z) = \alpha$  and therefore that  $y, z \in F$ .

ii) Let  $0 \in P^\circ$  and let  $P$  be bounded. Since  $X$  is the whole space, it is obvious that each  $\text{cone}(F) \subset X$ , and so we only need to show the other direction, that

$X \subset \bigcup_{F \in \mathcal{F}} \text{cone}(F)$ . Let  $0 \neq x \in X$ , as the case  $x = 0$  is trivial. Then  $\frac{x}{\|x\|} \in B_1$ . By Lemma 4.4, there is  $y \in \partial P$  such that  $\frac{y}{\|y\|} = \frac{x}{\|x\|}$ . By i), let  $F \in \mathcal{F}$  such that  $\frac{y}{\|y\|} \in F$ . Since  $\|x\|$  is a nonnegative scalar and  $y \in F$ ,  $x = \|x\|y \in \text{cone}(F)$ .

iii) For the same reasoning expressed in the beginning of ii), we only need to show that  $\overline{P} \subset \bigcup_{F \in \mathcal{F}} \text{hull}(F \cup \{0\})$ . Let  $x \in \overline{P}$ . Then by ii), let  $F \in \mathcal{F}$  so that  $x \in \text{cone}(F)$ . Then let  $y, z \in F$  and  $\alpha, \beta \in [0, \infty)$  so that  $x = \alpha y + \beta z$ . If  $\alpha + \beta = 1$  then we are done, as  $x$  would be a convex combination of points in  $F \subset F \cup \{0\}$ . If  $\alpha + \beta < 1$  then we are also done, as we can let  $\gamma = 1 - (\alpha + \beta)$ , then  $\alpha + \beta + \gamma = 1$  and we have that  $x = \alpha y + \beta z + \gamma 0$  and thus  $x$  is again a convex combination of points in  $F \cup \{0\}$ . Now, if it is the case that  $\alpha + \beta > 1$  then observe that

$$\frac{x}{\alpha + \beta} = \frac{\alpha}{\alpha + \beta}y + \frac{\beta}{\alpha + \beta}z$$

By the convexity of  $F$ ,  $\frac{x}{\alpha + \beta} \in F$ . But note that  $x, 0 \in \overline{P}$  and that

$\frac{x}{\alpha + \beta} = \frac{1}{\alpha + \beta}x + (1 - \frac{1}{\alpha + \beta})0$ . By the face property of  $F$ , both  $x$  and  $0$  are in  $F$ . From i) we know that  $F \subset \partial P$ . However,  $0$  is an interior point of  $P$ . This is a contradiction.  $\square$

CHAPTER 5  
 CONVERGENCE PROPERTIES OF MINKOWSKI FUNCTIONALS IN FINITE  
 DIMENSIONAL SPACES

In this chapter we will prove the main result of the paper, Theorem 5.9, which tells us that we may sample points in  $\overline{B_1}$  in order to approximate  $\|x\|$  as close as we wish using the Minkowski functional. Moreover, the  $\mathbb{R}^2$  case of regular inscribed polygons satisfies the geometric criterion for selecting points from  $\overline{B_1}$  that is outlined in the theorem. We show that if  $\{P_n\}$ , a sequence of polytopes, converges to  $B_1$  ‘from the inside’ (Definition 5.7) then in fact we obtain the pointwise convergence of  $\{\mu_{P_n}\}$  to  $\|x\|$ . Before we do this, we establish Lemma 5.8 which restricts the setting to  $\mathbb{R}^n$ . The first theorem is obvious by Figure 5.1, but we present the details regardless.

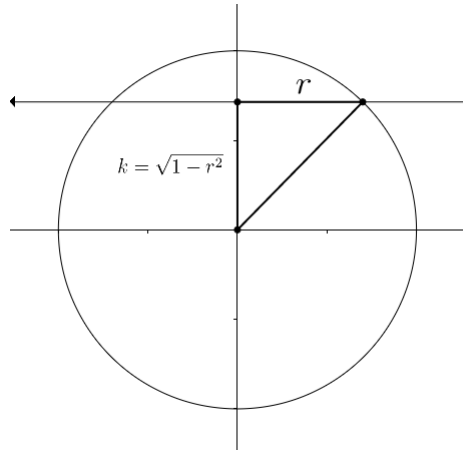


Figure 5.1: Profile of Hyperplane Intersecting  $\overline{B_1}$  in  $\mathbb{R}^2$ .

**Theorem 5.1.** *If  $H$  is a plane of distance  $k < 1$  from 0 which intersects  $\overline{B_1}$  in  $\mathbb{R}^n$ , then  $H \cap \overline{B_1}$  is an  $\mathbb{R}^{n-1}$  ball. Moreover, if the intersection is an  $\mathbb{R}^{n-1}$  ball, then the radius of the ball has the following relationship:*

$$k = \sqrt{1 - r^2}$$

Proof. Let  $H$  with distance  $k$  from the origin intersect  $\overline{B_1}$  in  $\mathbb{R}^n$ . Without loss of generality, write  $H$  as  $H = \{(x_1, \dots, x_n) : x_n = k\}$ . If  $\overline{B_1}$  is given as  $\overline{B_1} = \{(x_1, \dots, x_n) : x_1^2 + \dots + x_n^2 \leq 1\}$ , then

$$\begin{aligned} H \cap \overline{B_1} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1^2 + \dots + x_n^2 \leq 1) \wedge (x_n = k)\} \\ &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1^2 + \dots + x_{n-1}^2 \leq 1 - k^2) \wedge (x_n = k)\} \end{aligned}$$

We recognize this set to be an  $\mathbb{R}^{n-1}$  ball of radius  $r = \sqrt{1 - k^2}$ , and thus  $k = \sqrt{1 - r^2}$ .  $\square$

**Definition 5.2.** *The set  $P \in \mathbb{R}^n$  is a polytope if there exists a finite set  $S \subset \mathbb{R}^n$  so that  $P = \text{hull}(S)$ .*

**Theorem 5.3** (Unique Polytope Representation Theorem). *Let  $P$  be a convex polytope in  $\mathbb{R}^n$ . Then there exists a minimal representation of  $\overline{P}$  as the intersection of closed half spaces generated by a finite number of supporting hyperplanes  $H_1, \dots, H_k$  (supporting here means that each  $H_i$  meets  $\partial P$ , but misses  $P^\circ$ ). The proof of this theorem can be found in [2].*

In Chapter 5 we decomposed general convex sets into faces. In the final theorem of this chapter, we use a measurement of convex polytopes which depends on the facets (intuitively, the ‘largest’ flat spots on a polytope). In order to be able to apply our theorems from Chapter 5 we will need to connect faces to facets. In order to do this we borrow the definition of facet, edge, and part of the argument of Lemma 5.6 from [3].

**Definition 5.4.** *We say that  $F \in \mathcal{F}$  is a facet of  $P$  if there is a unique hyperplane containing  $F$ . Denote the set of facets  $\mathcal{F}'$ .*

**Definition 5.5.** *Let  $E_x$ , the edge determined by  $x$ , be the intersection of all facets containing  $x$ .*

**Lemma 5.6.** *If  $F$  is a face, then it is contained in an edge.*

Proof. For the sake of simple notation, let  $\mathcal{F}'_x$  denote the collection of facets which contain  $x$  as an element. Observe that Theorem 5.3 implies that every point in  $F$

belongs to at most finitely many facets. Therefore  $\{|\mathcal{F}'_y| : y \in F\} \subset \mathbb{N}$ , and hence this set has a least element. Let  $x \in F$  so that  $|\mathcal{F}'_x|$  is the least element. Let  $y \in F$  with  $y \neq x$  and let  $z \in (x, y)$ . If  $F'$  is a facet, hence also a face, then observe that if  $z \in F'$ , then  $x, y \in F'$  also by the definition of face. So  $\mathcal{F}'_z \subset \mathcal{F}'_x \cap \mathcal{F}'_y$  which also implies that  $|\mathcal{F}'_z| \leq |\mathcal{F}'_x|$ . However, by our choice of  $x$  we know that  $|\mathcal{F}'_x| \leq |\mathcal{F}'_z|$ . The double inequality gives that  $|\mathcal{F}'_x| = |\mathcal{F}'_z|$ , but since there is containment one way we also get that  $\mathcal{F}'_x = \mathcal{F}'_z$ . Thus we have  $\mathcal{F}'_x = \mathcal{F}'_z \subset \mathcal{F}'_y$ . Therefore

$$y \in \cap \mathcal{F}'_y \subset \cap \mathcal{F}'_x = E_x$$

We have shown that  $F \setminus \{x\} \subset E_x$ . Thus  $F = \{x\} \cup (F \setminus \{x\}) \subset E_x$ , and we are done.  $\square$

By the definition of edge, we already know that each edge is contained in a facet. Hence, by the preceding lemma, each face belongs to a facet. Let  $P$  be a convex polytope in  $\mathbb{R}^n$  with vertices in  $\overline{B_1}$ . Each facet of  $P$  gives rise to a plane which intersects  $\overline{B_1}$ , and so By Theorem 5.1, gives rise to the radius of the  $\mathbb{R}^{n-1}$  ball containing it, denoted  $r(F')$ . Let  $M(P) = \max\{r(F') : F' \in \mathcal{F}'\}$ . This measure is an important part of characterizing sequences of polytopes  $\{P_n\}$  for which  $\{\mu_{P_n^\circ}\} \rightarrow \|\cdot\|$  pointwise.

**Definition 5.7.** *let  $B \subset \mathbb{R}^n$ . We say that the sequence of sets  $\{P_n\}$  converge to  $B$  from the inside if each  $P_n$  is a subset of  $B$  and if for every  $x \in B$  there is a natural number  $N(x)$  so that for every  $n \geq N(x)$ ,  $x \in P_n$ .*

This next lemma is a statement which ‘ought to be true’ if someone were to look at it and think about it for a second, but I don’t believe anyone has proved it yet. The lemma is needed in Theorem 5.9. While in Theorem 5.9 it is used in order to show iii) $\Rightarrow$ i), a choice which I made in order to better showcase the analysis, I initially discovered that I would need it when I was working in the other direction of iii) $\Rightarrow$ ii).

**Lemma 5.8** (Intermediate Polytope Lemma). *For any  $\epsilon > 0$ , in  $\mathbb{R}^n$  there exists a finite set of points  $S \subset B_{1+\epsilon}$  so that  $P^* = \text{hull}(S)$  and  $B_1 \subset P^* \subset B_{1+\epsilon}$ .*

Proof. Let  $V$  represent the set of standard basis vectors in  $\mathbb{R}^n$  and let  $D$  be the convex hull of  $V \cup (-V)$ . Then  $\mathcal{C} = \{\frac{\epsilon}{2}D^\circ + x : x \in \overline{B_1}\}$  is an open cover for  $\overline{B_1}$ . Let  $\mathcal{C}' \subset \mathcal{C}$  be the finite open cover of  $\overline{B_1}$  guaranteed by compactness. Since  $\mathcal{C}'$  is finite, let  $\mathcal{C}' = \{\frac{\epsilon}{2}D^\circ + x_i\}_{i=1}^k$ . Let  $P^* = \text{hull}(\frac{\epsilon}{2}(V \cup (-V)) + \{x_i\}_{i=1}^k)$ . Then for each  $1 \leq i \leq k$ , since  $\frac{\epsilon}{2}D + x_i$  is the convex hull of  $\frac{\epsilon}{2}(V \cup (-V)) + x_i$  and since  $P^*$  is convex,  $\frac{\epsilon}{2}D + x_i \subset P^*$ . Since this is true for each such  $i$ ,  $\cup_{i=1}^k (\frac{\epsilon}{2}D + x_i) \subset P^*$ . Since  $\mathcal{C}'$  is a cover for  $B_1$  we get that

$$B_1 \subset \bigcup_{i=1}^k \left(\frac{\epsilon}{2}D^\circ + x_i\right) \subset \bigcup_{i=1}^k \left(\frac{\epsilon}{2}D + x_i\right) \subset P^*$$

Now, we need to show that  $P^* \subset B_{1+\epsilon}$ . Let the standard basis vectors be labeled as  $e_1, \dots, e_n$ . Then  $P^*$  is the convex hull of the set of points

$$S = \left\{ \frac{\epsilon}{2}(-1)^j e_l + x_i : j = 0, 1 \wedge l = 1, \dots, n \wedge i = 1, \dots, k \right\}$$

Let  $x \in P^*$ . Since  $P^*$  is the convex hull of  $S$ ,  $x$  admits a representation as

$$x = \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \left( \frac{\epsilon}{2}(-1)^j e_l + x_i \right)$$

where  $\sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} = 1$ . Now Observe that

$$\begin{aligned} \|x\| &= \left\| \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \left( \frac{\epsilon}{2}(-1)^j e_l + x_i \right) \right\| \\ &\leq \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \left\| \frac{\epsilon}{2}(-1)^j e_l + x_i \right\| \\ &\leq \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \left( \left\| \frac{\epsilon}{2}(-1)^j e_l \right\| + \|x_i\| \right) \\ &\leq \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \left( \frac{\epsilon}{2} + 1 \right) \end{aligned}$$



$$\begin{aligned}
&= \left(\frac{\epsilon}{2} + 1\right) \sum_{j=0}^1 \sum_{l=1}^n \sum_{i=1}^k \alpha_{j,l,i} \\
&= \frac{\epsilon}{2} + 1 < 1 + \epsilon
\end{aligned}$$

Hence  $x \in B_{1+\epsilon}$ . □

**Theorem 5.9.** *Let  $\{P_n\}$  be a sequence of convex polytopes with vertices in  $\overline{B_1}$  and with  $0 \in P_n^\circ$  for every  $n$ . The following are equivalent:*

- i)  $\{M(P_n)\} \rightarrow 0$
- ii)  $\{\mu_{P_n^\circ}\} \rightarrow \|\cdot\|$  pointwise.
- iii)  $\{P_n^\circ\} \rightarrow B_1$  from the inside.

Proof. i) $\Rightarrow$ ii). First we need to show that for each  $n \in \mathbb{N}$ ,  $B_{\sqrt{1-M(P_n)^2}} \subset P_n^\circ$ . Hence let  $n \in \mathbb{N}$  and let  $x \in B_{\sqrt{1-M(P_n)^2}}$ . By Theorem 4.8, let  $F$  be a face of  $P_n$  so that  $x \in \text{cone}(F)$ . Then there are elements  $y, z \in F$  and positive scalars  $\alpha, \beta$  so that  $x = \alpha y + \beta z$ . Suppose that  $\alpha + \beta \geq 1$ . It follows that  $\|x\| \geq \frac{\|x\|}{\alpha + \beta}$ . But observe that  $\frac{x}{\alpha + \beta}$  is an element in  $F$ :  $\frac{x}{\alpha + \beta} = \frac{\alpha}{\alpha + \beta}y + \frac{\beta}{\alpha + \beta}z \in [y, z] \subset F$ . Since  $F$  is contained in some facet  $F'$ , the norms of elements in  $F$  are as large as the minimum distance between 0 and the plane containing  $F'$ . That is, for all  $w \in F$ ,  $\|w\| \geq \sqrt{1 - r(F')^2}$  as demonstrated in Theorem 5.1. But note that  $M(P_n) \geq r(F')$  by definition. And therefore

$$\|x\| \geq \frac{\|x\|}{\alpha + \beta} \geq \sqrt{1 - r(F')^2} \geq \sqrt{1 - M(P)^2}$$

which contradicts  $x \in B_{\sqrt{1-M(P_n)^2}}$ , thus  $\alpha + \beta < 1$ . Since  $\frac{x}{\alpha + \beta} \in F$ , Theorem 4.8 tells us that  $\frac{x}{\alpha + \beta} \in \partial P_n$ . By Lemma 4.3, since  $\alpha + \beta < 1$ ,  $(\alpha + \beta)\frac{x}{\alpha + \beta} = x \in P_n^\circ$ . Now let  $x \in X$  and let  $\epsilon > 0$ . If  $x = 0$  we are done, so suppose otherwise. Since  $\{M(P_n)\}$  converges to 0,  $\{\sqrt{1 - M(P_n)^2}\}$  converges to 1 by continuity. Thus let  $n \in \mathbb{N}$  so that for all  $n \geq N$  we have  $\frac{\|x\|}{\|x\| + \epsilon} < \sqrt{1 - M(P_n)^2} \leq 1$ . Then for all  $n \geq N$ , by Lemma 2.8 and Lemma 2.9 we

also have

$$\begin{aligned}
|\mu_{P_n^\circ}(x) - \|x\|| &= \mu_{P_n^\circ}(x) - \|x\| \\
&\leq \mu_{B_{\sqrt{1-M(P_n)^2}}}(x) - \|x\| \\
&= \frac{\|x\|}{\sqrt{1-M(P_n)^2}} - \|x\| \\
&= \frac{\|x\| \left(1 - \sqrt{1-M(P_n)^2}\right)}{\sqrt{1-M(P_n)^2}} \\
&< \frac{\|x\| \left(1 - \frac{\|x\|}{\|x\|+\epsilon}\right)}{\frac{\|x\|}{\|x\|+\epsilon}} \\
&= \|x\| \left(1 - \frac{\|x\|}{\|x\|+\epsilon}\right) \frac{\|x\|+\epsilon}{\|x\|} \\
&= \|x\| \left(\frac{\|x\|+\epsilon}{\|x\|} - 1\right) = \|x\| + \epsilon - \|x\| = \epsilon
\end{aligned}$$

as desired.

ii) $\Rightarrow$  iii). Suppose that  $\{\mu_{P_n^\circ}\} \rightarrow \|\cdot\|$  pointwise and by way of contradiction suppose that  $\{P_n^\circ\}$  does not converge to  $B_1$  from the inside. By our assumptions, the vertices of  $P_n^\circ$  are in  $\overline{B_1}$ , and thus  $P_n^\circ \subset B_1$  for all  $n$ . Then let  $x \in B_1$  so that for all  $N \in \mathbb{N}$  there is an  $n \geq N$  so that  $x \notin P_n^\circ$ . Without loss of generality, up to extraction of a subsequence from  $\{P_n^\circ\}$ , suppose that for all  $n \in \mathbb{N}$   $x \notin P_n^\circ$ . Then for any  $n$ , if  $t < 1$ ,  $x \notin tP_n^\circ$ . Otherwise, if for some  $t < 1$  we had  $x \in tP_n^\circ$ , we would have  $\frac{x}{t} \in P_n^\circ$ . By convexity,  $[0, \frac{x}{t}] \subset P_n^\circ$ . Thus  $t\frac{x}{t} + (1-t)0 = x \in P_n^\circ$ , which isn't possible. Hence for any  $n$  we have  $\mu_{P_n^\circ}(x) \geq 1$ , and thus by assumption,  $\|x\| \geq 1$ . But  $x \in B_1$ , this is a contradiction.

iii) $\Rightarrow$  i). Suppose now that  $\{P_n^\circ\}$  converges to  $B_1$  from the inside but that, again by way of contradiction,  $\{M(P_n)\}$  doesn't converge to 0. Then there is some  $\epsilon > 0$  so that for all  $N \in \mathbb{N}$  there is an  $n \geq N$  so that  $M(P_n) \geq \epsilon$ . Again, without loss of generality assume that for each  $n$  we have  $M(P_n) \geq \epsilon$ . This means that each  $P_n^\circ$  has a supporting hyperplane which is at most distance  $\sqrt{1-\epsilon^2}$  from the origin. Then the closed half space which doesn't contain  $P_n^\circ$  meets the closed ball with radius  $\frac{1+\sqrt{1-\epsilon^2}}{2}$ . Hence, for each

$n \in \mathbb{N}$ , let  $x_n \in \overline{B_{\frac{1+\sqrt{1-\epsilon^2}}{2}}} \setminus P_n^\circ$ . Let  $x \in \overline{B_{\frac{1+\sqrt{1-\epsilon^2}}{2}}}$  be the accumulation point of  $\{x_n\}$  and let  $r \in \mathbb{R}$  so that  $\frac{1+\sqrt{1-\epsilon^2}}{2} < r < 1$ . By Lemma 5.8, let  $S \subset B_1$  be a finite set of points so that  $P^* = \text{hull}(S)$  with the property that  $B_r \subset P^* \subset B_1$ . By the convergence of  $\{P_n^\circ\}$  to  $B_1$  from the inside, for each  $y \in B_1$  let  $N_y \in \mathbb{N}$  denote the index for which if  $n \geq N_y$  then  $y \in P_n^\circ$ . Observe that  $\{N_s : s \in S\}$  is a finite subset of  $\mathbb{N}$ . Hence let  $N = \max\{N_s : s \in S\}$ . Then for all  $n \geq N$  we have  $S \subset P_n^\circ$ . By the convexity of each  $P_n^\circ$ , we have that  $P^* = \text{hull}(S) \subset P_n^\circ$ . Let  $V_x \subset B_r$  be a neighborhood of  $x$ . Since  $x$  is an accumulation point of  $\{x_n\}$ , let  $n_0 \geq N$  so that  $x_{n_0} \in V_x$ . Then we have  $x_{n_0} \in V_x \subset B_r \subset P^* \subset P_{n_0}^\circ$ , which is a contradiction.  $\square$

## CHAPTER 6

## FUTURE WORK: INFINITE DIMENSIONAL SPACES

The most natural question to ask now is this: How can we extend Theorem 5.9 to infinite dimensional vector spaces? Before we answer this question, we have to realize that convex polytopes, taken as in Definition 5.2, do not behave the same way in infinite dimensional spaces. The convex polytopes in our paper were always able to be chosen so that they contain 0 as an element of the interior. Once we knew this, we leveraged the geometry of the polytope in order to determine the size of the largest  $\mathbb{R}^n$  ball that lies inside. This fails in infinite dimensional spaces in the sense that if  $P$  in some infinite dimensional vector space  $X$  is the convex hull of finitely many points, then  $P$  will be completely contained in a finite dimensional subspace of  $X$ . Thus,  $P$ , even if it contains 0 as an element of the interior with respect to the subspace topology, will never contain a ball that has the same dimension as  $X$  itself. So, we simply cannot keep our definition of polytope as the convex hull of finitely many points. Attempts have been made to generalize polytopes so that they can exist in infinite dimensional spaces while containing 0 as an interior point. From [3]:

**Definition 6.1.** *Let  $P$  be a convex subset of  $X$ . Then  $P$  is a polytope if  $0 \in P^\circ$  and if for every collection of closed half spaces  $\{E_\alpha : \alpha \in A\}$  with  $P = \bigcap_{\alpha \in A} E_\alpha$  we have that  $x \in X$  implies that there is a finite collection  $A' \subset A$  with the property that  $x \in \bigcap_{\alpha \in A \setminus A'} E_\alpha$ .*

Maserick proves in [3] that polytopes defined this way have many of the same properties that polytopes do when defined as the convex hull of finitely many elements. Maserick shows that every point in the boundary of  $P$  is contained in finitely many hyperplanes from the collection  $\{H_\alpha : \alpha \in A\}$ , where  $H_\alpha = \partial E_\alpha$  from above. Maserick also proves a statement which would appear to give hope: If  $X$  is a separable space, then if  $P$  is a convex polytope it has at most countably many faces. This is interesting because there are separable infinite dimensional vector spaces which do not even have a countable basis, the vector space of sequences being an example.

Unfortunately for those wishing to study  $L^p$  polytopes with countably many faces, Maserick shows that if  $X$  is a separable infinite dimensional space, then it contains no bounded polytopes. To add insult to injury, it is also concluded that in infinite dimensional spaces where there are bounded convex polytopes, the unit ball is already a convex polytope. Here ends the journey in this line of questioning. If the unit balls in these spaces are already polytopes, then there is no reason to approximate them with polytopes. Without further generalization, this question is exhausted.

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