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Proportional reasoning in middle level mathematics textbooks

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PROPORTIONAL REASONING IN MIDDLE LEVEL MATHEMATICS TEXTBOOKS

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Introduction

Proportionality is a part of everyday life yet it is a concept that many people do not fully understand (Lamon, 2012). For example, when deciding whether or not it is a better deal to buy the regular size box or the value size box of granola bars, one would compare the price of each box to the number of granola bars. This is a proportion problem that is encountered in the real world, but how does one solve this problem? In the past, students were taught the cross-multiplication method to solve proportion problems. While effective, this strategy requires no understanding of proportional relationships. When cross-multiplying, students solve $\frac{a}{b} = \frac{c}{x}$, where *x* is an unknown value, by calculating $\frac{b \times c}{a} = x$. Since this method depends on students memorizing a procedure and elicits no understanding of proportionality, educators are questioning how mathematics, in this case proportionality, is taught.

The *Common Core State Standards for Mathematics* (National Governors Association, 2010) require sixth and seventh grade students to be able to describe the relationship between two quantities in a ratio and also between two ratios in a proportion. Additionally, these *Standards* require students to be able to compute a unit rate. Students who are only taught how to solve a proportion problem using cross-multiplication would most likely not be able to attain these mathematical standards. Therefore, it is increasingly important that students be able to reason proportionally, which cross-multiplication does not develop.

Since textbooks are common tools used by teachers to organize what they teach, including proportional reasoning, it would be beneficial to teachers if these materials could provide a useful framework to help students meet the *Common Core State Standards for Mathematics* (National Governor's Association, 2010). Therefore the purpose of my research is to critically analyze middle school textbooks. To be able to accomplish that task I will first identify critical components of proportional reasoning. Identifying the critical components will provide a framework in which to assess the tasks in the middle level mathematics textbooks. I will then evaluate the tasks in the middle level textbooks to determine if they contribute to children's understanding of proportional reasoning.

Research Questions

In order to evaluate proportional tasks it is first imperative to identify the critical components of proportional reasoning. I used the following research questions to focus my work.

1. Based on current research, what are the critical components of children's proportional reasoning?
2. What types of tasks are found in middle level mathematics textbooks to develop students' proportional reasoning?
    a. What are the characteristics of individual tasks?
    b. What is the potential of the tasks to provide students an opportunity to apply the critical components of proportional reasoning?

Identifying the critical components will provide a framework in which to assess the tasks in the middle level mathematics textbooks. By analyzing the tasks in middle level textbooks using the framework, the books can be ranked based on potential effectiveness of contributing to children's developing understanding of proportional reasoning.

Literature Review

Proportionality is a major concept in the middle school mathematics curriculum. The Common Core State Standards for Mathematics and the NCTM Principles and Standards for School Mathematics both include proportionality in their standards for the middle school mathematics curriculum (National Governors Association, 2010; National Council of Teachers of Mathematics, 2000). Not only is proportionality important in the classroom, but also in the real world. Proportionality has “been proved to be a universal mathematical tool of explaining and mastering phenomena in different fields of human activity” (Modestou & Gagatsis, 2010, p.36). The concept of proportionality is an idea that will continue to be present in everyday life, therefore making it extremely important. Additionally, a robust understanding of proportionality will lead students to success in many subsequent mathematics topics (Lobato & Ellis, 2010). An issue that arises is teaching students to solve proportions without understanding the relationships among the quantities, which is proportional reasoning. Proportional reasoning is a vital concept that many students do not understand. Not all individuals who can solve a proportion problem correctly are using proportional reasoning (Post, Behr, & Lesh, 1988).

Critical Components of Proportional Reasoning

The first research question is answered through a literature review. There is a vast body of literature that has identified important components of proportional reasoning. For the purpose of this study, the literature review will be presented as a response to the first research question. Four critical components of proportional reasoning were identified in the review of literature to apply to the work done on the second research question. In the following sections, the four components will be presented and situated within the body of research on proportional reasoning. These critical components were used as criteria when analyzing the effectiveness of textbook proportion tasks and their potential contributions to developing proportional reasoning for children.

Identifying proportional vs. non-proportional situations. First, in order to be able to reason proportionally, it is imperative that students distinguish between proportional situations and non-proportional situations. In most cases, the teacher tells students that they will be working on proportions. Immediately, students know that they will be presented with a proportion in each task and the need to differentiate between a proportional and non-proportional situation is obsolete. Students tend to overuse proportionality, meaning, students use proportion solution methods in non-proportional contexts (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010). This overuse of proportionality suggests that students may know how to perform a proportion solution method but they may not know when it is an appropriate method to use. A study of secondary mathematics students revealed that regardless of age, students are not distinguishing between proportional and non-proportional situations (Van Dooren, De Bock, De Bolle, Janssens, & Verschaffel, 2003). This means that the issue is not resolved as students progress through mathematics courses. While identifying areas of understanding that are critical to the development of proportional reasoning, research shows that the ability to distinguish between a proportional and a non-proportional situation is important.

Understanding a ratio as a composed unit. Another critical component of proportional reasoning is the ability to understand a ratio as a composed unit. This understanding is usually demonstrated through repeated iteration or partitioning. For example, if a student is able to understand the ratio "30 miles: 1 hour" as a composed unit of "30 miles per hour," and then use that unit to find other proportional speeds, they are demonstrating an understanding of a ratio as a composed unit. This component is not an indicator of proportional reasoning, but rather a necessary understanding before proportional reasoning can take place (Lobato & Ellis, 2010; Lamon, 1993). Since understanding a ratio as a composed unit leads to further sophistication of proportional reasoning, it is important to include as a critical component of this study.

Once students understand a ratio as a composed unit, they should be able to think about units flexibly when solving a proportion problem. For example, when finding the best deal on cereal one must compare the price to the number of ounces in each box. One way to find the best deal would be to find the price for one ounce in each box and compare the answers. For example, consider a problem about products where Box A weighs 16 ounces and costs $3.36, with Box B at 12 ounces and costing $2.64. Box A would cost $0.21 per ounce and Box B would cost $0.22 per ounce. In this example, price per ounce is the unit. This problem could be solved using different units, such as price per 2 ounces or price per 4 ounces, which would be easier to solve mentally than price per one ounce (Lamon, 2012). Using units flexibly is an important component of proportional reasoning because it indicates that the student understands the ratio as a composed unit. A student does not demonstrate understanding of a ratio as a unit by writing a/b or $a \div b$ because it does not mean that the student has mentally formed a ratio between a and b (Labato & Ellis, 2010). By providing students with tasks that allow them to use units flexibly, teachers can encourage students to find multiple solution strategies, which will help students to be able to use units flexibly (Lamon, 2012). Thinking about units flexibly is an important part of understanding a ratio as a composed unit because it helps to promote understanding of a ratio as a composed unit, which is one of the critical components of proportional reasoning.

Understanding multiplicative relationships. In order to develop proportional reasoning, students also need to be able to understand the relationships present in proportional situations. To achieve this understanding, students must first recognize the “multiplicative nature of situations involving ratio and proportion” (Lamon, 1993, p.58). When students are first introduced to proportional relationships, they tend to think that the relationships are additive rather than multiplicative. This type of thinking hinders their development of proportional reasoning.

There are two meanings to the word "more." When asking a question that asks about "which has more" one can answer additively or multiplicatively (Lamon, 2012). If one pitcher of lemonade contains 3 cups of lemon concentrate and 5 cups of water and the other pitcher contains 5 cups of lemon concentrate and 7 cups of water, which pitcher has more lemon flavor? In this example, students who reason additively use absolute comparisons. An absolute comparison would show that the second pitcher has more lemon flavor because there are 2 *more* cups of lemon concentrate in the second pitcher than in the first pitcher. Students who are proportional reasoners think multiplicatively and would make a relative comparison. Proportional reasoners answer the question "which pitcher has more lemon flavor?" by comparing the ratio of lemon concentrate to water in each pitcher of lemonade. Students who are able to understand this multiplicative relationship understand one of the critical components to proportional reasoning.

Once students understand and can use the multiplicative relationship, proportional reasoners need to be able to identify two types of relationships in a proportion problem: "the ratio of one quantity to another, and the ratio of a scaled version of the quantities to the original version" (Steinthorsdottir & Riehl, 2015, p.2). Students must be able to understand the multiplicative relationship between measures and within measures to understand the relationships between ratios in a proportion. Riehl and Steinthorsdottir (2014) explained within and between measure spaces using the Mr. Tall and Mr. Short problem (see Figure 1 for the problem). When measuring Mr. Tall's height in matchsticks and Mr. Short's height in matchsticks, the multiplicative relationship is the within measure space ratio. The relationship between Mr. Short's height in paperclips and Mr. Tall's height in matchsticks is the between measure space ratio. In a proportion, the within measure space ratios will always be equal and the between measure space ratios will be equal. In the context of the Mr. Tall and Mr. Short problem, the relationship between the matchsticks will be the same as the relationship between the paperclips (A:a = B:b). Additionally, the matchstick-to-paperclip ratio will be the same for both Mr. Tall and Mr. Short (A:B = a:b) as seen in Figure 2.

In the picture, you can see the height of Mr. Short measured with paperclips. Mr. Short has a friend, Mr. Tall. When we measure their heights with matchsticks, Mr. Short's height is 4 matchsticks and Mr. Tall's height is 6 matchsticks. How many paperclips are needed for Mr. Tall?

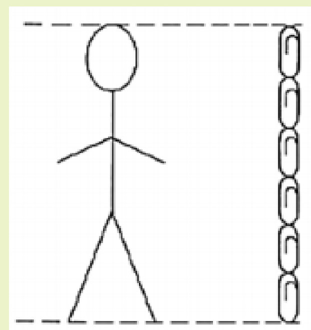

Figure 1: Mr. Tall and Mr. Short problem (Riehl & Steinthorsdottir, 2014, p. 222).

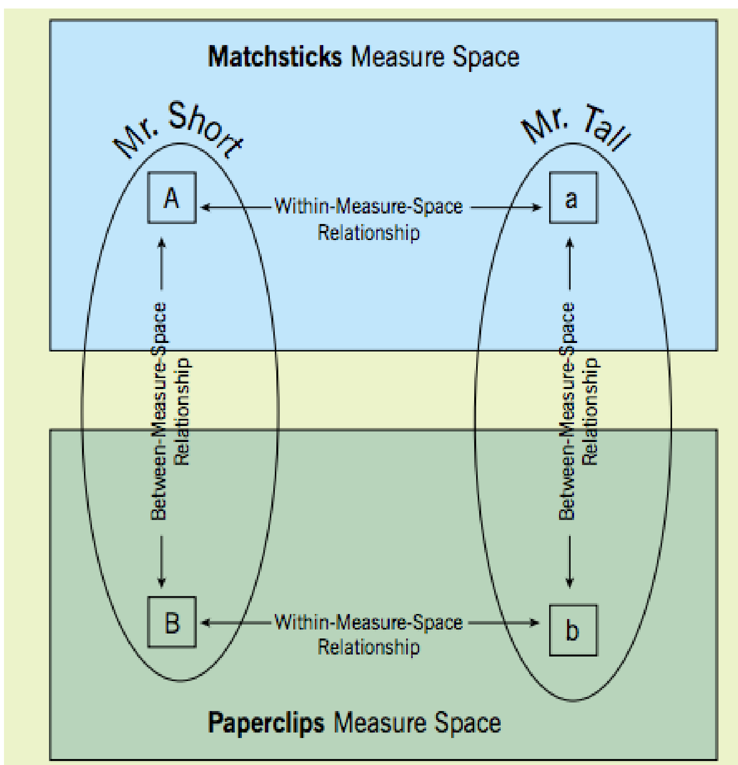

Figure 2: Within and between measure spaces (Riehl & Steinthorsdottir, 2014, p. 222).

Lamon (2012) identified characteristics of proportional thinkers. In her list, Lamon states that proportional thinkers can unitize, or mentally regroup, without losing track of the unit. This means that students should be able to work with any scale factor to find proportional ratios. Lamon also stated that proportional thinkers can “determine relationships that remain unchanged” (2012, p. 260). Without understanding that the multiplicative relationships called within measures, also known as the scale factor, and between measures, or the invariant, remains unchanged, students will not develop a robust understanding of proportions.

Using proportional reasoning flexibility. The last critical component of proportional reasoning is flexibility. There are many different semantic types of proportion word problems, and students should be able to reason proportionally about all problem types. Lamon (1993) identified four semantic types of proportion problems. The first type, “well-chunked measure problems,” (p.42) involves the comparison of two ratios in which the rate is familiar (e.g. miles per hour, cost per pound, etc.). “Part-part-whole” (p.42) is the second type of proportion problem. An example of a part-part-whole problem would be a problem related to making lemonade, considering the ratio of lemon juice concentrate to water. The third type, “associated sets” (p.42), presents two objects with a relationship that would initially be unclear but that is then defined in the problem (e.g. people to pizzas). The last type is “stretcher and shrinker” (p.43) problems. In this problem type, a ratio representing a measurement is scaled up (stretching) or down (shrinking.) For example, scaling on a map or similar triangle problems require students to enlarge or shrink a ratio.

Lamon (1993) found that students frequently use proportion concepts with associated-sets problems, but fail to be able to apply the same concepts to part-part-whole problems and stretcher and shrinker problems. This information indicates that students can be proficient in solving some types of proportion problems and not proficient in solving other types of proportion problems. This could be an issue when attempting to identify students who are proportional reasoners, specifically if students are only introduced to one or two semantic types. If only one or two semantic types appear in middle level textbooks, students will not be able to develop a full understanding of proportional reasoning without supplemental materials that include the remaining semantic types. This issue makes semantic type a critical component of proportional reasoning that is important to include when analyzing the usefulness of middle level textbooks in developing proportional reasoning.

Proportional reasoning in middle level mathematics textbooks

The Report of the 2012 National Survey of Science and Mathematics showed that in approximately four out of five mathematics lessons, students are completing textbook and worksheet problems (Banilower et al., 2013). Therefore, much of what students learn in mathematics class is coming directly from the textbooks. With this knowledge, it is important to ensure that mathematics textbooks are worthwhile tools for teachers to use with their students. Proportional reasoning has been an area of weakness for many middle school students (National Center for Education Statistics, 2013). Whereas textbooks are prominent tools used to develop students' mathematical thinking, one could ask if textbooks are lacking variety of tasks to addressing some critical components of proportional reasoning.

There are a variety of textbooks available today, including those thought of as traditional and those thought of as reform based. In traditional textbooks the teacher is encouraged to teach algorithms first, based on the organization of content in the text. When students are fluent in using the algorithm, they will then be expected to apply this knowledge to practical word problems (Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998). In contrast, reform textbooks generally encourage the teacher to let students problem solve to create their own solution method to practical word problems (Ben-Chaim et al., 1998).

Methodology

Recall the research questions I defined to guide my study. They are listed below.

1. Based on current research, what are the critical components of children's proportional reasoning?
2. What types of tasks are found in middle level mathematics textbooks to develop proportional reasoning?
    a. What are the characteristics of individual tasks?
    b. What is the potential of the tasks to provide students an opportunity to apply the critical components of proportional reasoning?

Due to the nature of the research questions, it was necessary to answer the first research question in order to be able to answer the second research question. I built upon the review of literature to address my first research question. I needed to obtain deeper knowledge of the critical components of proportional reasoning in order to create a tool to evaluate the middle level mathematics textbooks, which was completed in the literature review. The tool was developed by identifying the elements in the tasks that would determine the presence of each of the critical components. The findings from the first research question and the tool developed based on the information from the first research question helped me to answer the second research question.

To address the second question, I applied what I learned about the first research question to do an analysis of two different types of textbooks: reform textbooks and traditional textbooks. I included both types of textbooks since both are widely used in the classroom. From these two categories I picked a reform textbook series and a traditional textbook series. The reform textbook series chosen for this research was Connected Mathematics 3 (Lappan, Phillips, Fey, & Friel, 2014). The traditional textbook series chosen for this research was Glencoe Mathematics (Bailey et al., 2006). These textbooks are both from well-known, and commonly used publishers for mathematics textbooks.

The *Common Core State Standards for Mathematics* (National Governors Association, 2010) recommends that students learn about proportionality in sixth and seventh grades. Eighth grade standards do not specifically address proportionality but the topics in the eighth grade standards require students to apply their previous knowledge about proportionality (National Governors Association, 2010). Since this study is focused on the foundation of the development of proportional reasoning, the focus of the analysis will be on sixth and seventh grade textbooks.

Each task analyzed for this study was found in the ratios and/or proportions unit of each textbook. Although there may be components of proportional reasoning in other units in the textbook, this study is focused on the development of proportional reasoning. The section of the textbook about proportions is the main resource within the book for developing proportional reasoning; therefore the proportion unit of each textbook is the only unit analyzed for the critical components of proportional reasoning.

In order to identify the characteristics of the tasks, I organized the data I collected for each task using the headings found in Figure 3. For each task, I recorded whether or not the situation is proportional, non-proportional, or inversely proportional. I also recorded the semantic type of each task. Each task was identified as being associated sets, part-part-whole, stretcher and shrinker, well-chunked, or no context. An example of a problem with no context would be $\frac{1}{2} = \frac{x}{4}$. Since the numbers are not associated with a specific situation, there is no context. The quantity of the answer, the quantities in the problem, the quantity of the within measure, and the quantity of the between measure were all identified as being a whole number (an integer) or a non-integer. A non-integer is a number that is not a whole number. The information for quantities in problem and quantity in answer were collected but not used in the analysis because that information was not helpful in identifying any of the critical components. By recording these task characteristics, I was able to analyze the tasks to identify the presence of the critical components.

Within Measure Quantity	Between Measure Quantity	Quantity in Answer	Quantities in Problem	Problem Context	Relationship
Whole/ Non-Integer	Whole/ Non-Integer	Whole/ Non-Integer	Whole/ Non-Integer/ Both	Associated Sets/ Part-Part-Whole/ Stretcher and Shrinker/ Well-Chunked	Proportional/ Non-Proportional/ Inversely Proportional

Figure 3. Headings to Organize Data.

The first critical component is identifying proportional and non-proportional situations. I recorded whether each task was proportional, non-proportional, or inversely proportional. This data showed whether or not there was a mixture of proportional and non-proportional situations in the text. In order for students to be able to discern between a proportional and a non-proportional situation, they need to be exposed to both. Therefore, a textbook should include tasks with both proportional and non-proportional situations in order to develop the first critical component of proportional reasoning. Although identifying inversely proportional situations is not specifically identified as a critical component, it is a proportional situation that could be more difficult to identify as proportional. Students need to be able to identify an inversely proportional situation as proportional in order to attain mastery of the first critical component. Therefore, this data was collected as well.

The second critical component, identifying a ratio as a composed unit, is impossible to identify in a textbook task. In order to know whether a student was able to think about a ratio as a composed unit, it would be necessary to know what that student was thinking as they solved the task. Since this study does not include students' thoughts and focuses on only the task itself, this critical component was not included in the analysis of textbooks.

To identify the presence of the third critical component, multiplicative thinking, in the textbook tasks, I looked at the within measure space and between measure space ratio. Similar to the first critical component, it is not possible to measure a student's understanding of multiplicative thinking through a textbook task, but it is possible to identify elements that would encourage multiplicative thinking in textbook tasks. In order for students to develop multiplicative thinking it is imperative that students are able to practice identifying both the within and between measure space to fully understand the multiplicative nature of proportion problems. If a student understands the multiplicative nature of the within measure space, he or she understands that proportional ratios are scaled up or scaled down. If a student understands the multiplicative nature of the between measure space, he or she understands the invariance. Recording whether the within and between measure spaces are whole numbers or non-integers is important because students are generally more comfortable working with whole numbers. Since they are more comfortable working with whole numbers, students tend to choose between finding the scale-factor and finding the invariance based on which one will produce a whole number. Tasks that encourage multiplicative thinking will have a variation of whole and non-integer values for both the within and between measure space. The quantities in the problem and the quantity for the answer were collected for identifying elements of this critical component but did not prove to be useful during the analysis of these problems.

Lastly, I recorded the semantic type of the task to identify the variety of tasks being presented to students in the textbook. Students will be unable to develop flexibility, the fourth critical component, if they are not provided with a variety of semantic types. Each task was identified as one of the four semantic types or as a no context task. The semantic types are associated sets, part-part-whole, stretcher and shrinker, and well-chunked. The data collected could show lack of variety or an abundance of variety of semantics types and representations of proportions, which would show whether or not the textbook tasks allow students to think about proportional reasoning flexibly.

Limitations

The findings of this study are limited due to the resources used. Only two textbook series, one traditional and one reform, were used for this study. Since only one series for each type of textbook was analyzed, a generalization about reform or traditional textbooks cannot be made.

Although textbooks are oftentimes the basis of instruction in the mathematics classroom, it is not the only source of information for students in the classroom. Teachers can add to the material in the textbooks with instruction or supplemental materials to further develop students' proportional reasoning. Therefore, this study measures only the potential that each textbook has for developing students' proportional reasoning and cannot accurately measure students' actual understanding of proportionality.

Another limitation to this study is the lack of access to students' thoughts. This study would benefit from student interviews. This information would help to supplement the data from the textbooks and provide a better picture of the development of students' understanding of proportional reasoning. As previously stated multiplicative thinking was identified in the tasks through elements that would potentially develop that critical component since student understanding could not be measured directly through the tasks. Also, another critical component, identifying a ratio as a composed unit, was not analyzed at all because this critical component cannot be assessed without access to student's thought. If students were interviewed for this study, the development of these components could be measured more accurately.

Results

The results of the first research question were reported in the literature review. The critical components of proportional reasoning are identifying proportional and non-proportional situations, understanding a ratio as a composed unit, thinking multiplicatively, and flexibility. These identified critical components were used to create the tool that was used to record the characteristics of the textbook tasks.

There were 411 textbook tasks analyzed during this study. The sixth grade reform textbook (Lappan, Phillips, Fey, & Friel, 2014a) had 57 proportionality tasks, the sixth grade traditional textbook (Bailey et al., 2006a) had 105 proportionality tasks, the seventh grade reform textbook (Lappan, Phillips, Fey, & Friel, 2014b) had 75 proportionality tasks, and the seventh grade traditional textbook (Bailey et al., 2006b) had 174 proportionality tasks. Since each textbook has a different number of proportionality tasks, the results will be reported in percentages for comparison purposes.

To measure the first critical component of proportional reasoning, identifying proportional and non-proportional situations, each task was classified as being proportional, non-proportional, or inversely proportional. Table 1 shows the percentage of tasks that are proportional, non-proportional, and inversely proportional in each textbook. The textbooks analyzed for this study did not have many tasks in which students were asked to identify a situation as proportional or non-proportional. Although there were very few non-proportional tasks, it is noteworthy that each textbook had tasks that were non-proportional. No textbook in this study had tasks that were inversely proportional. As seen in Table 2, there is also in increase of non-proportional tasks from 6th grade to 7th grade.

Table 1

Percentage of Proportional, Non-Proportional, and Inversely Proportional Tasks in each Textbook

Textbook	Proportional	Non-Proportional	Inversely Proportional
7th Grade Traditional	96%	4%	0%
7th Grade Reform	93%	7%	0%
6th Grade Traditional	99%	1%	0%
6th Grade Reform	98%	2%	0%

Table 2

Percentage of Proportional, Non-Proportional, and Inversely Proportional Tasks in each Grade Level

Textbook	Proportional	Non-Proportional	Inversely Proportional
7th Grade Textbooks	95%	5%	0%
6th Grade Textbooks	99%	1%	0%

To measure the third critical component, multiplicative thinking, the within and between measure space values were found for each task. The within and between measure space was recorded as being a whole number or a non-integer. Table 3 shows the percentage of whole number and non-integer values in the within measure space for each task. The within measure space is a whole number in the majority of tasks for all textbooks. A whole number within measure space is most common in the 6th grade traditional textbook. A non-integer within measure space is most common in the 7th grade reform textbook. Table 4 shows an increase in non-integer within measure space values between 6th grade and 7th grade.

Table 3

Percentage of Non-Integer and Whole Number Values for the Within Measure Space

Textbook	Non-Integer Within Measure	Whole Number Within Measure
7th Grade Traditional	43%	57%
7th Grade Reform	49%	51%
6th Grade Traditional	18%	82%
6th Grade Reform	40%	60%

Table 4

Percentage of Non-Integer and Whole Number Values for the Within Measure Space in each Grade Level

Textbook	Non-Integer Within Measure	Whole Number Within Measure
7th Grade Textbooks	44%	56%
6th Grade Textbooks	26%	74%

Table 5 shows the percentage of whole number and non-integer values for the between measure space in each task. The between measure space is a non-integer in a majority of tasks for all textbooks. In addition to having the highest percentage of whole number within measure space values the 6th grade traditional textbook also has the highest percentage of non-integer between measure space values. Table 6 shows the percentage of whole number between measure space values increases between 6th and 7th grade.

Table 5

Percentage of Non-Integer and Whole Number Values for the Between Measure Space

Textbook	Non-Integer Between Measure	Whole Number Between Measure
7th Grade Traditional	69%	31%
7th Grade Reform	48%	52%
6th Grade Traditional	79%	21%
6th Grade Reform	63%	37%

Table 6

Percentage of Non-Integer and Whole Number Values for the Between Measure Space for each Grade Level

Textbook	Non-Integer Between Measure	Whole Number Between Measure
7th Grade Textbooks	63%	37%
6th Grade Textbooks	73%	27%

The last critical component, flexibility, requires students to be able to solve all four types of problems. Each task was identified as being one of the four semantic types or having no context. Table 7 shows the percentage of tasks found for each semantic type in each textbook. Stretcher and shrinker problems are the least represented semantic type. There are no stretcher and shrinker problems in the reform textbooks and only 13% of the tasks in the traditional textbooks are stretcher and shrinker problems. Both traditional textbooks have a higher percentage of no context tasks than any other percentage of semantic types. In contrast, reform textbooks have the significantly lower percentage for no context tasks.

Table 7

Percentage of Semantic Types Represented by Textbook Tasks

Textbook	Associated Sets	Well-Chunked	Part-Part-Whole	Stretcher and Shrinker	None
7th Grade Traditional	5%	19%	26%	13%	38%
7th Grade Reform	11%	44%	36%	0%	9%
6th Grade Traditional	12%	10%	13%	13%	50%
6th Grade Reform	46%	28%	26%	0%	0%

Another semantic type that is underrepresented in these tasks is associated-sets. This semantic type is only well represented in the 6th grade reform textbook. Associated-sets represents a smaller percentage of the tasks than the stretcher and shrinker problems in both the 7th grade traditional textbook and the 6th grade traditional textbook. This shows that well-chunked and part-part-whole are the most represented semantic types. Well-chunked tasks are common in every day life; for example, speed limits are given in miles per hour. In order to find out how many hours it would take you for a trip, you could use proportional reasoning. Associated-sets is also a common semantic type found in everyday situations. For example, if I know that one pizza feeds three people, I can use a proportional relationship to figure out how many pizzas I would need to feed twelve people. Since Associated-sets is a semantic type used in everyday life, like well-chunked problems, it is equally important that these types of problems are well represented in the tasks students are solving.

If students are only exposed to certain semantic types they may not be able to identify other semantic types as being a proportional situation. This would affect a student’s ability to discern between a proportional and non-proportional situation, which is the first critical component. This makes it is even more important for textbook tasks to represent all four types of proportionality problems.

Discussion

To analyze whether or not there is evidence of the critical components of proportional reasoning in the textbook tasks, it is necessary to compare the results of the first research question and the characteristics of each task. The individual tasks were analyzed to find the characteristics and now the tasks will be analyzed across textbook series to find whether or not the tasks provide students with an opportunity to apply the critical components of proportional reasoning.

The first critical component, identifying proportional and non-proportional situations is evident in the textbook tasks. The data showed that each textbook has tasks that are non-proportional, however it is a low percentage of all of the tasks. Although it should not be expected that half the tasks should be non-proportional, it is questionable whether or not students would be able to identify a proportional or non-proportional situation based on a handful of tasks. It is encouraging that the number of non-proportional tasks increases from 6th to 7th grade because this means that as students mature in their proportional reasoning they are being exposed to more non-proportional situations.

The non-proportional tasks in these textbooks were comparison problems. This means that the task was presented as two ratios and the students had to determine if the ratios were proportional. In this type of task, students are determining if the situation is proportional based on the numbers in the problem. There were no tasks that asked students to identify a situation as non-proportional based on the context. This means that these tasks have some evidence of this critical component of proportional reasoning, but this critical component could be more evident if the texts included more variety of tasks and tasks that asked students to identify a situation as non-proportional based on context as well as number structure. This may be an area in which the textbooks are lacking to support students in their development of proportional reasoning.

It is also interesting that no textbook included tasks that were inversely proportional. Although one would not expect a large number of inversely proportional tasks, it is concerning that there were none in any of the four textbooks that were analyzed. Unless the textbooks are supplemented with other materials, students using these textbooks would not be exposed to inversely proportional situations at all. This is an area of proportional reasoning that is clearing lacking in these textbooks.

Multiplicative thinking, the third critical component of proportional reasoning, is evident in the tasks. There is a variety of whole number and non-integer values for both the within and between measure spaces. Students can use and understand a multiplicative relationship even if the value is not a whole number. Whole numbers can encourage students to use that measure space because whole numbers are easier to work with than a non-integer. This variety of whole number and non-integer values helps students use both the within measure and between measure spaces when solving proportional tasks. It encourages students to switch between using the between measure space and the within measure space to solve their problems. Using both the within measure space and between measure space helps students to understand the multiplicative relationship in a proportional situation.

Although there is a variety of whole number and non-integer values for the within and between measure space in the problems, the within measure space is a whole number a majority of the time and the between measure space is a non-integer a majority of the time. This means that students may be more inclined to find the within measure space on a majority of problems. While this is still multiplicative thinking, it means that students may be thinking about the scale factor more often as being proportional than thinking about the invariance as being proportional. If a student is given a problem with a whole number within measure space and a non-integer between measure space, which the data shows is the structure of a majority of the problems in these texts, it is most likely that the students will solve the problem by finding the scale factor because it is a whole number. This type of number structure does not encourage students to use the between measure space and therefore can limit students' understanding of the multiplicative structure of proportionality.

The fourth critical component, flexibility, is somewhat, but not fully evident in the textbook tasks. All four semantic types were present in the traditional textbook, but one of the four semantic types, stretcher and shrinker problems, was not present in either of the reform textbooks. This means that students working through the reform textbook series may not have any exposure to stretcher and shrinker problems, which could make them less flexible in their proportional reasoning. The data also showed a high percentage of purely numerical tasks without a context compared to other problem types in the traditional books. A high percentage of "no context" tasks in a textbook reduces students' exposure to the semantic types drastically, which could cause them to be less flexible in their proportional reasoning. If students are developing proportional reasoning skills, it is necessary for them to solve problems with context so that they can make sense of the proportional situation.

Conclusion

The purpose of this study was to first identify the critical components of proportional reasoning and then to analyze whether or not textbook proportionality tasks provide students with the opportunity to apply those critical components of proportional reasoning. Defining the critical components and identifying missing elements of the critical components in the textbook tasks allowed me to analyze the types of proportionality tasks that are found in middle level textbooks.

The results of this study suggest that middle level textbook tasks show evidence of the critical components of proportional reasoning, but there are also key elements of these critical components that could be strengthened. It is possible that there may not be a perfect textbook for developing proportional reasoning in the middle level grades. Even if there is a perfect textbook for developing proportional reasoning, it is likely that not all schools will have this perfect textbook. Because not all textbooks have the perfect combination of tasks to develop proportional reasoning, it is important for teachers to recognize limitations of the textbook tasks in the textbook they use. With this analysis, teachers can add supplemental material to the textbook tasks in order to fully address all of the critical components of proportional reasoning.

Based on the results of this study, it is suggested that teachers using these textbooks, and possibly others, add materials covering inversely proportional situations since they are not covered in any of the tasks in the textbooks. Including inversely proportional tasks into the curriculum will help students to be able to understand proportionality better and prepare them for future mathematics concepts. It would also be wise for teachers to include tasks that require students to determine if a situation is proportional or non-proportional based on the context of the problem instead of the number structure. This would help students to be able to know when to use proportional thinking in real life situations, not just in a textbook task. Teachers should choose tasks that vary in whole number and non-integer in both the within and between measure spaces. Some textbook tasks provided this variety, but some extra tasks may need to be added to ensure that students can work with both the scale factor and the invariance. Teachers should also consider adding a variety of semantic types to the tasks they use in the classroom, since the tasks in these textbooks seem to be lacking in some areas. It would be beneficial to expose students to a wider variety of contexts to develop the proportional reasoning.

Proportionality is an important mathematical concept that is used in everyday life and in subsequent mathematics topics. To meet educational standards, succeed in subsequent mathematics topics, and be able to solve real world proportionality problems, students need to have an understanding of proportionality and be able to reason proportionally. Textbooks are one of the most common resources for teachers to use in the mathematics classroom so a lot of what students know about proportionality may be coming from textbook problems. Textbooks may not always have the perfect combination of tasks to fully teach proportional reasoning based on the critical components of proportional reasoning as they are identified by research. Teachers should be critical and thoughtful when selecting proportionality tasks for their students to complete to ensure that students understand this important mathematical concept that can be used in everyday life and can lead to success in future mathematics concepts.

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