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Tessellations: An artistic and mathematical look at the work of Maurits Cornelis Escher

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TESSELLATIONS:

AN ARTISTIC AND MATHEMATICAL LOOK AT THE WORK OF

MAURITS CORNELIS ESCHER

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Introduction

I first became interested in tessellations when my fifth grade mathematics teacher placed multiple shapes that would tessellate at the front of the room and we were allowed to pick one to use to create a tessellation. Students traced the outlines of these shapes, one right after the other and fitting them together. I chose one I thought looked like a fish. After my teacher explained how to create the tessellations with this template, I traced the shape over and over, fitting it into itself on my paper, similar to what is seen in Figure 1 below. I enjoyed coloring my shapes in with different shades of blue while adding scales and eyes to make them look like real fish. This was one of my favorite mathematics lessons because of the opportunity to be creative in geometry and make artwork, combining my two favorite subjects. After that one tessellation lesson in fifth grade, I do not recall seeing or learning more about tessellations in school.

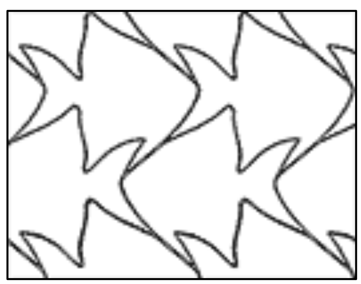

Figure 1. Fish tessellation (Tessellations.org).

The purpose of this study was to learn more about the mathematics of tessellations and their artistic potential. Whenever I have seen tessellations, I always admire them. It is baffling how such complicated shapes can be repeated infinitely on the plane. This research aimed to increase the amount of tessellation information and activities available to secondary mathematics teachers by connecting the tessellations of Maurits Cornelis Escher with their underlying mathematics in order to use them for teaching secondary mathematics. The overarching premise of this study was to create something that would also be beneficial in my career as a secondary mathematics educator.

Regular shapes, such as squares and hexagons, look the same from multiple sides so it is clear to see that they tessellate the plane, as seen in Figure 2 below. What intrigues me the most are tessellations that somehow fit into themselves repeatedly when the tile is more complex, such as the fish I used in elementary school. It is fascinating how a tessellated shape, when it is all by itself, can appear interesting and misshapen, but once there are many of the same shape in a repeated pattern, it creates something very orderly, consistent, and beautiful.

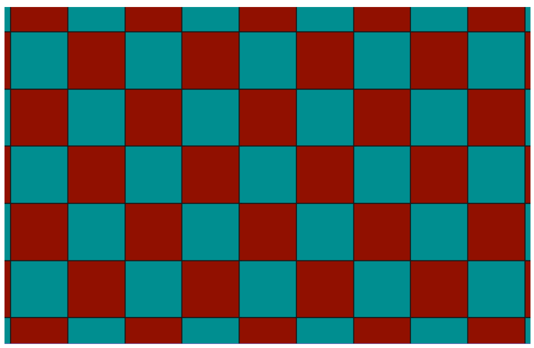

Figure 2. Tessellating squares. Created using *TesselManiac!* (Lee, 2014).

Through this research, I mathematically described the artistic tessellations of Maurits Cornelis Escher. Many of Escher's most popular works of art started with the foundation of a tessellation; I sought to understand the mathematics of these foundational tessellations as well as how Escher transformed polygons to create his designs. I applied what I learned about Escher's mathematical approach to art and created my own version of a tessellation similar to Escher's. The mathematics I researched served as a guide guide for my creation and provided a basis for including tessellations in my future teaching curriculum. The following research questions were used to focus my work and provide what I needed to create my own Escher-like tessellation.

1. How did Maurits Cornelis Escher create his artworks based on tessellations?
2. What kinds of mathematical tessellations serve as the foundation for artworks created by Maurits Cornelis Escher?
3. How can one create an artwork containing Escher-like tessellations?

Literature Review

There were two central themes to my thesis. The first theme was Escher's work connecting mathematics and art. Though Escher did not originally intend to do mathematics with his tessellations, many of his artworks today are clear examples of properties of multi-dimensional geometry and tessellations. The second main theme of my thesis was to create my own Escher-like tessellation. Using what I learned through research about the life of Escher, his methods and his media, and then combining this knowledge with research explicitly on tessellations throughout history, I was able to create my own variation on Escher's mathematical artwork.

Maurits Cornelis Escher

Maurits Cornelis (M.C.) Escher was a Dutch graphic artist born on June 17, 1898, in Leeuwarden in the Netherlands (Locher, 1971). As a young student, he did not excel in school but was excellent at drawing (Cooper, Spitzer, & Tomayko, 2013). Escher's father encouraged him to enroll in the School for Architecture of Haarlem after high school to pursue architecture as a career, since none of the other subjects interested Escher (Bool et al., 1981). After discovering that he preferred art, and at the encouragement of graphic artist Samuel Jessurun de Mesquita, Escher soon switched his major to graphic arts (Locher, 1971). After graduation, Escher traveled to Italy, lived in Rome from 1923 to 1935, and met his future wife, Jetta Umiker (Locher, 1971).

During his travels in the spring and summer while he lived in Rome, Escher drew whatever caught his eye and made prints of the drawings during the winter months (Bool et al., 1981). He continued creating prints and artwork based on landscapes and the beauty of Italy (Bool et al., 1981). Escher first became interested in tessellations on one of his last trips to Spain

when he visited the Alhambra in 1936 (Locher, 1971). The Alhambra is a fourteenth century Moorish castle located in Granada, Spain (Clemens & O'Daffer, 1977). Escher became fascinated with tile artwork he saw there, such as the one below in Figure 3 (Locher, 1971). The Moors are credited with creating the first known tessellations and were soon followed by the Egyptians (Clemens & O'Daffer, 1977). In both ancient cultures, evidence of tilings covering the entire floor of a structure with no open spaces were found in houses and ancient architecture (Clemens & O'Daffer, 1977). Based on these artworks, Escher concluded, "A surface can be regularly divided into, or filled up with, similar-shaped figures (congruent) which are contiguous to one another without leaving any open spaces" (Clemens & O'Daffer, 1977, p. 76). These were some of the earliest forms of tessellations that Escher was exposed to and these led to his enchantment with repetition in his artwork. This inspired many of his works involving tessellations.

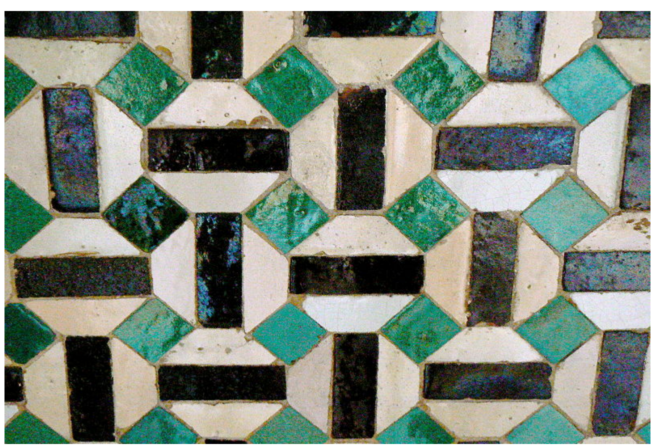

Figure 3. Moorish tiling from the Alhambra (Wikipedia.org).

Soon after 1936, Escher and his wife returned to the Netherlands where he started making prints that were no longer based on nature and real-life scenes; he began creating prints of patterns and abstract images (Bool et al., 1981). This new period in his work continued into 1937 when Escher began representing structures while playing with perspective; he even included a

possible reference to infinity in some artworks (Locher, 1971). He continued creating art using various media including prints, woodcuts, and pencil drawings until the end of his life in 1972 (The M.C. Escher Foundation, 2013).

Escher's tessellations. Starting in 1936, Escher created a total of 128 works of art based on tessellations. Most of his works were done in pencil, ink, or watercolor and then made into prints (Locher, 1971). He disliked finiteness and loved the idea of repetition, especially in artwork (Locher, 1971). To start, Escher based his early works on filling the two-dimensional plane completely, modeling this after tilings seen in Egyptian and Moorish architecture (Bool et al. 1981). However, most of these patterns were based on regular polygonal shapes in the plane (Clemens & O'Daffer, 1977). Escher wanted to create new shapes he could use to tessellate the plane that were not simply designs or patterns of regular polygons (Clemens & O'Daffer, 1977). He started creating shapes representing animals, insects, and other objects from the natural world. Moreover, Escher sought to link the continuous infinite pattern with real world shapes and ideas, specifically animals (Locher, 1971). In 1937, Escher's first original construction with his own 'real-world' shapes involving surface-filling patterns (tessellations) was created. Then, in 1944, Escher began to add a three-dimensional focus to his two-dimensional prints and drawings, as can be seen below in Figure 4, by manipulating space and perspective while playing with how the human eye perceives a work of art (Bool et al., 1981). These additions were very mathematical in nature; Escher even remarked once, "I often seem to have more in common with mathematicians than with my fellow artists" (Locher, 1971, p. 51).

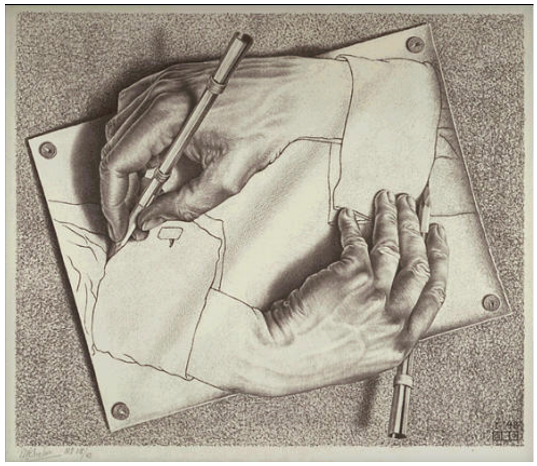

Figure 4. Drawing Hands – 1948 *Lithograph.* (The M.C. Escher Foundation, 2013).

The most prominent and popular pieces done by Escher depict recognizable beings. Figure 5 contains a tessellation-based work of art done in his later years. His love of the natural world is evident in the use of lizards to create the illusion of infinity in this piece as the lizards grow smaller and smaller down the page (Ranucci, 2007). As Cooper, Spitzer, and Tomayko (2013) stated, "One of his greatest 'games' was to manipulate a polygon while maintaining its ability to tessellate, so that the resulting shape would be recognized as a lizard, bird, fish, horse, or other figure" (p. 378). This particular work in Figure 5 was created using a triangle as the base polygon for the tessellation, then altering the triangle using some of the geometric transformation techniques that will be discussed later. The triangle foundation can easily be seen if the vertices (the apex where two adjacent sides of a polygon meet) from the lizard's nose, tail, and the extremely pointy leg of the lizard are connected. Mathematically, this is not a tessellation because the tile does not stay the same size throughout. Instead, Escher is illustrating what is known as the Poincare disk to illustrate the infinite, an idea he frequently revisited in his later years.

Figure 5. Lizard (No.101) – 1956 *Ink, Pencil, Watercolor.* (The M.C. Escher Foundation, 2013).

The Mathematics of Tessellations

A tessellation, as previously mentioned, is a specific type of pattern that involves the repetition of a shape multiple times that never ends. For the purpose of this study, the focus was on two-dimensional tessellations that cover a plane completely by repeating a pattern of polygons where there are no gaps or overlap, in a way that continues infinitely (Cooper, Spitzer, and Tomayko, 2013). Mathematically, this is done by completely covering 360° around a specified vertex point. A vertex in a tessellation is where the corners of polygons meet. Figure 6 depicts the initial drawings Escher sketched before completing Figure 7, which gives an illustration of a tessellation done in Escher's early years. As seen in Figure 6, Escher started this tessellation with a quadrilateral and then transformed it into a bird and a fish, depending on how he illustrated the shapes. Examples of this can be found in nearly all of his tessellation artworks where Escher used one polygon or another to form the foundation of the repeating pattern. Note that the ends can be filled with more of the shape by following the pattern; this is how Escher represented infinity in these works of art.

Figure 6. Flying Fish/Bird (No. 80) – 1950 *India Ink, Paint, Pencil.* (The M.C. Escher Foundation, 2013).

Figure 7. Flying Fish/Bird (No. 80) – 1950 *India Ink, Paint, Pencil.* (The M.C. Escher Foundation, 2013).

There are three mathematical categories of tessellations using polygons, regular, semi-regular, and Penrose. Additionally, all triangles and all quadrilaterals can tessellate the plane. Note that semi-regular tessellations and Penrose tilings will not be discussed here since Escher did not use them in his artworks. A regular polygon encloses a portion of the plane and is formed by line segments joined end-to-end where all of the angles are of equal measure and all of the sides are of equal length. Tessellations involving shapes such as these are called regular, or simple, tessellations (Clemens & O'Daffer, 1977). It is known that in a two-dimensional plane the only polygons that tessellate by themselves are all triangles, all quadrilaterals, and regular hexagons (Clemens & O'Daffer, 1977). It is important to note that not all regular polygons

tessellate the plane; for example, consider the regular pentagon in Figure 8. Regardless of how the shape is rotated, there will always be overlap or gaps so a regular pentagon will never cover the entire plane (Clemens & O'Daffer, 1977).

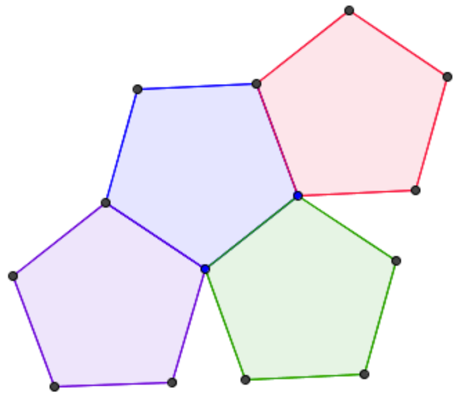

Figure 8. Regular Pentagons – created using Geogebra (Hohenwarter, 2001).

To understand why regular pentagons do not tessellate the plane, as seen in Figure 8 above, consider the three triangles that can be created when drawing diagonals from one vertex in a regular pentagon, in Figure 9. Since all triangles have a total angle measure of 180°, then the sum of the interior angles of a regular pentagon will equal 540°. Dividing 540° by 5 (since there are 5 interior angles on a pentagon and each has the same measure), then each interior angle measure is equal to 108°. In order for this pentagon to tessellate, 360° must be a multiple of (divisible by) 108° and leave no remainder. Since this is not the case, a regular pentagon cannot tessellate the two-dimensional plane. By using a similar argument, it can be seen that regular triangles, squares, and hexagons can be used to form regular tessellation.

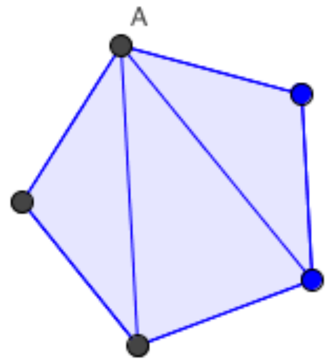

Figure 9. Regular Pentagon with diagonals from corner A – created using Geogebra (Hohenwarter, 2001).

While tessellations based on these regular polygons are some of the easiest to create and recognize, tessellations involving non-regular polygons also exist. A non-regular polygon is any polygon that does not have all sides *and* all angles of equal measure. Contrary to the above example of a regular pentagon, an example of a non-regular tessellating polygon is an irregular pentagon, as seen in Figure 10. In this figure, it is easy to see that all of the sides of the pentagon are not the same length, just as all of the angles are not the same measure. Since the sum of the three angles surrounding a vertex is 360°, the pentagons in Figure 10 tessellate the plane, versus the non-tessellating pentagon in Figure 8. To be certain, we would have to establish a pattern that would allow us to do this infinitely on the plane since the vertices in the tessellation below vary. Nonetheless, there are non-regular polygons that can tessellate the plane.

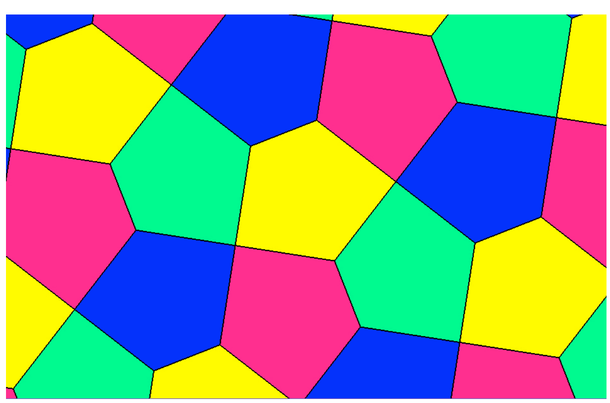

Figure 10. Non-regular Pentagons – created using *TesselManiac!* (Lee, 2014).

Geometric Transformations and Tessellations

Many of Escher's tessellation-based artworks do not look like polygonal tessellations, as seen in much of Escher's work (Clemens & O'Daffer, 1977). The tiles used by Escher are animals or abstract shape with no evidence of a polygon being used as the tile. These types of tessellations involve starting with a pattern made of polygons and then altering the sides and vertices of the polygon to form more exotic shapes (Clemens & O'Daffer, 1977). These alterations can consist of rotating a section of the shape around a chosen corner of the polygon or translating new sections along the side of the polygon and replacing certain sections with the transformations. All of these transformations must be done while taking into account the vertices and interior angle measures of each shape to ensure the resulting figure fits together to cover the plane infinitely (Clemens & O'Daffer, 1977).

As mentioned before, in order for a shape to tessellate, it must be able to completely cover 360° surround each vertex, meaning the total angle measure around the point is 360°. This can be done only if the sum of the interior angles of a shape divided by the number of sides found in this shape can divide 360°. In mathematical notation, $n =$ number of sides or vertices, and therefore 360 must be able to be divided by $\frac{(n-2)\cdot 180}{n}$ with no remainder. For example, consider the regular tessellation of equilateral triangles shown in Figure 11. If one side of the triangle in this tessellation is replaced with a curved side, this new curved side must then be translated, or slid, to each of the other sides in order for the shape to fit into itself, as can be seen in Figure 12. This would involve translating the new curved side onto each of the two other sides. The whole process of translating can be approximately seen, very basically, in Figure 13. Some other alterations may involve rotating the change or addition around a vertex and not among sides expressly (Clemens & O'Daffer, 1977).

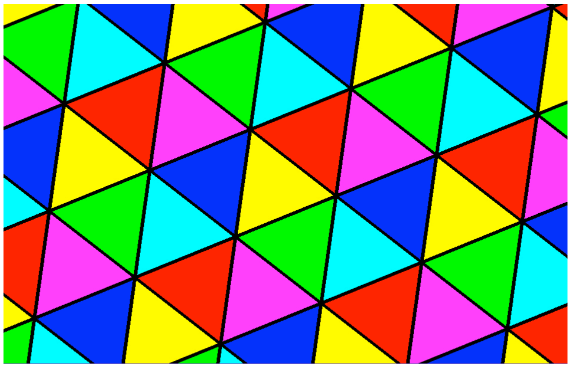

Figure 11. Equilateral Triangles – created using *TesselManiac!* (Lee, 2014).

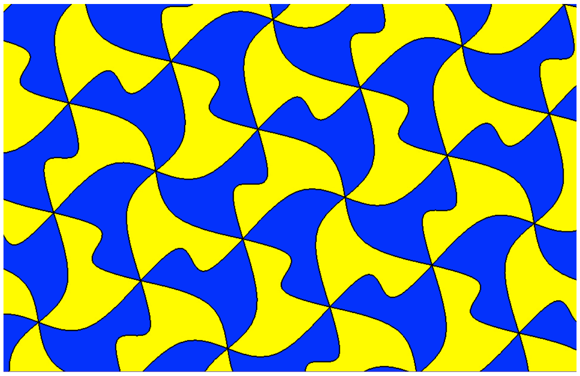

Figure 12. Tessellated Equilateral Triangles – created using *TesselManiac!* (Lee, 2014).

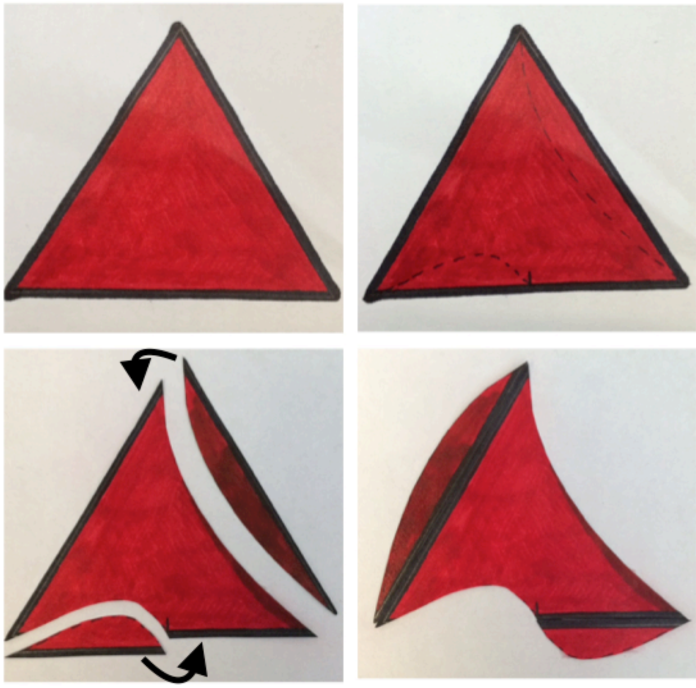

Figure 13. Translation process (Bachmeier, 2016).

In this way, Escher created his own fantastical tessellations. Escher knew that since regular polygons tessellate the plane, simply altering each side so it fits with another while still maintaining the same arrangement and number of vertices would result in a shape that would also tessellate (Locher, 1971). Throughout his life, Escher discovered how to create tessellating patterns without using mathematical properties of tessellations and transformational geometry; he probably used trial-and-error. This allowed Escher to develop a general sense about which shapes would tessellate better than others. It was through this methodical process that Escher's famous tessellations were created.

Methodology

In order to be able to create a tessellation just as M.C. Escher did, it was first necessary to understand what tessellations are and how they are created. To do this, I studied the mathematics of tessellations and applied what I learned to create my own design. I began with regular tessellations and then searched for patterns among the graphical representations of each regular polygon to be able to apply these same patterns among non-regular polygons and their tessellations. In doing so, I thus figured out which tessellations served as the best foundations to build on for my own Escher-like tessellation.

After investigating tessellations in general, I described the mathematics behind these tessellations in order to determine how a tessellation can be mathematically created. I developed proofs to show the regular polygons that tessellate the plane (triangles, quadrilaterals, and hexagons) as well as provided counterexamples of regular polygons that do not tessellate. I also proved that *all* triangles can be used to tessellate the plane and that the same is true of *all* quadrilaterals. Additionally, I described how transforming the sides of polygonal tiles, in terms

of rotations, reflections, or translations, will still retain the shape's properties allowing it to tessellate. This is the artistic component used by Escher to make his tessellation-based art.

Finally, I delved deeper into the works of Escher by examining how artistic tessellations were physically created. To do this, I used the software program *TesselManiac!* (Lee, 2014) in combination with paper and pencil exploration to experiment with Escher's techniques to create tessellations. I created a tessellation of my own based on Escher's methodology and the mathematics that I have studied. This was created with the help of technology and, at some points along the way, actual paint, ink, pencils, and paper as well to fill in the details.

Mathematical Proofs for Tessellating Polygons

Many of Escher's tessellations included several uses of varying shapes that tessellate the plane. Often, his works did not stick with just one shape, but instead combined more than one shape to produce an overall tessellating design, similar to the image found in Figure 14 taken from the Alhambra. However, this research focused on those tessellations involved only one geometric shape, not many different ones. For this reason, the following proofs are about geometric shapes that will tessellate the two-dimensional plane without the use of other shapes. Moreover, I will focus on triangles, quadrilaterals, and hexagons since they are the most mathematically interesting.

Figure 14. Tile from the Spanish Alhambra (Cooper, Spitzer, and Tomayko, 2013).

Conjecture I: *Equilateral triangles tessellate the plane.*

To prove this claim, first note that any triangle's interior angle measures total 180°. Given an equilateral triangle, divide 180° (the total interior angle measures) by 3 (total number of interior angles) to result in 60° for each interior angle measure. Pick any corner of the triangle and the corresponding interior angle will be 60°. Recall that 360° need to be covered at any vertex on the tessellation; dividing 360° by 60° is equal to six. Therefore, six of the same equilateral triangles around a chosen vertex will complete 360°. Likewise, around any arbitrary vertex in this tessellation, vertices in the tessellation can be surrounded by six triangles. Thus, equilateral triangles tessellate the plane.

Conjecture II: *All quadrilaterals tessellate the plane.*

To prove this claim, first consider any arbitrary quadrilateral, such as $DEFG$ in Figure 15 below. Dividing the quadrilateral using a diagonal creates two triangles, each with interior angle sums of 180°. Thus, the sum of the interior angles of any quadrilateral will be 360°. Choosing any single corner on the quadrilateral to form a vertex in a tessellation with one copy of each of the other angles of the quadrilateral will result in exactly 360°. This can be seen in Figure 16. Therefore, all quadrilaterals can be used to tessellate the plane.

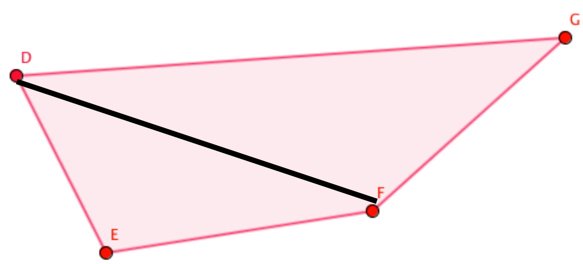

Figure 15. Created using Geogebra (Hohenwarter, 2001).

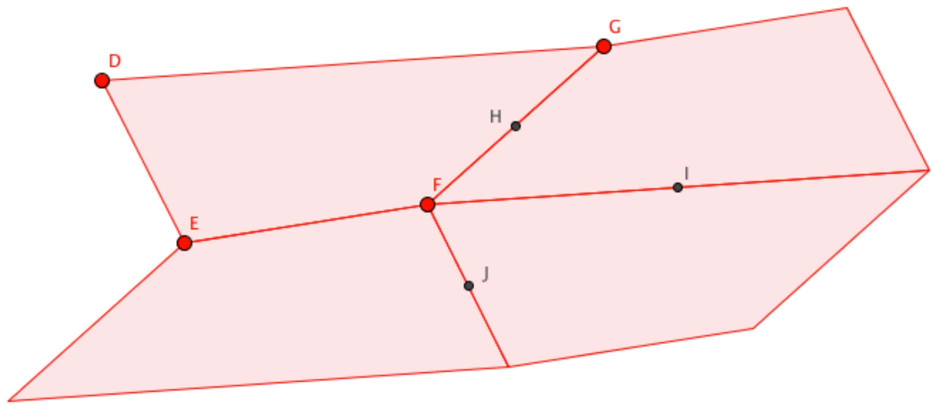

Figure 16. Created using Geogebra (Hohenwarter, 2001).

More specifically, to rotate this quadrilateral to form a tessellation involves using the midpoints of each side. Find the midpoint of $\overline{FG}$, point H, and then rotate the entire quadrilateral $180°$ around point H to produce a copy of the quadrilateral at the same vertex. Take the midpoint of side $\overline{DG}$ on the new quadrilateral, point I, followed by a rotation of $180°$ around this midpoint to produce a third copy of the same quadrilateral. Finally, take the midpoint of side $\overline{DE}$ on the new quadrilateral, point K, and rotate this most recently rotated quadrilateral $180°$ to surround the vertex at point F (refer to Figure 16). By repeating this process of finding midpoints and rotating the quadrilateral, we can tessellate the plane. Therefore, any quadrilateral can be tessellated (Clemens & O'Daffer, 1977).

Escher used a variety of quadrilaterals (trapezoids, rectangles, squares, parallelograms, etc.) in his tessellations. Since this proof shows that any quadrilateral will tessellate the plane, it can be shown that any of the different quadrilaterals used by Escher in his works will therefore tessellate the plane.

Conjecture III: *All triangles tessellate the plane.*

A direct result of now knowing that all quadrilaterals tessellate is the fact that all triangles, not only regular ones, tessellate the plane. To prove this corollary, consider the proof

involving quadrilaterals and note that any quadrilateral can be split into two triangles. Therefore, any triangle (not only equilateral triangles discussed earlier) has interior angles with a sum of 180° and additionally all quadrilaterals have two triangles contained within them. As such, then all triangles will tessellate the plane because all quadrilaterals tessellate the plane.

Conjecture IV: *All regular hexagons tessellate the plane.*

To prove this claim, consider the regular hexagon $LMNOPQ$ in Figure 17 in which each interior angle measure $120°$ ($\frac{(n-2)\cdot 180}{n}$ where $n = 4$). Constructing $\overline{LO}, \overline{MP}, and\ \overline{NQ}$ finds their intersection at point X in the center of the hexagon, forming six equilateral triangles. Because each interior angle of the hexagon measures $120°$, constructing these segments bisects each of these angles, forming angles of $60°$, as seen below. Since 360° can be divided by 120°, leaving no remainder, then we know that a hexagon with interior angle measure of 120° will tessellate. Therefore, a tessellation of a regular hexagon then becomes the actual tessellation of 6 equilateral triangles. Because we already proved that all triangles tessellate the two-dimensional plane and a regular hexagon is made of six equilateral triangles, then we can conclude that all regular hexagons tessellate the two-dimensional plane.

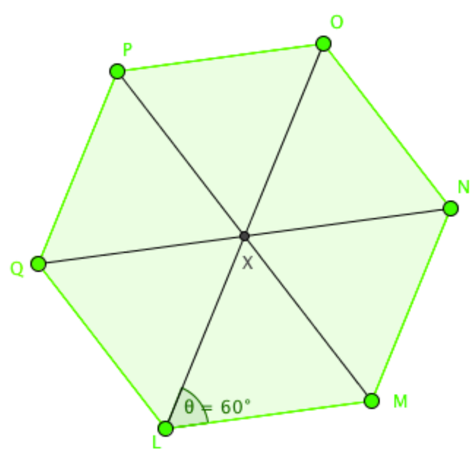

Figure 17. Created using Geogebra (Hohenwarter, 2001).

My Tessellation Creation

To this day, Escher's tessellations are still mystifying; it is amazing how Escher created his own tessellations through pencil and paper. How he did this makes sense mathematically, but the method he used, without technology, is a difficult and tedious process. In this day and age, technology is used to gain the precision that Escher achieved in his works. Additionally, speed is constantly a factor and computer software can accomplish many of the same tessellations at a much faster rate than that of a human hand.

Disregarding speed initially, I began simply by picking out shapes around me that I thought would tessellate. The first image explored was the beginning letter of my name, E. I found a particular pattern in which a capital letter E would tessellate and drew these out by hand. I initially used a ruler and pencil, but soon found that more precision was needed in order to depict a true tessellation. I then began cutting out shapes I knew would tessellate, such as a triangle or hexagon, from cardstock and then physically went about transforming the pieces by cutting off certain sections and translating these sections to corresponding parts of the shape. This also proved to be tedious and not quite as easily precise as I initially thought it would be. At this point, I turned to technology to assist in my tessellation creation.

Prior to finalizing my tessellation creation, I reviewed some of Escher's artworks to gain inspiration and ideas to draw from. Many of Escher's pieces depict animals; based on this fact, I decided to try to create an animal that was from the aquatic world. In trying to do this, many tessellations turned out to actually look like other things and not animals at all, such as leaves, random countries, or the sun and moon. I created many, many tessellation options before deciding on a final one, including several others involving aquatic creatures, some that looked like monsters, and others, which are located in Appendix B.

I utilized the software *TesselManiac!* (Lee, 2014) to assist in the drawing and constructing of my own tessellations. I created a tessellation entitled *Octopus Ocean* to display as the final piece for this project. My exploration with *TesselManiac!* (Lee, 2014) began by choosing which shape was desired to tessellate. The options include a triangle, quadrilateral, pentagon, and hexagon. After selecting a shape, the next step was to decide how to tessellate the tile. These options include simply translating (sliding) the shape so it fits into itself, rotating the shape around a fixed vertex, using a series of rotations and reflections to make the shape to fit into itself, or translating the shape followed by a 180° reflection. Combining translations, reflections, and rotations in varying degrees and orders resulted in very different tessellations each time. The last step when creating tessellations in *TesselManiac!* (Lee, 2014) was to manipulate the tessellating tile into the desired shape, rather than simply leaving it as the triangle or quadrilateral that it started out as. The manipulation was done by extending or shortening sides of the shape, adding more curved or sharp edges, or adding vertex points. Each time a specific side or point was adjusted or changed, the corresponding sides of the many other tessellating shapes changed accordingly. Refer to Figure 18 as an illustration of one of the steps in this process.

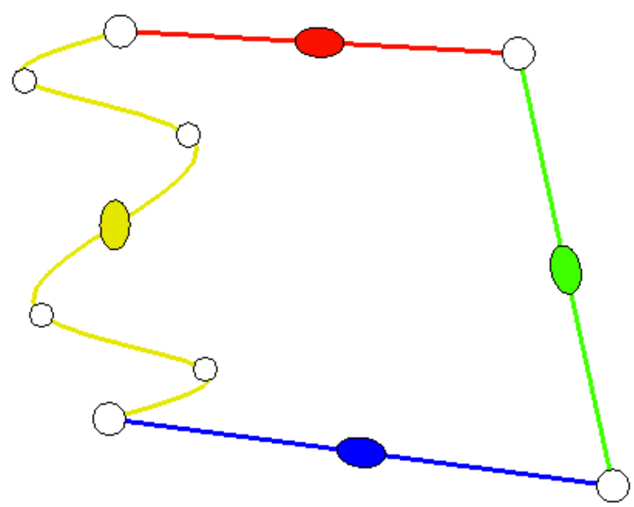

Figure 18. Created using *TesselManiac!* (Lee, 2014).

For my artistic application of the mathematics presented, I began with a rectangular tiling of the plane. My tessellation pattern involved rotating the entire tile around each midpoint of each side, as seen in the section where I proved all quadrilaterals can tessellate the plane. I began altering my tile to look like an aquatic animal, adding tentacles, tails, and curves by transforming sides of the tile until I thought the tile resembled an octopus. Next, I printed the black and white outline design of my tessellation I created using *TesselManiac!* (Lee, 2014) and added an artistic flair using paint, permanent markers, and other materials. The black and white version of my creation can be seen below in Figure 19. In order to get a better idea of the octopi I was envisioning in this piece, the final product was produced on a larger scale with physical materials.

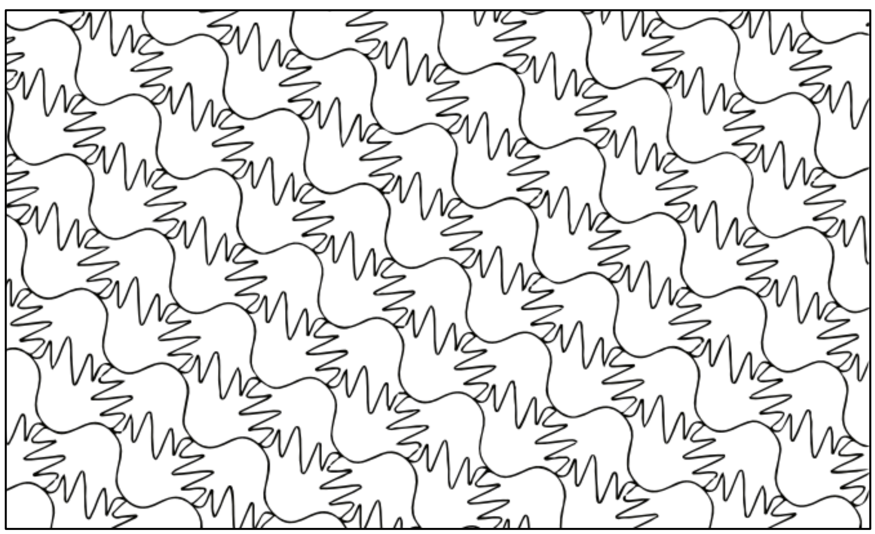

Figure 19. Octopus Ocean (Bachmeier, 2016).

Experimenting and utilizing *TesselManiac!* (Lee, 2014) was very beneficial for the creation of my own tessellation. The software made it possible to experiment with many of the different options for patterns of tessellating in order to develop ideas about which might work best for my own tessellation. I created several different tessellations throughout the whole process. However, in order to fully utilize *TesselManiac!* (Lee, 2014) to its full potential, prior

background research, proofs, and understanding tessellations mathematically was necessary. Completing this extensive research allowed me to better construct accurately tessellating patterns and to understand the algorithms behind the *TesselManiac!* (Lee, 2014) software.

Conclusion and Discussion

Maurits Cornelis Escher was a man of many talents, both mathematically and artistically. Examining and pulling apart his work was what helped to further understand the mathematics behind his tessellations and to, consequently, create my own Escher-like tessellation. Through my research and artistic exploration, I gained insight into the mind, life, and work of Escher. I learned about Escher's career and the experiences in his life that led him to create some of the finest works of art known today. Though Escher started out simply creating patterns of repetitions that he found intriguing, he eventually discovered the mathematical side to his creations and began incorporating that into his precise works of art. This mathematical component led to eluding towards infinity, something not quite explored extensively during Escher's time, especially by an artist.

For another part of this research, I investigated and proved mathematical theorems that would be useful in order to create an Escher-like tessellation using the many connections Escher made between mathematics and art. By doing this, I satisfied the curiosity that resulted in this research; tessellations do belong in mathematics classes, uniting art and mathematics. Just as I was able to do long ago in an elementary mathematics class, I had the opportunity with this study to be creative while also doing mathematics and eventually creating something beautiful and worthwhile. However, contrary to my tessellation of ten years ago, I now have a much better understanding of how tessellations mathematically work, the many different forms and varieties of tessellations, and how to effectively create a mathematically precise tessellation. Though I

mainly explored creating tessellations with the help of software, I also investigated with pencil, paper, and compass, just as Escher might have done with his tessellations. The hands-on approach helped to more fully understand how the process of creating a unique tessellation is completed and the many places for error along the way if precision is not carefully executed. Simply picking random shapes of objects or letters and trying to see if they might tessellate was an integral part of this research.

While researching the life and work of Escher was interesting, and creating my own tessellation helped make connections between mathematics and Escher's work, the original purpose of this study was to gain information that would meaningfully benefit my future career as a secondary mathematics teacher. To this end, I plan to use this research with my students and apply the knowledge I have gained to my future lessons. By using art to motivate the learning of mathematics, as it originally motivated me, it is my hope that future students will be able to connect more with mathematics in the classroom. A lot can be learned in a high school Geometry class, or even Algebra or Calculus classes, from Escher and the work he did that continues to be popular today. As mentioned in the beginning, the only tessellations I encountered in my formal education were in a fifth grade mathematics classroom. After researching Escher's tessellations and expanding on proofs of regular polygon tessellations, I now know that much more can be taught and investigated with respect to tessellations.

A key lesson that should be explored and taught to middle school students involves discovering which regular shapes tessellate by examining the number of triangles created in each shape and the ability of the interior angle measures of a shape to divide 360°. This is very simple for students to discover on their own, and coming up with a general "rule" to figure out if a shape tessellates the two-dimensional plane or not is mathematically appropriate for these students.

Additionally, I also believe that allowing students time to explore and make his or her own creations, whether using *TesselManiac!* (Lee, 2014) or by hand, is a way to connect mathematics to the creative parts of their lives. To this end, I will incorporate tessellation lessons in some form into my curriculum, whether at the high school or middle school level, in order to broaden students' horizons and to engage more students in the study of mathematics by bringing an artistic component into the classroom. Making tessellations is not a lost art, forgotten in the times of Escher; they are still relevant, exciting, and just as applicable today as they were when Maurits Cornelis Escher first explored and created tessellations.

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Appendix A

The first initial steps in creating my own tessellation involved selecting a regular shape, selecting a tessellation pattern, and transforming my shape, as shown below in figures A1 through A3.

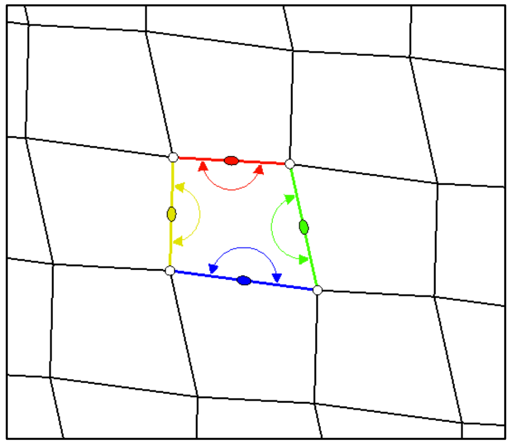

Figure A1. Octopus Ocean Phase 1 (Bachmeier, 2016).

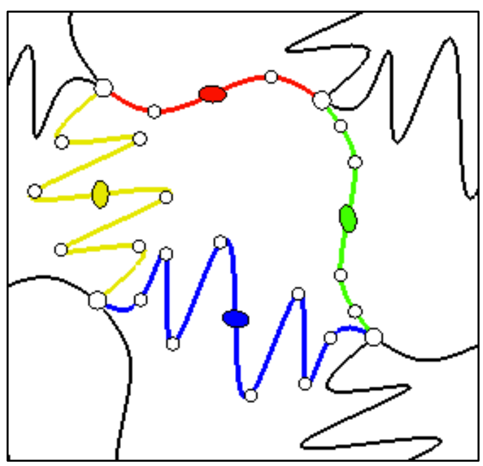

Figure A2. Octopus Ocean Phase 2 (Bachmeier, 2016).

Figure A3. Octopus Ocean Final Phase (Bachmeier, 2016).

Appendix B

When I was exploring the functionality of *TesselManiac!* (Lee, 2014). I discovered that it was enjoyable, relaxing, and entertaining to create tessellations. Below in figures B1 through B7 are various other tessellations that were created along the path to my final tessellation creation.

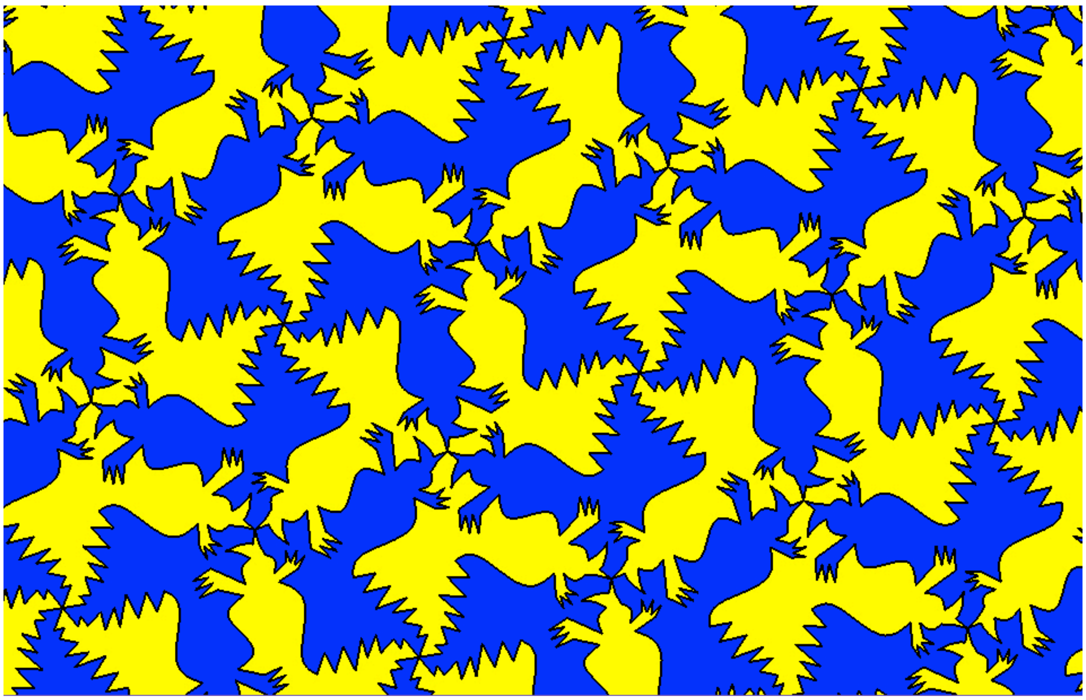

Figure B1. Monster #1 (Bachmeier, 2016).

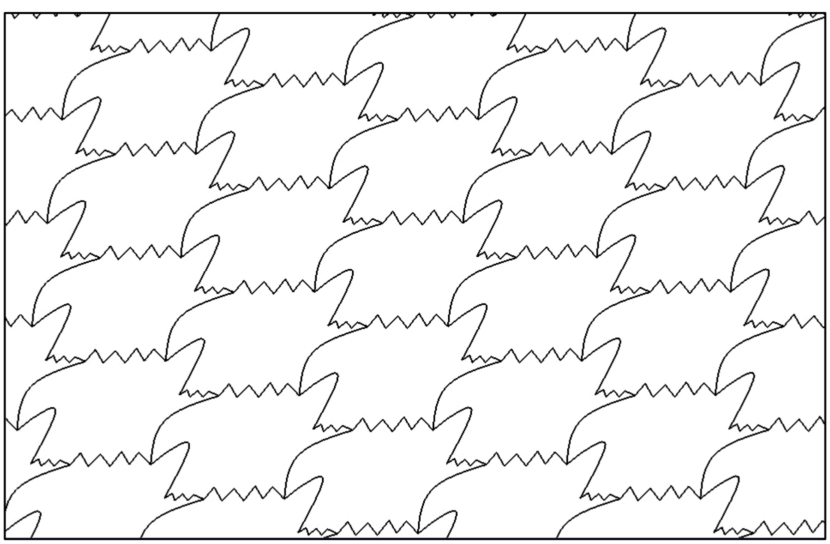

Figure B2. Monster #2 (Bachmeier, 2016).

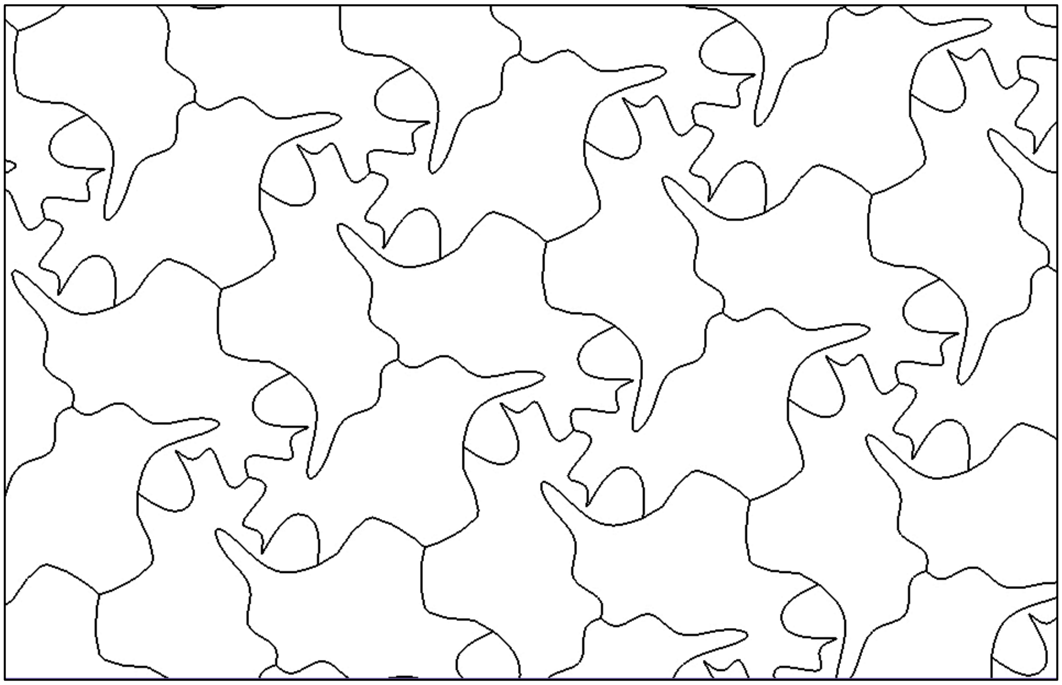

Figure B3. Leaves of Change (Bachmeier, 2016).

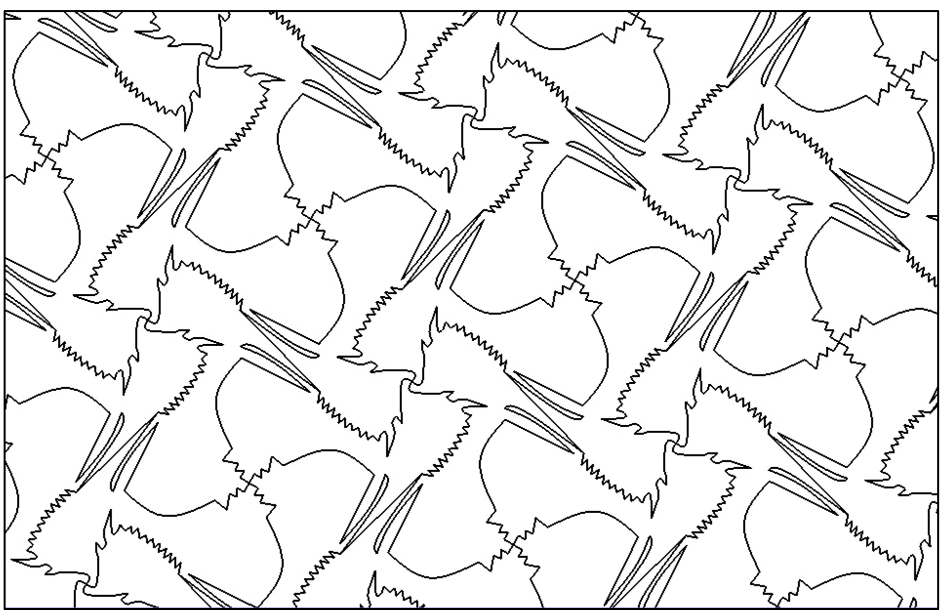

Figure B4. Monster #3 (Bachmeier, 2016).

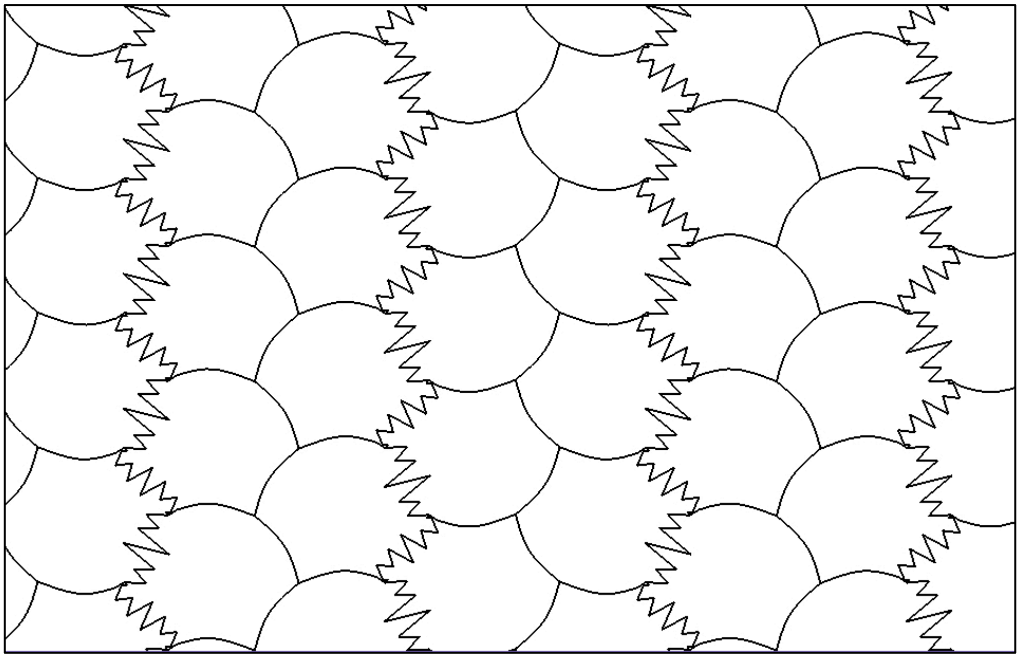

Figure B5. Sun and Moon (Bachmeier, 2016).

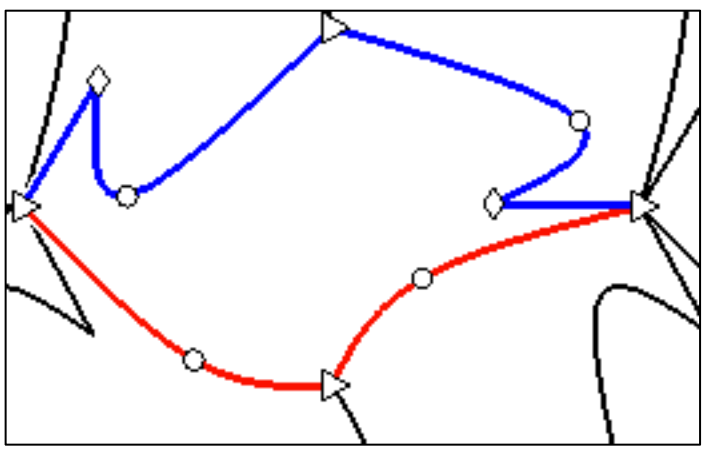

Figure B6. Whale Phase 1 (Bachmeier, 2016).

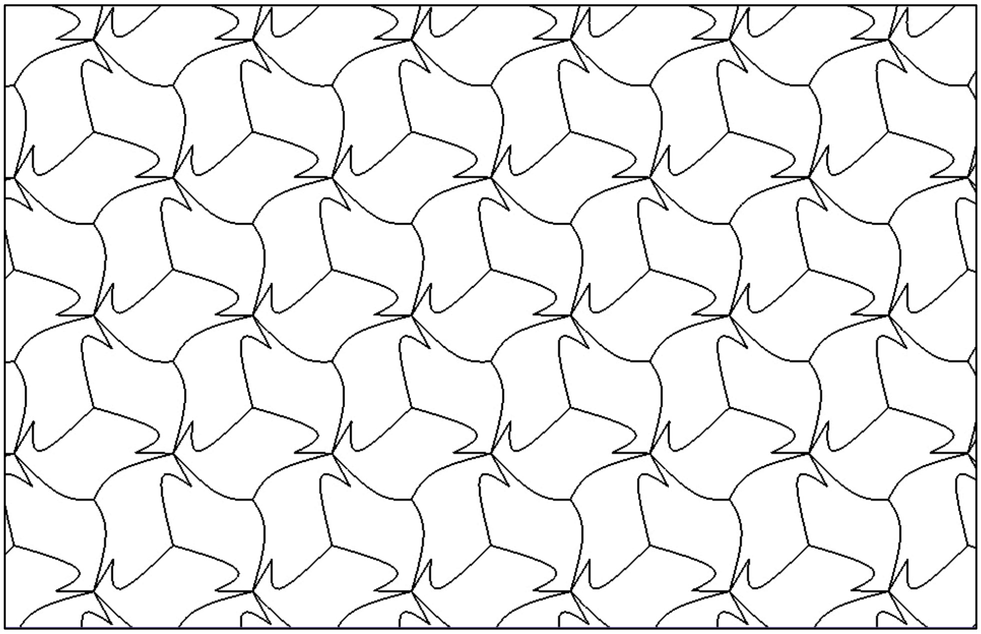

Figure B7. Whale Phase 2 (Bachmeier, 2016).