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K dimension continued fractions and k dimension golden ratios

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K DIMENSION CONTINUED FRACTIONS

AND

K DIMENSION GOLDEN RATIOS

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The following is an investigation dealing with continued fractions based on research conducted by Professor John C. Longnecker at the University of Northern Iowa.

CONTINUED FRACTIONS AND THE GOLDEN RATIO

An expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}} \qquad (1)$$

is called a regular or simple continued fraction. Standard continued fractions are very well behaved. The variables $a_1, a_2, a_3, \dots$ represent constants that may be allowed to take on values only in specific domains during certain applications of the continued fraction. For the purpose of this paper they will represent non-negative integers. The number of these variables, called elements, may be either finite or infinite. In the first case the fraction will be written in the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n}}} \qquad (2)$$

and called a finite continued fraction or to be specific an n^{th}-order continued fraction. The n^{th}-order continued fraction can also be denoted by $[\, a_1, a_2, \dots, a_n \,]$. In the second case the fraction will be written in the form (1) and is called an infinite continued fraction.

An example of a simple, or what will be called a two-dimensional, continued fraction using the four-tuple [1,2,3,4] follows.

$$[\,1, 2, 3, 4\,] = 1 + \frac{1}{[\,2,3,4\,]}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{[\,3,4\,]}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

$$= \frac{43}{30}$$

If an infinite continued fraction is filled with 1's then it can be represented by

$$[\underbrace{1, 1, 1, 1, 1}_{\infty}] = \quad 1 + \cfrac{1}{[\underbrace{1, 1, 1, 1, 1}_{(\infty-1)}]}$$

If we use the following

$$[\underbrace{1, 1, 1, 1, 1}_{\infty}] = A$$

A satisfies the equation

$$A = 1 + \frac{1}{A}$$

or $$A^2 = A + 1$$

which has the positive solution G. G is a well known number often referred to as the Golden Ratio (see Appendix B) and is equal to

$$= \frac{1 + \sqrt{5}}{2} \cong 1.62.$$

K DIMENSION CONTINUED FRACTIONS

It was felt by John C. Longnecker that there should exist infinite k-dimensional continued fractions which when filled with 1's would produce their own "Golden Ratios". This lead to the birth, in 1988, of k-dimensional continued fractions which, although ill behaved, produce those and many other things.

For convenience we will introduce the notation for the function $g_k(A)_n = [a_1, a_2, a_3, \ldots, a_n]_k$ where k identifies the dimension, or order, of the continued fraction, A is the collection $a_1, a_2, \ldots, a_n$, and n identifies the cardinality of A.

Now we will show the same four-tuple A = [1,2,3,4] as a 3-dimensional continued fraction. Because it is defined recursively, as will be shown in detail later, we will start with the parts that will be needed to calculate the continued fraction.

$$g_3(A)_1 = [4]_3 = 4$$

$$g_3(A)_2 = [3, 4]_3$$

$$= 3 + \frac{1}{[4]_3} = \frac{13}{4}$$

$$g_3(A)_3 = [2, 3, 4]_3$$

$$= 2 + \frac{1}{[3,4]_3} + \frac{1}{[3,4]_3[4]_3}$$

$$= 2 + \frac{4}{13} + \frac{1}{13} = \frac{31}{13}$$

$$g_3(A)_4 = [1, 2, 3, 4]_3$$

$$= 1 + \frac{1}{[2,3,4]_3} + \frac{1}{[2,3,4]_3[3,4]_3}$$

$$= 1 + \frac{13}{31} + \frac{13}{31} \cdot \frac{4}{13} = \frac{48}{31}$$

The reader should note that the answer, 48/31, is different than 43/30 which was obtained with the same numbers using a 2-dimensional evaluation.

Now returning to the general function defined to represent k-dimensional continued fractions, $g_k\,(A)_n$ is defined as follows

$$g_k\,(A)_n = g_n\,(A)_n \qquad \text{for } n < k$$

$$g_k\,(A)_n =$$

$$a_1 + \frac{1}{[a_2, a_3, \ldots, a_n]_k} + \frac{1}{[a_2, a_3, \ldots, a_n]_k\,[a_3, a_4, \ldots, a_n]_k} + \ldots$$

$$+ \frac{1}{[a_2, a_3, \ldots, a_n]_k\,[a_3, a_4, \ldots, a_n]_k \ldots [a_{n-k}, \ldots, a_n]_k} \qquad \text{for } n >= k$$

It might be noted that if k = 2, then this will produce two terms; $g_k\,(A)_n$ has k terms in the sum (for n >= k). If we let

$$R_{1,k} = [a_2, a_3, \ldots, a_n]_k\,,\ R_{2,k} = [a_3, a_4, \ldots, a_n]_k\,, \ldots,\ R_{n-k-1,k} = [a_{n-k}, \ldots, a_n]_k$$

then

$$g_k\,(A)_n = a_1 + \frac{1}{R_{1,k}} + \frac{1}{R_{1,k}\,R_{2,k}} + \ldots + \frac{1}{R_{1,k}\,R_{2,k} \ldots R_{n-k-1,k}}$$

If the reader wishes they may easily prove to themselves that this produces

$$g_k\,(A)_n = a_1 + \frac{1}{R_{1,k}}\left(1 + \frac{1}{R_{2,k}}\left(1 + \frac{1}{R_{3,k}}\left(1 + \left(\ldots + \frac{1}{R_{n-k-1,k}}\right)\ldots\right)\right)\right)$$

which is a preferable expression when performing calculations. The degree k to which the continued fraction is carried out is called the dimension.

We will examine one behavior (fortunately controlable) of a particular family of these k-dimensional continued fractions that produces strange results.

SPECIAL CONTINUED FRACTIONS

For the purposes of this paper we will not be working with any continuous fractions of degree less than 3. We will confine our attention to a specific sequence of continued fractions defined by the function $f_3\,(x)_n$ where n identifies the number of zeros in front of the final positive variable, x, and 3 is the degree or dimension of the continued fraction.

The following give several representatives for the sequence with n increasing.

$$f_3\,(x)_0 = [\,x\,]_3 = x$$

$$f_3\,(x)_1 = [\,0\,x\,]_3 = 0 + \frac{1}{[\,x\,]_3}$$

$$f_3\,(x)_2 = [\,0\,0\,x\,]_3 = 0 + \frac{1}{[\,0\,x\,]_3} + \frac{1}{([\,0\,x\,]_3)([\,x\,]_3)} = 0 + \frac{1}{[\,0\,x\,]_3}\left(1 + \frac{1}{[\,x\,]_3}\right)$$

$$f_3\,(x)_3 = [\,0\,0\,0\,x\,]_3 = 0 + \frac{1}{[\,0\,0\,x\,]_3} + \frac{1}{([\,0\,0\,x\,]_3)([\,0\,x\,]_3)} = 0 + \frac{1}{[\,0\,0\,x\,]_3}\left(1 + \frac{1}{[\,0\,x\,]_3}\right)$$

The general case for 3 dimensions would be expressed:

$$f_3\,(x)_n = [\,\underbrace{0\,0\ldots0}_{n}\,x\,]_3 = 0 + \frac{1}{[\,\underbrace{0\,0\ldots0}_{(n-1)}\,x\,]_3} + \frac{1}{([\,\underbrace{0\,0\ldots0}_{(n-1)}\,x\,]_3)([\,\underbrace{0\,0\ldots0}_{(n-2)}\,x\,]_3)} \qquad (3)$$

$$= 0 + \frac{1}{[\,\underbrace{0\,0\ldots0}_{(n-1)}\,x\,]_3}\left(1 + \frac{1}{[\,\underbrace{0\,0\ldots0}_{(n-2)}\,x\,]_3}\right)$$

This is a recursive sequence because in order to find $f_3\,(x)_n$, it is necessary to find $f_3\,(x)_{n-2}$ and $f_3\,(x)_{n-1}$.

If we examine the results that arise from the calculation of $f_3\,(x)_n$ as n grows, we might begin to see a pattern. Here is the sequence, as calculated through n = 9:

$$f_3(x)_0 = x$$

$$f_3(x)_1 = \frac{1}{x}$$

$$f_3(x)_2 = x + 1$$

$$f_3(x)_3 = 1$$

$$f_3(x)_4 = \frac{x + 2}{x + 1}$$

$$f_3(x)_5 = \frac{2x + 2}{x + 2}$$

$$f_3(x)_6 = \frac{2x}{2x + 2}$$

$$f_3(x)_7 = \frac{3x + 4}{2x + 3}$$

$$f_3(x)_8 = \frac{4x + 5}{3x + 4}$$

$$f_3(x)_9 = \frac{5x + 7}{4x + 5}$$

LIMIT OF SPECIAL CONTINUED FRACTIONS

By observing the pattern in the previous set of fractions, it might suggest convergence to a limit; the existence of such a limit is herein assumed to be A_3. Let us now find what condition A_3 must meet. Using the definition of a 3-dimensional continued fraction:

$$A_3 = \lim_{n\to\infty} f_3(x)_n = \lim_{n\to\infty} \{[\underbrace{0\,0\ldots0}_{n}\,x]_3\}$$

$$= \lim_{n\to\infty} 0 + \frac{1}{[\underbrace{0\,0\ldots0}_{n-1}\,x]_3} + \frac{1}{[\underbrace{0\,0\ldots0}_{n-1}\,x]_3[\underbrace{0\,0\ldots0}_{n-2}\,x]_3}$$

As n goes to infinity, $\lim_{n\to\infty} [\underbrace{0\,0\ldots0}_{n-1}\,x]_3$ and $\lim_{n\to\infty} [\underbrace{0\,0\ldots0}_{n-2}\,x]_3$ are the same as $\lim_{n\to\infty} [\underbrace{0\,0\ldots0}_{n}\,x]_3$. This is because as n goes to infinity there are an infinite number of zeros before x and the removal of one or two zeros from the series does not change it. Consequently,

$$A_3 = \lim_{n\to\infty}\{[\underbrace{0\,0\ldots0}_{n}\,x]_3\} = \lim_{n\to\infty}\{[\underbrace{0\,0\ldots0}_{n-1}\,x]_3\} = \lim_{n\to\infty}\{[\underbrace{0\,0\ldots0}_{n-2}\,x]_3\}$$

and

$$\lim_{n\to\infty} f_3(x)_n = A_3$$

then A_3 satisfies the equation

$$A_3 = \frac{1}{A_3} + \frac{1}{A_3 \cdot A_3}$$

or more clearly $$A_3^3 = A_3 + 1. \qquad (4)$$

Another form of this, $A_3^3 - A_3 - 1 = 0$, will be utilized later.

For another useful form, the equation (4) can be rewritten using the following:

Given $$S = A^n + A^{n-1} + \ldots + 1 \qquad (5)$$

then $$A \cdot S = A^{n+1} + A^n + \ldots + A$$

$$A S - S = A^{n+1} - 1$$

$$S = \frac{A^{n+1} - 1}{A - 1}$$

hence $$A_3^3 = \frac{A_3^2 - 1}{A_3 - 1} .$$

Also if the function $f_3(A)$ is defined as

$$f_3(A) = A^3 - A - 1 \qquad (6)$$

and we use the fact given earlier that $A_3^3 - A_3 - 1 = 0$, it then follows that $f_3(A_3) = 0$.

All of the properties demonstrated with the series of third-dimension equations can be illustrated in the general case for the k-dimension.

Each of the functions $f_k(x)_n$ will have their own limit A_k which satisfies the equation

$$f_k(A_k) = 0$$

or $$A^k = A^{k-2} + A^{k-3} + \ldots + 1$$

and each have their own function

$$f_k(A) = A^k - A^{k-2} - A^{k-3} - \ldots - 1. \qquad (7)$$

If we observe that

$$f_3(A) = A^3 - A - 1$$

and $$f_4(A) = A^4 - A^2 - A - 1$$

$$= A(A^3 - A - 1) - 1$$

$$= A f_3(A) - 1$$

then we can see that $f_4(A)$ can be found recursively. It can be shown inductively that the functions $f_k(A)$ are recursive in the following way

$$f_k(A) = A\, f_{k-1}(A) - 1. \qquad (8)$$

These functions can also be represented in another form that is derived from (7). Remember that

$$f_k(A) = A^k - A^{k-2} - A^{k-3} - \ldots - 1 .$$

It can easily be seen that

$$f_k(A) = A^k - (A^{k-2} + A^{k-3} + \ldots + 1)$$

which by applying (5) yields

$$f_k(A) = A^k - \frac{A^{k-1} - 1}{A - 1} \quad \text{for } A \neq 1 \qquad (9)$$

LIMIT OF THE LIMIT OF SPECIAL CONTINUED FRACTIONS

If we plot a portion of the $f_k(A)$ functions for values of x between 1 and 2 we see the graph that appears in Appendix A. It appears from this graph that the functions are all increasing and that there is a limit to the A_k's. The proof of these two claims follows. The limit will be called L and it will be defined as follows:

L is the limit as k increases without bound of A_k. (10)

If we remember that $f_k(A_k) = 0$, $(A_{k,0})$ is where each of the functions cross the x-axis; we are finding the limit of where they cross.

First of all, we are interested in the slope to see if the functions are all increasing. Looking at the recursive definition of the function $f_4(A)$ as given in (8) at a specific place, say A_3, we find the interesting result that

$$f_4(A_3) = A_3 f_3(A_3) - 1$$

$$= -1$$

since A_3 is, by definition, the number that satisfies the equation $f_3(A) = 0$.

Further, for each k,

$$f_k(A_{k-1}) = -1$$

or as the graph in Appendix A demonstrates, the function $f_k(A_{k-1}) = -1$ at the same time that $f_{k-1}(A)$ crosses the x-axis at A_k.

To determine the slope, we look at the derivative of the function f_3 (A) as represented in (6), it will be

$$f_3'(A) = 3A^2 - 1$$

which establishes that

$$f_3'(A) > 0 \qquad \text{for } A > 1 \qquad\qquad (12)$$

and calculus guarantees that the function f_3 (A) is an increasing function at all points A > 1.

Now using (8) we can progress to the function of the next higher degree

$$f_4(A) = A\, f_3(A) - 1$$

$$f_4'(A) = A\, f_3'(A) + f_3(A)$$

If we substitute to find the slope of f_4 (A) where $A = A_3$ (i.e. the point shown earlier where the function f_3 (A) crosses the x-axis and $f_4(A_3) = -1$) we obtain

$$f_4'(A_3) = A_3\, f_3'(A_3) + f_3(A_3)$$

$$= A_3\, f_3'(A_3)\,.$$

Since we are working in the positive range of numbers A_3 is always positive and $f_3'(A_3)$ is always positive as shown in (12). The multiplication of two positive numbers always produces a positive number so the slope of f_4 (A) is positive at $A = A_3$.

Now if we look at the slope for all points $A > A_3$ we get

$$f_4'(A) = A\, f_3'(A) + f_3(A) > 0 \qquad (A > A_3)$$

establishing that f_4 (A) is increasing at all points $A > A_3$.

It can also be shown inductively that f_k (A) is increasing for $A > A_{k-1}$. This then forces $A_4 > A_3$ and in general $A_{k+1} > A_k$. This then increases the credibility of the accuracy of the graph as shown in Appendix A and establishes the sequence of A_k's as an increasing sequence.

The graph strongly suggests that the functions cross the x-axis once. The interval of that crossing for all f_k (A) is somewhere between the points 1 and 2, shown next.

We can observe from (8) that $f_k(0) = -1$ for all n. If we substitute 1 for A we find

$$f_k(1) = 1\, f_{k-1}(1) - 1. \tag{11}$$

If we start with the fact that

$$f_3(A) = A^3 - A - 1$$

we therefore have

$$\begin{aligned} f_3(1) &= 1^3 - 1 - 1 \\ &= -1. \end{aligned}$$

If we continue using recursive definition (11) then

$$f_4(1) = (-1) - 1 = -2$$

$$f_5(1) = (-2) - 1 = -3$$

$$\vdots$$

It is then possible to prove inductively that

$$f_k(1) = -(k-2)$$

which proves that each of the series of functions $f_k(A)$ are always negative at 1.

From (9) if we look at $f_k(2)$ we obtain

$$\begin{aligned} f_k(2) &= 2^k - \frac{2^{k-1} - 1}{2 - 1} \\ &= 2^k - 2^{k-1} + 1 \\ &= 2 \cdot 2^{k-1} - 2^{k-1} + 1 \\ &= 2^{k-1} + 1 \end{aligned}$$

which gives positive values for all k. Since each $f_k(A)$ is a polynomial and hence continuous and using the facts that f_k (1) is always negative and f_k (2) is always positive, then the mean value theorem establishes that the functions must cross the axis between those two values or A_k will always be between 1 and 2. This, along with (10), establishes that L also must lie between 1 and 2. Since we also know that, by definition, $f_k(A_k) = 0$, A_k is the point where it crosses. Therefore f_k for $A < A_k$ must be negative and f_k for $A > A_k$ must be positive.

To further refine the range where the limit L lies we will turn once again to G, the Golden Ratio. As explained earlier, G satisfies the following equation

$$G^2 - G - 1 = 0$$

$$\text{or} \qquad G^2 - 1 = G \qquad (13)$$

Therefore if we look at f_3 (A) from (6) at G we obtain

$$\begin{aligned} f_3(G) &= G^3 - G - 1 \\ &= G(G^2 - 1) - 1 \\ &= G . G - 1 && [\text{by } (13)] \\ &= G^2 - 1 \\ &= G && [\text{by } (13)] \end{aligned}$$

If any $f_k(G) = G$ then by (8)

$$\begin{aligned} f_{k+1}(G) &= G \cdot f_k(G) - 1 \\ &= G \cdot G - 1 \\ &= G^2 - 1 \\ &= G \end{aligned}$$

hence $f_k(G) = G$ for all $k \geq 3$ by induction. (14)

Since G is a positive number less than 2 and $L = \lim_{k \to \infty} (A_k)$ then $L \leq G$. This says that the limit L must exist between 1 and G.

Since $L \geq A_k$ for all k (because A_k is an increasing sequence) then $f_k(L) \geq f_k(A)$ for all k because

f_k is an increasing function in the range. But $f_k(A) \geq 0$ for $A \geq A_k$ so $f_k(L) \geq 0$ for all k.

If we look at $f_k(L)$ by (9)

$$f_k(L) = L^k - \frac{L^{k-1} - 1}{L - 1} \geq 0 \qquad \text{for all } k$$

$$\frac{L^{k+1} - L^k - L^{k-1} + 1}{L - 1} \geq 0$$

$$L^{k-1}(L^2 - L - 1) + 1 \geq 0$$

$$L^{k-1}(L^2 - L - 1) \geq -1$$

$$L^2 - L - 1 \geq -\frac{1}{L^{k-1}} \qquad \text{for all } k$$

and since $\lim_{k \to \infty} -\dfrac{1}{L^{k-1}} = 0$ (since $L > 1$) then

$$L^2 - L - 1 \geq 0 \quad \text{for all } k$$

noticing that this is in the form of the Golden Ratio (13) we can obtain $L \geq G$.

We now have $L \leq G$ and $L \geq G$ which mean that it must be true that

$$L = G = \frac{1 + \sqrt{5}}{2} \quad .$$

It is rather interesting to note that not only is the limit of the series A_k equal to G but that all of the functions $f_k(A)$ at that limit G are equal to G (shown in 14). It is also interesting to further note that A_k is independant of the value of x in the initial definition of this series of special continued fractions. Major questions about the general behavior of k-dimensional continuous fractions remain unanswered at the time of this investigation.

K DIMENSION GOLDEN RATIOS

Now that the reader has some idea of what k-dimensional continued fractions are, we will expand on the exsitance of k-dimensional golden ratios. The development of k-dimensional continued fractions offers the ability to complete a mathematical pattern that previously had a gap in it. As was explained earlier, the 2-dimensional golden ratio can be obtained by filling a 2-dimensional, infinite continued fraction with 1's. There are other ways to obtain the golden ratio, one way, using the Fibinacci Numbers will be demonstrated next.

The Fibinacci Numbers are a series of numbers obtained through the following:

Let F be a series of numbers defined by $f_k = f_{k-1} + f_{k-2}$. If we let $f_1 = 0$ and $f_2 = 1$ then we develop the traditional Fibinacci series that follows:

0 1 1 2 3 5 8 13 21 34 55 . . .

If we form fractions by using this sequence as follows

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{8}{5} \quad \frac{13}{8} \quad \frac{21}{13} \quad \frac{34}{21} \quad \frac{55}{34} \quad \ldots$$

the limit of this sequence of fractions is the golden ratio $= \dfrac{1+\sqrt{5}}{2}$.

It is interesting to note that the limit of a sequence produced in this manner will be the golden ratio regardless of the non-negative integers f_1 and f_2 used to start the series.

We can find the 3-dimensional golden ratio in a similar manner by using the Tibinacci Numbers. Let F be a recursive sequence of numbers defined by the following: $f_k = f_{k-1} + f_{k-2} + f_{k-3}$. If we let $f_1 = 0$, $f_2 = 0$ and $f_3 = 1$ we will develop the sequence that follows:

0 0 1 1 2 4 7 13 24 44 81 . . .

If we form fractions in the same way as done before, the limit of those fractions is the 3-dimensional golden ratio. This is extended to k generations leading to k-dimensional golden ratios but only the 2-dimensional golden ratio had a continuous fraction form that was easy to derive. Using k-dimensional continued fractions gives the ability to easily express k-dimensional golden ratios.

If we take an 3-dimensional, infinite continued fraction and fill it with 1's, we get the following equation:

$$[\underbrace{1,1,1,1,1}_{\infty}]_3 = 1 + \frac{1}{[\underbrace{1,1,1,1,1}_{(\infty-1)}]_3} + \frac{1}{[\underbrace{1,1,1,1,1}_{(\infty-1)}]_3 \, [\underbrace{1,1,1,1,1}_{(\infty-2)}]_3}$$

Once again, all of the continued fraction terms are equal so that if we let $A = [\underbrace{1,1,1,1,1}_{\infty}]_3$, the equation produces:

$$A = 1 + \frac{1}{A} + \frac{1}{A^2}$$

which can be simplified to

$$A_3^3 = A_3^2 + A_3 + 1$$

where A_3 is the 3-dimensional golden ratio.

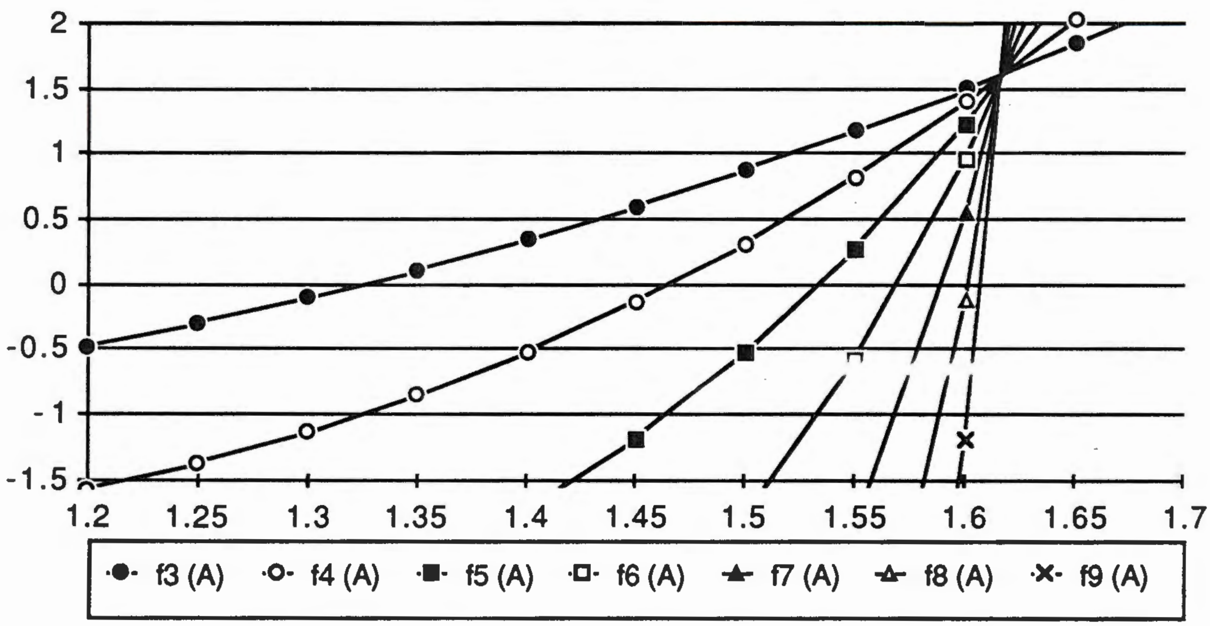
This can easily be extended into k-dimensions where each A_k is the golden ratio for the k^{th} dimension and satisfies the equation:

$$A_k^k = A_k^{k-1} + A_k^{k-2} + A_k^{k-3} + \ldots + 1.$$

This form leads to many manipulations leading to notation for each golden ratio much easier than that which was previously used.

It is also interesting to note that the limit of the k-dimensional golden ratios is 2.

APPENDIX A



2
1.5
1
0.5
0
-0.5
-1
-1.5
1.2
1.25
1.3
1.35
1.4
1.45
1.5
1.55
1.6
1.65
1.7
f3 (A)
f4 (A)
f5 (A)
f6 (A)
f7 (A)
f8 (A)
f9 (A)

APPENDIX B: THE GOLDEN RATIO

The golden ratio is the number, A, where A is the positive solution to the quadratic equation

$$A^2 = A + 1$$

$$A = \frac{1 + \sqrt{5}}{2}.$$

The golden ratio was first discovered and used by the Pythagoreans and was called the Divine Proportion. It shows up in many places throughout mathematical and geometrical relationships. A rectangle that is made so that the proportion of its sides are equal to the golden ratio is pleasing to the eye and has been used in ancient and contemporary architecture, including the Parthenon at Athens.

With this number being utilized as much as it is, there is a possibility that if the derivation were made easier, golden ratios in other dimentions would also be utilized.