Game theory and its practical applications

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Game Theory and its Practical Applications

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by

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Game theory, once considered a purely mathematical science, has evolved much in recent years. Although still considered a branch of mathematics because of its need for rigor and precise analysis, game theory has several applications to the social and biological sciences. This paper provides a survey of the historical evolution of game theory, defines game theory and the principle theorem serving as its cornerstone, and discusses practical applications of game theory in both economic and social situations.

I. Historical Overview

Although some scholars believe that the historical forerunner of game theory was probability theory, probability theory did not originate until the seventeenth century. Probability theory, or the theory of games of chance, is traced by most historians to 1654, when French mathematicians Blaise Pascal and Pierre de Fermat, inspired by the gambling misfortunes of French nobleman Chevalier de Mere, made efforts to solve practical problems of gambling (Colman, 1982, p. 11). Historical evidence points out that game theory, or the theory of games of strategy, can be dated to much earlier times.

I. A. Early Contributions to Game Theory

The beginning of game theory can be traced to ancient times. The Babylonian Talmud, written during the first five centuries A.D., is the compilation of ancient law and tradition which serves as the basis of Jewish religious, criminal, and civil law. One dilemma discussed in the Talmud is the so-called marriage contract problem: A man has three wives whose marriage contracts specify that, in the case of his death, they are to receive estates worth 100, 200, and 300
[units] each. The Talmud apparently gives contradictory recommendations. In the case where the man dies leaving an estate of only 100, the Talmud recommends equal division among the wives. If the estate is worth 300, however, then the Talmud recommends the proportional division of (50, 100, 150). For an estate worth 200, the recommendation is a mysterious (50, 75, 75). This particular dilemma has baffled Talmudic scholars for two millennia. In 1985, in the *Journal of Economic Theory*, it was recognized that the Talmud anticipates the modern theory of cooperative games. Each solution corresponds to the nucleolus of an appropriately defined game (Walker).

Centuries later, in a letter dated 13 November 1713, James Waldegrave provided the first known minimax mixed strategy solution to a two-person game. Waldegrave’s letter, written to Pierre-Remond de Montmort, included discussion about a two-person version of the card game *le Her*. Waldegrave’s solution is a minimax mixed strategy equilibrium, but Waldegrave made no general extension of his result to other games. He was concerned that a mixed strategy “does not seem to be in the usual rules of play” of games of chance (Walker).

It was not until 200 years later that the first major contribution to game theory was made. In 1912, German mathematician Ernst Zermelo proved that every strictly competitive game of perfect information possesses either a guaranteed winning strategy for one of the players or guaranteed drawing strategies for both. This result applies to games such as chess and tic-tac-toe in which each player knows what moves have been made previously, but Zermelo failed to provide a systematic method for determining the winning or drawing strategies. It has been found that tic-tac-toe has drawing strategies for both players; if it is played rationally it must end in a draw. In the case of chess, however, it is still unknown as to whether one of the players has a winning strategy or whether both players have drawing strategies, although many speculate that the latter is true (Colman, 1995, p. 12 - 13).

A decade later much of the groundwork of game theory was laid by French mathematician Emile Borel. Between 1921 and 1927 Borel published several notes on strategic games. Borel gave the first modern formulation of mixed strategy, and he found the minimax
solution for two-person games with three or five possible strategies (Walker). But even Borel could not prove the fundamental Minimax Theorem (see Section III), a theorem which serves as the cornerstone of formal game theory (Colman, 1995, p. 13).

I. B. John von Neumann, the RAND Corporation, and the Arms Race

In the mid 1920s, Austro-Hungarian mathematician John von Neumann began studying the mathematical structure of poker and other games; he wondered if there is a rational way to play every game. Von Neumann, also popularly known for his major contributions to the design of high-speed electronic computers that were important in the development of the hydrogen bomb, was the first major figure in game theory. He approached his “poker problem” with mathematical rigor, and in 1928 he succeeded in proving the Minimax Theorem applicable to two-person zero-sum games. Von Neumann succeeded in proving that there is always a right, or optimal, way to play such games. He believed that the Minimax Theorem would pave the way for a more formal game theory that would eventually include other types of games, including those of $n$-players and those where the players’ interests partially overlap. So expanded, he believed, game theory could be applied to any type of human conflict (Poundstone, p. 7).

It is worth noting that although von Neumann is often given credit for “inventing” game theory, strong parallels between the papers of von Neumann and Borel do exist. For instance, both used poker as an example and both examined the problem of bluffing. And just as von Neumann appreciated the potential economic and military applications of game theory, so did Borel. As minister of the French Navy in 1925, Borel even warned against overly simplistic applications of game theory to warfare. Perhaps most importantly, Borel asked the two basic questions of game theory even though he did not develop these issues very far: For which games
is there a best strategy? and How does one find such a strategy? Many believe that von Neumann was jealous of prior claims to his innovation, and von Neumann's early works make but scant mention of Borel. Lest there be any doubt, von Neumann's longtime friend, mathematician Stanislaw Ulam, reported that von Neumann's work was in fact inspired by one of Borel's papers (Poundstone, pp. 41-42).

In 1944, von Neumann and Princeton economist Oskar Morgenstern published their analysis of game theory in *Theory of Games and Economic Behavior*. Their purpose for writing this book was to analyze problems about how people behave in economic situations. In their words, these problems "have their origin in the attempts to find an exact description of the endeavor of the individual to obtain a maximum of utility, or, in the case of the entrepreneur, a maximum of profit" (Morris, p. viii). Von Neumann and Morgenstern presented their game theory as a mathematical foundation for economics, and their book stimulated much interest among mathematicians and mathematically-inclined economists (Colman, 1995, p. 13). A review of their book in the "American Mathematical Society Bulletin" predicted, "Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science -- the science of economics" (Poundstone, pp. 7-8).

One of the places where game theory was immediately accepted was at a prototypic think tank called the RAND (Research AND Development) Corporation. The RAND Corporation is a nonprofit research organization that was started by the U.S. Air Force after World War II. The Corporation studies various policy problems of the U.S., especially those involving national defense (World Book Encyclopedia, Q-R, p. 134). The RAND Corporation was originally developed to perform strategic studies on intercontinental nuclear war, and for that reason RAND
took on a large orbit of stellar thinkers as full-time researchers and outside consultants. RAND even considered game theory important enough to hire von Neumann as a consultant and to devote a great deal of effort not only to studying military applications of game theory but also to doing basic research in the field (Poundstone, p. 8).

Interestingly, in the years following von Neumann and Morgenstern’s publication, game theory acquired a strange reputation among the general public. Many of the early researchers in game theory were supported partially or entirely by the U.S. Department of Defense or were somehow connected to the RAND Corporation. These researchers worked on problems involving nuclear confrontation with the Soviet Union and wrote about these problems as if they were simply interesting, complex games. Because the bloody realities of nuclear war were hardly mentioned, game theory was often identified with war gaming and was thought of as cold and inhuman. At the same time, the power of game theory was overstated. It was believed that game theory could solve problems that were, in fact, far too difficult for it and for the technology (namely computers) of that time. Later, in a reaction to this, there was a tendency to undervalue game theory. In reality, game theory is neither all-powerful nor merely a mathematician’s toy without relevance to the real world (Morris, p. vii); this becomes more apparent the deeper one delves into the topic of game theory.

In 1949 the Soviet Union exploded its first atomic bomb in Siberia thus ending the U.S. nuclear monopoly. The Soviet bomb sparked a nuclear arms race and the world now had two atomic powers. For the first time in the history of the world, it was now possible to launch a surprise nuclear attack that would completely wipe out an enemy nation. By 1950 some U.S. citizens decided that the U.S. should consider an immediate, unprovoked nuclear attack on the Soviets; this idea went by the euphemistic name of preventive war. Today it may not seem
conceivable that the preventive war movement would find much more support than with a lunatic fringe; in reality, however, many of the undeniably intelligent of this country supported the movement. Among these supporters were two of the brightest mathematicians of the time: Bertrand Russell and von Neumann. And while these two men differed in many aspects, they did agree that the world did not have room for two atomic superpowers (Poundstone, p. 4).

Russell spoke publicly in favor of issuing an ultimatum to the Soviets threatening them with nuclear devastation unless they surrendered sovereignty to a U.S.-dominated world state. In a 1947 speech, Russell said: “I am inclined to think that Russia would acquiesce; if not, provided this is done soon, the world might survive the resulting war and emerge with a single government such as the world needs” (Poundstone, p. 4). Von Neumann took an even tougher stance in favoring a surprise nuclear first-strike. He was quoted in Life magazine as saying, “If you say why not bomb them [the Soviets] tomorrow, I say why not today? If you say today at five o’clock, I say why not one o’clock?” (Poundstone, p. 4).

Clearly, neither Russell nor von Neumann had any love for the Soviets, and they believed that preventive war was foremost a matter of logic. They saw preventive war as the only rational solution to the deadly dilemma of nuclear proliferation. As Russell put it in a 1948 New Commonwealth article advocating preventive war: “The argument that I have been developing is as simple and as unescapable as a mathematical demonstration” (Poundstone, p. 4). But logic itself can go awry. Perhaps nothing captures the whole bizarre episode of preventive war better than the unintentionally Orwellian words of U.S. Secretary of Navy Francis P. Matthews, who in 1950 urged the nation to become aggressors for peace (Poundstone, pp. 4 - 5).

In January of 1950, two RAND scientists made what some scholars believe is the most influential discovery in game theory since its inception. Melvin Dresher and Merrill Flood
carried out an experiment which introduced the game now known as the Prisoner’s Dilemma (Walker). This Prisoner’s Dilemma is a simple, baffling game that challenged part of the theoretical basis of game theory back in 1950 and is an intellectual riddle that still puzzles many today (Poundstone, p. 8). Although theorists now realize that the Prisoner’s Dilemma is a universal concept that has practical applications to biology, psychology, sociology, economics, law, and other disciplines where a conflict of interests may exist, Prisoner’s Dilemma was “discovered” just as nuclear proliferation and the arms race were becoming serious concerns. The tensions of the early nuclear era are a classic illustration of a Prisoner’s Dilemma (see Section V) (Poundstone, p. 9).

Perhaps author William Poundstone best summed up the concept of the Prisoner’s Dilemma when he wrote:

In the last years of his life, von Neumann saw the realities of war becoming more like a fictional dilemma or the abstract games of his theory. The perils of the nuclear age are often attributed to “technical progress outstripping ethical progress.” This diagnosis is all the more disheartening for the suspicion that there is no such thing as ethical progress, that bombs get bigger and people stay the same. The Prisoner’s Dilemma has become one of the premier philosophical and scientific issues of our time. It is tied to our very survival (Poundstone, p. 9).

Study of the Prisoner’s Dilemma has great power for helping to explain why human and animal societies are organized as they are. Indeed, it is one of the great ideas of the twentieth century, simple enough for anyone to grasp and of fundamental importance.

I. C. John Nash

After von Neumann, the next major figure in game theory was RAND consultant and mathematician John Nash. In the early 1950s, Nash took game theory to a level that von Neumann and Morgenstern had not; Nash studied non-cooperative games where coalitions are
forbidden. Von Neumann and Morgenstern's handling of games of more than two players focused on coalitions, or groups of players who choose to act in concert. These game theorists supposed that rational players would consider the consequences of joining every possible coalition and choose the one most advantageous. This approach seems reasonable given von Neumann and Morgenstern's grand aim, which was to treat economic conflicts as n-person games (Poundstone, pp. 96 - 97). For example, businesses collaborate to fix prices or to drive a competitor out of the market and workers join unions to bargain collectively. In each instance it is logical to expect that parties will form coalitions whenever it is advantageous. In effect, this is the basis of a laissez-faire, or free-market, economy (Poundstone, p. 97).

The only type of non-cooperative games that von Neumann concerned himself with were two-person zero-sum games, which are necessarily non-cooperative. Because one player's gain is the other player's loss, there is no point in forming a coalition in two-person zero-sum games. That case, however, was already covered by von Neumann's Minimax Theorem, and so Nash's work was primarily concerned with non-zero-sum games of three or more players (Poundstone, p. 97).

Von Neumann's Minimax Theorem proved that any two rational beings who find their interests diametrically opposed can nonetheless settle on a rational course of action in confidence that the other will do the same. This rational solution of a zero-sum game is an equilibrium reached as a result of self-interest and mistrust, where the mistrust is reasonable in view of the opposing aims of the players (Poundstone, p. 97).

Nash expanded on von Neumann's findings by showing that equilibrium solutions also exist for non-zero-sum two-person games. In two papers entitled "Equilibrium Points in N-Person Games" (1950) and "Non-cooperative Games" (1951), Nash proved the existence of a strategic equilibrium for non-cooperative games, which is referred to as the Nash equilibrium (see Section IV) (Walker). Although it might seem that when two players' interests are not completely opposed, it would be even easier to find a rational solution, but this is not the case. In fact it is often harder, and such solutions are often less satisfying (Poundstone, p. 97).
I. D. Other Contributions to Game Theory

In 1957, RAND alumni Duncan Luce and Howard Raiffa published the text *Games and Decisions*. The book begins: “In all of man’s written record there has been a preoccupation with conflict of interest; possibly only the topics of God, love, and inner struggle have received comparable attention” (Poundstone, p. 236). They gave the Prisoner’s Dilemma much emphasis in this book, and wrote: “The hopelessness that one feels in such a game as this cannot be overcome by a play on the words *rational* and *irrational*; it is inherent in the situation” (Poundstone, p. 121). As a whole, this book made game theory accessible to a wide range of social scientists and psychologists (Colman, 1995, p. 13).

In 1962, Martin Shubik wrote “Incentives, Decentralized Control, the Assignment of Joint Costs and Internal Pricing,” which turned out to be one of the first applications of game theory to cost allocation. An article describing an early use of game theory in insurance was written by Karl Borch in 1962. In “Application of Game Theory to Some Problems in Automobile Insurance,” Borch indicated how game theory can be applied to determine premiums for different classes of insurance when required total premium for all classes is given. In 1994, Douglas G. Baird, Robert H. Gertner, and Randal C. Picker published *Game Theory and the Law*, one of the first books in law and economics to take an explicitly game theoretic approach to the subject (Walker). In a very short time it had become obvious to scholars in a wide range of disciplines that game theory could be applied not only to mathematics, but also to economics, politics, foreign policy, as well as other areas of the social and biological sciences (Poundstone, pp. 5 - 7).

II. Defining Game Theory

Game theorist and author Anatol Rapoport defined game theory in an unconventional yet effective manner when he wrote:
Game theory is largely concerned with the classification of games, and in this it has much in common with other sciences which at a certain stage of their development were concerned mainly with classification. For example, biology was for many centuries a classification science (a taxonomy). Biologists sought a "proper" way to classify plants and animals. It would seem at first that what the "proper" principles of classification are depends largely on what the classifier is interested in. For instance, someone coming into frequent contact with animals in a primitive life environment might classify animals into large and small, or into dangerous and harmless, or into edible and inedible. There comes a time, however, when observation and description of nature becomes more or less separated from immediate functional interests. Accordingly, biologists soon recognized that although mice and lizards were both small animals while horses and crocodiles were both large animals, nevertheless mice were more closely related to horses than to lizards while crocodiles were more closely related to lizards than to horses (Rapoport, pp. 15 - 16).

Rapoport continued by saying:

The principle according to which game theory classifies games is best understood if game theory is viewed as the branch of mathematics concerned with the formal aspect of rational decision. The emphasis here is on the word formal, which in this context means devoid of content (Rapoport, p. 16).

As stated earlier, game theory is considered a branch of mathematics because of its need for rigor and precise analysis, and mathematics does not consider content in situations of formal relations. For example, arithmetic is not concerned with apples, candy bars, or tuition dollars; rather, it is concerned with the relationship among numbers. Geometry is not concerned with land tracts or shapes of objects but only with spatial relationships. Analogously, a mathematical theory of rational decision is concerned not with the dilemma of making wise decisions but with the logical structure of problems which arise in connection with the necessity of making decisions (Rapoport, p. 16).

The mathematical theory of games is concerned with the logic of decision making in conflict situations. Game theory does not deal with any particular game but with all of them, not
with technical but with theoretical matters. For example, “What is the best way to play Chess?”
is not a game-theoretical question, whereas “Is there a best way to play Chess?” is a game-
theoretical question (Rapoport, p. 14).

Game theory is applicable to conflict situations where the following three properties hold:

(1) There are two or more decision makers, called players;
(2) Each player has a choice of two or more ways of acting, called strategies, such that the outcome of the interaction depends on the strategy choices of all of the players;
(3) Each player has well-defined preferences among the possible outcomes so that numerical payoffs reflecting these preferences could be assigned to all players for all outcomes (Colman, 1995, p. 3).

An essential feature of these conflict situations is that each player only has partial control over
the outcome and that each player is perfect in reasoning and in mental skills. Games in which
the outcome depends solely or partially on dexterity or physical strength are therefore not
relevant to the matter at hand (Bacharach, p. 1). Thus, it is immediately obvious that games such
as chess and poker fall within the bounds of game theory, while other activities that are
commonly referred to as games (i.e. hopscotch, space invaders, and solitaire) are not really
games in the technical sense (Colman, 1982, p. 3).

The primary objective of mathematical game theory is to determine, through logical
reasoning alone, what strategies the players should decide upon and what outcomes will result.
Strictly mathematical game theory is therefore normative rather than descriptive in that it seeks
to discover how players should act in order to pursue their own interests most effectively as
opposed to predicting how players will act in a social situation; thus, strictly mathematical game
theory cannot be tested by experimental means (Colman, 1995, p. 4).
Game theory as it is applied to the social and biological sciences, on the other hand, has proved useful in helping to explain and predict behavior in a wide range of situations. Predictions derived from informal, or non-mathematical, game theory can be tested through empirical research, and even if satisfactory formal situations cannot be found, non-mathematical game theory can provide illuminating insights. Many will argue that certain important features of individual and collective rationality, cooperation and competition, trust and suspicion, threats and commitments cannot be clearly described or explained, without the framework of game theory (Colman, 1995, p. 4). Thus, game theory can be thought of as the study of conflict between thoughtful and potentially deceitful players who are assumed to be perfectly rational and who wish to maximize their objective functions given their perceptions about the environment. This goal of understanding why players in competitive situations act as they do is especially relevant when the game is on a large scale with many players and complicated rules. The economy and international politics are two good examples (Morris, p. vii).

II. A. Formal Representations of Games

Understanding the terminology and notation used in game theory is key in deciphering difficult theorems and evaluating conflict situations. Thus, it is worth the reader’s time to carefully study the ideas presented in this section.

The three most common ways of formally representing games are described below:

(1) Extensive Form. The most complete description of games is called extensive form. It details the various stages of the interaction, the conditions under which a player has to move, the information a player holds at different stages, and the motivation of the players.
(2) **Normal Form** (or **Strategic Form**). This is a more abstract representation of a game. Here one notes all of the possible strategies of each player together with the payoff that results from strategy choices of the players. In the normal form, many details of the extensive form are omitted. This form allows one to concentrate on the strategic aspects of a game, but it neglects the dynamic structure of the game.

(3) **Characteristic Function Form** (or **Coalitional Form**). Whereas the normal form can be viewed as a reduced version of the extensive form, the characteristic function form of a game represents more than just a further abstraction from details of the game. It is a description of social interactions where binding agreements can be made and enforced. Binding agreements allow groups of players, or coalitions, to commit themselves to actions that may be against the interest of individual players once the agreement is carried through. This representation of a game is particularly useful for analyzing distributional questions. The distribution of payoffs from the joint outcome among the members of a coalition will determine what kind of agreement can be reached and what kind of coalitions will form (Eichberger, pp. 1 - 2).

For the purposes of this paper, both the **extensive form** and the **normal form** are discussed in greater detail.

**II. A. 1. Extensive Form**

The extensive form gives the most detailed description of a game. It tells exactly which player should move, when the player should move, what the possible moves are, the outcomes corresponding to each move, and the information of the players at each stage (Handbook of Game Theory, p. 20).

In any game, the set of players can be denoted by $I$. In most applications, it is described by listing its elements as $I = \{1, 2, ..., I\}$, where $I$ is a finite set. A given player from the set $I$ is denoted by $i$, where $i \in I$ (Eichberger, p. 3).
The extensive form of a game is perhaps best represented diagrammatically by means of a game tree, or a tree graph. In such a tree, each point, or node, represents a point in the game where a decision and thus a move must be made; this move is either the decision of a player or a chance event (Thomas, p. 23). A game can be characterized by a set of nodes \( N \). A move, or action, of a player takes him from one node to the next. The set of possible actions in a game is denoted by \( A \), regardless of which player takes them (Eichberger, p. 3).

If \( a, b, c, \ldots \) denote the nodes (positions) of a game and if \( \alpha, \beta, \chi, \ldots \) denote the actions of the same game, then it is possible to specify various orders of nodes indicating how one position arises from another. In the game tree below (Figure 1), the set of nodes is \( N = \{a, b, c, d, e, f, g, h\} \) and the set of actions is \( A = \{\alpha, \beta, \chi, \delta, \varepsilon, \phi, \gamma\} \). Players can reach the nodes of a game tree by moving along the branches from left to right. Note that while each node in the left tree has a unique predecessor, node \( e \) in the right tree has two predecessors, namely \( b \) and \( c \) (Eichberger, p. 4).

![Game Trees](https://via.placeholder.com/150)

**Figure 1**: Game Trees Representing Sequences of Moving
(Courtesy: Eichberger, p. 3)

In many games, such as checkers or chess, it is not unheard of for different sequences of moves to lead to the same position of the figures. In the right tree of Figure 1, for example,
position $e$ can be reached by moving from node $b$ by way of action $\delta$ or by moving from node $c$ by way of action $\epsilon$. Representing a game using such a tree, therefore, leads to uncertainty as to how node $e$ was reached. To eliminate this confusion, if a player can arrive at the same point in a game by two or more different sequences of moves, the moves will always be distinguished by different nodes. Thus, for the purposes of game theory, the situation shown in the right tree of Figure 1 cannot occur.

In this representation of the extensive form, examples will necessarily involve finite sets of actions $A$ and finite sets of nodes $N$. This restriction allows the game theorist to diagrammatically represent games in game tree form. In the formal definitions, however, the restrictions of finite sets of actions and nodes can be relaxed (Eichberger, p. 4).

The initial node of a game tree is denoted by $o$, for origin. Let $\sigma: N \rightarrow N$ be the function that associates with each node other than the origin $o$, its predecessor, and for the origin, define $\sigma(o) = o$ (Eichberger, p. 4). For any node $n \in N$ and for any positive integer $k$, $\sigma^k(n)$ indicates the $k$th iteration of the function $\sigma$, that is,

$$\sigma^k(n) = \frac{\sigma(\sigma(\ldots \sigma(n) \ldots)))}{k\text{-times}}.$$  

By definition, then, a game tree is a set of nodes $N$ and a function $\sigma: N \rightarrow N$, $\sigma(o) = o$, such that, for all nodes $n \in N$, $\sigma^k(n) = o$ for some positive integer $k$ holds. Here the condition $\sigma^k(n) = o$ for all $n$ is necessary to ensure that all nodes are connected to the origin, that is, that the nodes actually form a tree (Eichberger, pp. 4 - 5).
By considering Figure 1 once again, the reader will notice that the set of nodes is \( N = \{a, b, c, d, e, f, g, h\} \), and the origin is node \( a \) where \( o = a \). In the left tree of Figure 1, the following is true:

\[
\sigma(h) = \sigma(g) = \sigma(f) = c, \quad \sigma(e) = \sigma(d) = b, \quad \sigma(b) = \sigma(c) = a.
\]

Additionally, the condition that all nodes must be connected with the origin is satisfied in the left tree since the following is true:

\[
\sigma^2(h) = \sigma(\sigma(h)) = a, \quad \sigma^2(g) = \sigma(\sigma(g)) = a,
\]

\[
\sigma^2(f) = \sigma(\sigma(f)) = a, \quad \sigma^2(e) = \sigma(\sigma(e)) = a,
\]

\[
\sigma^2(d) = \sigma(\sigma(d)) = a.
\]

The right tree does not satisfy the definition of \textit{game tree} since \( e \) has two predecessors and the function \( \sigma \) is not defined for node \( e \) (Eichberger, p. 5).

In addition to wanting to know at which node a player is positioned, a game theorist generally wants to know how actions lead the player from one node to the next. This can be expressed through the \textit{predecessor action function} \( \alpha: N \setminus \{o\} \rightarrow A \), which associates with each node \( n \) (except the origin \( o \)) the action \( \alpha(n) \) leading from the predecessor node \( \sigma(n) \) to the node \( n \) (Eichberger, p. 5). Thus, the function \( \alpha \) for the left tree of Figure 1 can be explicitly written as:

\[
\alpha(b) = \alpha, \quad \alpha(c) = \beta, \quad \alpha(d) = \chi, \quad \alpha(e) = \delta,
\]

\[
\alpha(f) = \varepsilon, \quad \alpha(g) = \phi, \quad \alpha(h) = \gamma.
\]
Here the action leading from $\sigma(n)$ to $n$ is uniquely defined for all of the nodes in $N$ except for the initial node $o = a$. Recall that by definition there is no action leading to the initial node $o = a$ (Eichberger, pp. 5 - 6).

When analyzing a game, a game theorist may be interested in knowing in which situations the players are required to make decisions and in which situations the game must end. Thus, nodes are described as either decision nodes or terminal nodes. Let $\sigma^{-1}(n)$ be the inverse image of the function $\sigma$, that is, the set of nodes that have node $n$ as a predecessor (Eichberger, p. 6). A node is labeled a terminal node if it is the predecessor of no other node (i.e. if $\sigma^{-1}(n) = \emptyset$ holds). Any nodes that are not terminal nodes are decision nodes. Hence, the set of all terminal nodes can be written as $\tau(N) = \{n \in N \mid \sigma^{-1}(n) = \emptyset\}$ and the set of all decision nodes as $\mathcal{D}(N) = N \setminus \tau(N)$. Furthermore, the terminal nodes and decision nodes partition the set of nodes by the following two conditions: $\mathcal{D}(N) \cup \tau(N) = N$ and $\mathcal{D}(N) \cap \tau(N) = \emptyset$; that is, each node must belong to either the set of terminal nodes $\tau(N)$ or to the set of decision nodes $\mathcal{D}(N)$, but no node can belong to both (Eichberger, p. 6). By considering the case of the left game tree in Figure 1, the reader will notice that the set of terminal nodes is $\tau(N) = \{d, e, f, g, h\}$ and the set of decision nodes is $\mathcal{D}(N) = \{a, b, c\}$.

With the predecessor action function $\alpha$, a game theorist can identify the actions available at any decision node of a game tree. For any node $n \in \mathcal{D}(N)$, let $A(n)$ be the set of actions $\alpha(m)$ from nodes $m$ that have $n$ as their predecessor node; that is, let $A(n) = \{\alpha(m) \mid \sigma(m) = n\}$. Here the reader may notice that the set of actions at terminal node $n$ is not well defined; this justifies the terminology of calling nodes in $\mathcal{D}(N)$ decision nodes (Eichberger, p. 7).
The set of decision nodes $\mathcal{D}(N)$ can be partitioned into mutually exclusive subsets $N_i$, $i \in I$ where $N_i$ denotes the set of decision nodes at which player $i \in I$ must choose an action. A list of mutually exclusive sets of decision nodes for each player, $(N_i), i \in I$, is called a player partition of $\mathcal{D}(N)$ (Eichberger, p. 7). Because a player partition assigns nodes to players who have to make a decision there, it is possible to define the set of all possible actions of some player $i$, $A_i$, as the union of all sets $A(n)$ over nodes $n$ in $N_i$.

The game tree shown below (Figure 2) shows a player partition.

![Figure 2: A Player Partition](https://example.com/figure2)

(Courtesy: Eichberger, p. 7)

Notice that the decision nodes in Figure 2 are nodes $a$, $b$, and $c$. Suppose that in this game, Player 1 makes the first move at decision node $a$ and Player 2 makes the next move at either decision node $b$ or $c$, depending upon Player 1’s move. Then the set of decision nodes $\mathcal{D}(N)$ can be split up into decision nodes of Player 1, $N_1 = \{a\}$, and of Player 2, $N_2 = \{b, c\}$. Thus, the player partition of this game is $(N_1, N_2)$ (Eichberger, p. 7).

In Figure 2 the decision nodes are labeled with a 1 or a 2 to reflect which player is making a move. And the following sets of actions at the decision nodes $\mathcal{D}(N) = \{a, b, c\}$ exist:

$$A(a) = \{\alpha, \beta\}, \quad A(b) = \{\chi, \delta\}, \quad A(c) = \{\varepsilon, \phi, \gamma\}.$$
Here, the set of actions for Player 1 is $A_1 = \{\alpha, \beta\}$ and the set of actions for Player 2 is $A_2 = \{\chi, \delta, \epsilon, \phi, \gamma\}$ (Eichberger, p. 8).

Also of interest to the game theorist are the payoffs of the players at the conclusion of the game. These payoffs are indicated at the terminal node of the game tree. The payoff function associates with each terminal node $n \in \mathcal{T}(N)$ a payoff vector $(r_1(n), r_2(n), \ldots, r_i(n))$ that specifies the payoff for each player. Thus, for each player $i \in I$, $r_i(n)$ is a number that indicates the payoff to Player $i$ if terminal node $n$ is reached. By definition, then, the payoff function $r: \mathcal{T}(N) \to \mathbb{R}^I$ associates with each terminal node a vector of real numbers, the payoff to each of the players for each terminal node (Eichberger, p. 8). Figure 3 indicates how payoffs are shown on a game tree. For game trees in general, the first number indicates the payoff of Player 1 at that terminal node, the second number indicates the payoff of Player 2 at that same node, and so on. The payoffs in Figure 3 are as follows:

for Player 1: $r_1(d) = 1, \quad r_1(e) = 2, \quad r_1(f) = 0, \quad r_1(g) = -2, \quad \text{and} \quad r_1(h) = 4.$

for Player 2: $r_2(d) = 0, \quad r_2(e) = -1, \quad r_2(f) = 1, \quad r_2(g) = 5, \quad \text{and} \quad r_2(h) = 5.$

![Figure 3: Player Payoffs](Courtesy: Eichberger, p. 8)
Up to this point, a game theorist can gather much information about a game from the game trees already presented. But the description of the extensive form already presented has failed to account for the situation where a player cannot observe the moves of his opponent(s). Since actions lead players from node to node, a player who cannot observe the action of his opponent(s) will not know at which node he is positioned (Eichberger, p.11). The following Matching Pennies example illustrates the aforementioned information problem:

**Matching Pennies**: Two players, Player 1 and Player 2 each put a coin on the table but keep their moves hidden from each other. Player 1 puts her coin down first, then Player 2 does the same. Finally, they reveal to each other the sides of the coins lying face up on the table. If the sides match, Player 1 wins a dollar from Player 2; if the sides do not match, Player 2 wins a dollar (Eichberger, pp. 11 - 12).

Let $H$ denote “heads” and $T$ denote “tails” for Player 1. Similarly, let $h$ denote “heads” for and $t$ denote “tails” for Player 2. Now consider the two game trees below (Figure 4):

While the two game trees in Figure 4 are nearly identical in appearance, the game tree on the left fails to indicate the fact that Player 2 cannot observe Player 1’s move and thus does not know if she is at node $b$ or node $c$ when making his move. In order to show that a player is
unable to distinguish nodes, the decision nodes that are indistinguishable in the set are joined into what is called an information set. In a game tree diagram, information sets are often indicated by joining relevant nodes with a dotted line (Eichberger, p. 12). Because the right tree in Figure 4 indicates the information set of Player 2, it is the correct representation of the Matching Pennies game.

An information set, denoted by \( u \), is a set of decision nodes for a given player. In the right tree of Figure 4, the information set for Player 2 is \( u = \{b, c\} \). And because it is possible for an information set to contain a single node, the information set for Player 1 is \( u' = \{a\} \) (Eichberger, p. 12).

In order for an information set to reflect the idea that a player is not certain as to at which node he is positioned, the following conditions must hold:

1. Information sets of Player \( i \in I \) contain only decision nodes of Player \( i \).
2. Each decision node of Player \( i \) is contained in one and only one information set of this player.
3. The same choices must be available to a player at all nodes of an information set (Eichberger, p. 13).

Conditions (1) and (2) indicate that information sets partition the set of decision nodes of a player, say \( N_j \). Nodes in the same information set are distinguishable for this player (Eichberger, p. 13). Additionally, \( U_i \) denotes the set of all information sets for each player \( i \in I \).

Condition (3) indicates that for any information set \( u \), \( A(x) = A(y) \) for all \( x, y \in u \). The set of choices at \( u \), denoted \( A(u) \), is therefore unambiguously given by the set of actions available at some node of this information set, \( A(u) = A(x) \) for some \( x \in u \) (Eichberger, p. 13).
The right game tree of the Matching Pennies example illustrates these concepts. In the Matching Pennies example, each player has a single information set, namely $U_1 = \{\{a\}\}$ and $U_2 = \{\{b, c\}\}$. The set of actions at the three decision nodes a, b, and c are:

$$A(a) = \{H, T\}, \quad A(b) = \{h, t\}, \quad A(c) = \{h, t\}.$$

The information set of Player 1 contains only one node, \{a\}, and the set of choices at this node is $A(\{a\}) = A(a) = \{H, T\}$. The information set of Player 2, however, contains two nodes. The actions at these two nodes must be the same: $A(\{b, c\}) = A(b) = A(c) = \{h, t\}$. These properties can be summarized as follows: $U_i$ is the set of information sets of Player $i \in I$. For any $u \in U_i$, the set of choices at $u$ is denoted by $A(u)$. The list of all of these sets $\{U_i\}_i$ is called information partition (Eichberger, p. 13).

**Figure 5**: Equivalent Representations of the Matching Pennies Example  
(Courtesy: Eichberger, p. 14)

If the players of a game do not know the decisions of their opponents when making their moves, then a situation of simultaneous moving occurs. Thus, in the case of simultaneous moving it does not matter if the choices literally occur at the same moment; rather, it only matters that the players are lacking information about their opponents’ decisions. The sequence
of moves is immaterial in a simultaneous move game; the information that the players have when they make their choices is what is important (Eichberger, p. 14). And because this is true, there is often more than one way to formally represent a simultaneous move game in game tree form. The two game trees in Figure 5 are equivalent representations of the Matching Pennies example where each player puts down a coin at the same time.

The following summarizes the new terminology and notation that has been presented in this section:

- the set of players $I$
- a set of situations $N$ and the sequence of their appearance $\sigma$, the game tree $(N, \sigma)$
- the set of actions and a function that relates actions to nodes $(A, \alpha)$
- who comes to move at the decision nodes, the player partition $(N_i)_{i \in I}$
- which decision nodes each player can distinguish, the set of information sets $U$
- what choices a player has at each information set, the set of choices at information sets $(A(u))_{u \in U}$
- what the payoffs to the players are when the game ends, the payoff function $r$

(Eichberger, p. 15)

By definition, a game in extensive form is completely defined by the following list $\Gamma$:

$$\Gamma = (I, (N, \sigma), (A, \alpha), (N_i)_{i \in I}, U, (A(u))_{u \in U}, r).$$

Consider the following game in extensive form (see Figure 6).

Figure 6: A Game in Extensive Form

(Courtesy: Eichberger, p. 15)
The above game tree can be described as follows:

\[ I = \{1, 2\}, \]

\[ N = \{a, b, c, d, e, f, g\} \]

\[ \sigma(a) = a, \quad \sigma(b) = a, \quad \sigma(c) = a, \quad \sigma(d) = b, \]

\[ \sigma(e) = b, \quad \sigma(f) = c, \quad \sigma(g) = c, \]

\[ A = \{U, D, T, B, t, b\}, \]

\[ \alpha(b) = U, \quad \alpha(c) = D, \quad \alpha(d) = T, \quad \alpha(e) = B, \quad \alpha(f) = t, \quad \alpha(g) = b, \]

\[ N_1 = \{a\}, \quad N_2 = \{b, c\}, \]

\[ U_1 = \{\{a\}\}, \quad U_2 = \{\{b\}, \{c\}\}, \]

\[ A(\{a\}) = \{U, D\}, \quad A(\{b\}) = \{T, B\}, \quad A(\{c\}) = \{t, b\}. \]

The reader is referred to any book on Graph Theory for further details about game trees.

\[ II. \ A. \ 2. \ \text{Normal Form} \]

In the normal, or strategic, form of a game, the players can make all of the moves that they could make in the “original” game, and they receive the same payoffs, but the sequential nature of the moves is lost, as is the idea of perfect information. However, it is the same game, no matter which form is taken (Thomas, p. 28).

When games are classified and conclusions about them are drawn according to the properties of the payoff matrix described below, the games are said to be in normal form. Because a game theorist assumes that the number of strategies available to each player in a given game is finite, he can therefore suppose that the strategies available to Player 1 are numbered
from 1 to \( N \) and those available to Player 2 are numbered from 1 to \( M \). A game theorist can additionally describe each pair of strategies chosen by \((i, j)\), where the \( i \)-th strategy can be chosen by Player 1 and the \( j \)-th strategy by Player 2. This pair determines an outcome, say \( O_{ij} \). Mathematically speaking, the variable outcome \( O_{ij} \) is a function of the two variables \( i \) and \( j \). If all of the possible outcomes \( O_{ij} \) are arranged in a rectangular array of \( N \) rows \((i = 1, 2, \ldots, N)\) and \( M \) columns \((j = 1, 2, \ldots, M)\), then the game at hand is in matrix form and the entire strategic structure of the game is depicted (see Figure 7) (Rapoport, p. 46). Practically, the theory of two-person zero-sum games is often stated as a theory of games in normal form.

![Figure 7: Normal Form in Matrix Representation]( COURTESY: Rapoport, p. 46)

The greatly simplified structure of games in normal form makes analysis of the behavior of the players much easier (Eichberger, p. 29). For that reason, many of the examples in this paper are represented in matrix form.

**III. The Minimax Theorem**

The genesis of game theory is often traced to John von Neumann's proof of the Minimax Theorem. Most books on game theory mention the Minimax Theorem, but it is seldom proved.
Non-mathematicians find von Neumann’s proof to be too lengthy and complex to grasp, and for that reason the proof is often presented in text in a simplified form. A simplified version of the proof for the Minimax Theorem was developed by Andrew M. Colman, a Reader in Psychology at the University of Leicester and can be found in the appendix of his book entitled *Game Theory & its Applications* (1995). Colman’s proof is intended to be more explicit and therefore easier to follow than those presented in mathematical textbooks, although a basic knowledge of elementary algebra and geometry is assumed. A complete understanding of the proof requires knowledge of probability theory and mathematical analysis.

Preliminary comments about the Minimax Theorem and the theorem itself are presented below. See Colman’s aforementioned book for more details.

**III. A. Preliminary Comments**

A finite, two-person, zero-sum game is specified by a rectangular array of numbers \([a_{ij}]\), called a *payoff matrix*, with \(m\) rows and \(n\) columns. The numbers are the payoffs to Player I, and because the game is zero-sum, Player II’s payoffs are simply the negatives of these numbers. According to the rules of the game, Player I chooses a strategy corresponding to one of the rows, and simultaneously... Player II chooses a strategy corresponding to one of the columns. The number at the intersection of the chosen row and column is the payoff to Player I. Thus if Player I chooses row \(i\) and Player II chooses column \(j\), then the number \(a_{ij}\) at the intersection is the amount gained by Player I and lost by Player II; in other words, the amount \(a_{ij}\) is transferred from Player II to Player I.

Instead of deliberately selecting a *pure strategy* - a specific row or column - a player may use a randomizing device to choose among the strategies. A player with two pure strategies, for example, may choose one of them by tossing a coin. A player who chooses in this way is said to be using a *mixed strategy*. In general, a mixed strategy assigns a predetermined probability to each available pure strategy; in the coin-tossing example, for example, the assigned probabilities are 1/2 and 1/2. A mixed strategy can be represented by a string of non-negative numbers of length \(m\) (for Player I) or \(n\) (for Player II) that sum to 1. A mixed strategy for Player I can accordingly be written \((x) = (x_1, x_2, ..., x_m)\) while a mixed
strategy for Player II \( (y) = (y_1, y_2, \ldots, y_n) \), where \( x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0; y_1 \geq 0, y_2 \geq 0, \ldots, y_n \geq 0; x_1 + x_2 + \ldots + x_m = 1 \); and \( y_1 + y_2 + \ldots + y_n = 1 \). A pure strategy can viewed as a special case of a mixed strategy in which a probability of 1 is assigned to one of the \( x_i \) or \( y_j \) and 0 to each of the others.

If Player I uses a mixed strategy \( (x) \) and Player II uses a mixed strategy \( (y) \), then row \( i \) will be chosen with probability \( x_i \) and column \( j \) with probability \( y_j \). Because these events are independent, the payoff \( a_{ij} \) will occur with probability \( x_i y_j \). The expected payoff is then simply a weighted average of all of the payoffs \( a_{ij} \), each one occurring with probability \( x_i y_j \), and it can be written \( \Sigma a_{ij} x_i y_j \), where \( i = 1, 2, \ldots, m \), and \( j = 1, 2, \ldots, n \) (Colman, 1995, p. 317 - 318).

III. B. The Theorem

Since the expected payoff indicates Player I’s average gain and Player II’s average loss, Player I wishes to maximize it while Player II wishes to minimize it. If Player I knew in advance that Player II was going to use the mixed strategy, say \( (y') \), then Player I’s best counter-strategy would be to maximize the expected payoff against \( (y') \); the expected payoff would then be

\[
\max_{x} \Sigma a_{ij} x_i y_j
\]

If Player II knew Player I’s mixed strategy, say \( (x') \), in advance, then Player II could use a counter-strategy that minimizes the expected payoff of

\[
\min_{y} \Sigma a_{ij} x_i y_j
\]

In reality, however, these counter-strategies cannot be implemented because neither player has perfect information about the other’s plans. Player I can, however, ensure a maximum security level by assuming that Player II will approach any strategy \( x \) with the counter-strategy that minimizes the payoff in that situation and by choosing \( (x) \) so as to maximize the expected
payoff under this pessimistic assumption. Player I thus guarantees that the expected payoff will be no less than

\[
\max_{(x)} \min_{(y)} \sum a_{ij} x_i y_j.
\]

Similarly, Player II’s security level is maximized by the use of a strategy (y) that minimizes the expected payoff against Player I’s maximizing counter-strategy against it. Thus, the expected payoff will not exceed

\[
\min_{(y)} \max_{(x)} \sum a_{ij} x_i y_j.
\]

Taking all of the above into consideration, we arrive at the following theorem:

**THE MINIMAX THEOREM:** If \([a_{ij}]\) is any \(m \times n\) payoff matrix, then

\[
\max_{(x)} \min_{(y)} \sum a_{ij} x_i y_j = \min_{(y)} \max_{(x)} \sum a_{ij} x_i y_j,
\]

where \((x) = (x_1, x_2, ..., x_m)\) and \((y) = (y_1, y_2, ..., y_n)\) represent all strings of non-negative numbers of length \(m, n\), and sum 1 (Colman, 1985, pp317 - 319).

**IV. Two-person Zero-sum Games**

As mentioned earlier, a two-person zero-sum game is one in which one person wins and the other loses. That is, the payoffs to the players must add up to zero. With a two-person zero-sum game, the interests of the two players are diametrically opposed -- an outcome that is favorable for one player is necessarily unfavorable for the other. Because each player can gain
only at the expense of the other, they are involved in a strictly competitive game; there are no prospects of mutually profitable collaboration (Colman, 1995, p. 53). Because two-person zero-sum conflicts are amenable to formal analysis, many significant contributions to mathematical game theory relate to them.

Many economic, political, military, and interpersonal conflicts correspond to strictly competitive games. Examples of these conflicts, if the opponents are diametrically opposed, may include the following: two television networks competing for audiences; two retailers competing for market shares; two politicians competing for votes; or two parents competing for the custody of their children after a divorce. It is a serious mistake to regard all competitive interactions as zero-sum games, however, because in reality the protagonists' interest are seldom strictly opposed (Colman, 1995, pp. 53-54).

One example of a simple two-person zero-sum game is referred to by military historians as the Battle of Bismarck Sea, an incident occurring during the Second World War. The following account is based on the paper “Military Decision and Game Theory” written by O. G. Haywood, Jr. in 1954.

In February 1943, during the critical phase of the struggle in the south-western Pacific, the Allies received intelligence reports indicating that the Japanese were planning a troop and supply convoy to reinforce their army in New Guinea. The convoy could sail either north of the island of New Britain where rain and poor visibility were almost certain, or south of the island, where the weather would probably be fair. By either route, the trip would take three days. General George C. Kenney, commander of Allied forces in the South Pacific was ordered by his supreme commander, General MacArthur, to attack the convoy with the objective of inflicting maximum destruction. General Kenney had to decide whether to concentrate the bulk of his reconnaissance aircraft on the northern or southern route. Once the convoy was sighted, it would be bombed continuously until its arrival in New Guinea.

The players in this game were General Kenney and the Japanese commander, Hitoshi Imamura. The strategies from which each had to choose were the
northern and southern routes. The outcomes were the number of days of bombing
that would result from each possible combination of choices. Kenney’s staff
estimated that if the reconnaissance aircraft were concentrated mainly on the
northern route, then the convoy would probably be sighted after one day, whether
it sailed north or south, and would therefore be subjected to two days of bombing
in either case. If the aircraft were concentrated mainly on the southern route, on
the other hand, then either one or three days of bombing would result depending
on whether the Japanese sailed north or south respectively. The number of days
of bombing may be interpreted as Kenney’s gains and the Japanese commander’s
losses. Because the Japanese payoffs are just the negatives of Kenney’s, the game
is obviously zero-sum (Colman, 1995, pp. 54-55).

Although there are many ways to represent the above game, the game tree and payoff
matrix are two of the simplest. In Figure 8, the extensive form of the game is represented by
means of a game tree. Recall that each possible situation in a game is referred to as a node, and a
move (or action) of a player leads from one node to the next (Eichberger, p. 3). The nodes of the
game tree below are labeled with the names of the players whose choices they represent, and the
branches of the tree represent the strategies between which the players must choose. The top­
most node is labeled “Imamura,” and it refers to the Japanese commander who has the first
move. The nodes at the second level refer to Kenney, the commander of the Allied forces in the
South Pacific. The bottom-most nodes represent the outcomes that are reached after each player
has moved in accordance with the rules of the game (Colman, 1995, pp. 55-56).

![Figure 8: Extensive Form of the Battle of Bismarck Sea Game](image_url)

(Courtesy: Colman, 1995, p. 55)
The nodes are enclosed in dashed loops to indicate the information sets to which they belong; when making a move, a player cannot distinguish between choice points enclosed within a single information set. Notice that the node labeled “Imamura” is in an information set of its own, but both of the nodes labeled “Kenney” are enclosed in a single information set. This indicates that at the time of choosing, Kenney does not know whether he has reached the left-hand or right-hand node because he does not know in which direction Imamura has chosen to sail. Thus, this game is one of imperfect information, and the players are moving in ignorance of any preceding moves. [Note: The game would be strategically equivalent if the initial node were labeled “Kenney” and the succeeding “Imamura” nodes were enclosed in a single information set (Colman, 1995, p. 56).]

In Figure 9 the Battle of Bismarck Sea game is represented in a payoff matrix. Recall that each row of the payoff matrix corresponds to Player I’s strategies, and each column of the payoff matrix displays the normal form of the game. The normal form allows a game involving a sequence of moves to be depicted statically, with the players simultaneously choosing a row or a column. Any finite game in extensive form can be represented in normal form without the loss of strategically relevant information (Colman, 1982, p.50).

\[
\begin{array}{ccc}
 & \text{Imamura} & \\
N & 2 & 2 \\
S & 1 & 3 \\
\end{array}
\]

*Figure 9: Normal Form of the Battle of Bismarck Sea Game*
(Courtesy: Colman, 1995, p. 57)
In the Battle of Bismarck Sea game, players who are rational according to the Minimax Theorem will choose their northern strategies, and the value of the game to each player is two days of bombing because that is the payoff that results from rational play on both sides. These minimax strategies were in fact the ones employed by Imamura and Kenney, and the Japanese suffered a decisive defeat; the Japanese convoy of ten warships and twelve transports carrying approximately 15,000 men was destroyed. The outcome cannot be blamed on any strategic error of Imamura; it was inherent in the payoff structure of the game, whose value was positive and thus favorable to Kenney (Colman, 1995, p. 58).

Here the minimax strategies are optimal because they intersect in an equilibrium point of the game, or a Nash equilibrium. The Nash equilibrium concept has two particularly attractive characteristics: (1) It generalizes the dominant strategy equilibrium concept and the iterated dominance equilibrium concept; and (2) It yields a payoff for each player that is always at least as good as the maximum value each player could guarantee himself (Eichberger, p. 84). Thus, a key property of a Nash equilibrium is that no player has any incentive to deviate unilaterally from it; the players should have no reason to regret their strategy choices when the other players’ choices are made known (Colman, 1995, p. 59). In the Battle of Bismarck Sea game, the outcome would have been no better for the Japanese had they deviated unilaterally from the minimax strategy because they still would have suffered two days of bombing. If Kenney had deviated unilaterally, he also would have obtained a worse outcome.

At this point it is worth mentioning that a minimax or equilibrium strategy does not necessarily take full advantage of irrational play on the part of an opponent. Sometimes a non-minimax choice may serve as the best counter-strategy. For example, if Kenney would have had ample reason to believe that Imamura planned to sail south, then his best counter would have
been to sail south, even though this would not be a rational move according to formal game theory. Against an irrational opponent, it could be argued that the minimax principle loses some of its persuasive force, but in games against human adversaries the assumption of rationality is usually a safe one (Colman, 1995, p. 59).

The Nash equilibrium is easy to locate in the Battle of Bismarck Sea game because its payoff matrix contains a saddle point. A saddle point of a payoff matrix is a minimum in its row and a maximum in its column. If maximin and minimax are equal, then and only then, must there be a saddle point in the payoff matrix where its value occurs. When a game has a saddle point, the saddle point is the expected outcome of rational play; it is the solution of the game (Poundstone, p. 54). The fact that a saddle point necessarily corresponds to the intersection of the players’ equilibrium or minimax strategies is of particular usefulness when solving games that are more complicated than the Battle of Bismarck Sea game (Colman, 1995, pp. 59-60).

V. Two-person Mixed-motive Games

Games in which the players’ preferences among the outcomes are neither identical nor diametrically opposed are called mixed-motive games. In mixed-motive games, players are motivated partly to cooperate and partly to compete with one another. A player in a mixed-motive game must deal with both an intrapersonal and interpersonal conflict. In a mixed-motive game, it is not the case that what one player gains the other must lose. Mixed-motive games are sometimes referred to as variable-sum or nonzero games (Colman, 1995, p. 100).

Throughout the period from the 1950s to the 1970s, experimental gaming research centered largely on two-person mixed-motive games, especially the Prisoner’s Dilemma game.
Experimental researchers were attracted to the Prisoner’s Dilemma and other related games because they provide simple methods for investigating different aspects of strategic interaction (Colman, 1995, p. 134).

The following example illustrates the main points of the Prisoner’s Dilemma game:

Two criminals are caught by the police, and because of a lack of evidence, the prosecution needs a confession to convict. If neither prisoner confesses, they will each be charged and convicted for a minor offense, which is one year less than a conviction for the main crime. The prosecutor offers each prisoner a deal. If Prisoner 1 confesses and Prisoner 2 does not, Prisoner 1 will get three years off his sentence whereas Prisoner 2 will get an extra year (-1) in prison (and vice-versa). If both confess, they will be punished according to the law (Eichberger, p. 65).

The above situation can easily be translated into the following game in strategic form: There are two players, and each has a strategy set with two options, “to confess” (C) or “not to confess” (N). Figure 10 shows the payoff matrix for the possible strategies.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
</tr>
<tr>
<td>Player 1</td>
<td>1, 1</td>
</tr>
<tr>
<td>C</td>
<td>3, -1</td>
</tr>
</tbody>
</table>

*Figure 10: Prisoner’s Dilemma Payoff Matrix*
(Courtesy: Eichberger, p. 66)

It is clear that there is no dilemma for the prisoners in terms of how they should behave. If Prisoner 2 confesses, Prisoner 1 is better off cooperating or else he will serve an extra year in prison. If Prisoner 2 does not confess, it is best for Prisoner 1 to confess, since doing so will
remove three years from his jail term. No matter how Prisoner 2 acts, it is best for Prisoner 1 to cooperate with the authorities (Eichberger, p. 66)

By a similar argument, it is reasonable to conclude that Prisoner 2 will confess in any case. This leads one to predict (C,C) as the outcome of the game. The dilemma lies in the fact that this individually rational behavior precludes the best outcome for the prisoners, namely not to confess (N,N) and to get a lighter sentence. Here the source of the dilemma is not the lack of communication between the prisoners but their incentives. Even if they could talk and agree not to confess, each prisoner has an incentive to break the agreement at the other’s expense (Eichberger, p. 66).

The Prisoner’s Dilemma game is fascinating because it involves two of the major dilemmas in conflict situation, and also because it models problems as diverse as nuclear disarmament, wage negotiation, and the controversy of whooping cough vaccinations. The first dilemma focuses on the player’s objective: Should the player look out for his personal interests or the group’s interests? This conflict is between individual rationality (which would lead to a confession in Prisoner’s Dilemma) and group rationality (which would suggest keeping quiet) (Thomas, p. 55). Here lies the psychologists’ interest in the game.

The second dilemma focuses on how often the Prisoner’s Dilemma game will be played. If the game will only be played once, there is no reason to build up your opponent’s trust in you; here it would seem best to confess. It can be shown that if the game is played a fixed number of times, any equilibrium pair of strategies will result in (C,C) being played all of the time (Thomas, p. 56). The argument proceeds as follows: Think of the last game. Since there will be no more games, both prisoners will choose to confess as in the one-game situation. Having decided on what happens in the final game, consider the second-to-last game. Since the last game’s
strategies are now fixed one can think of this as really the last game, and so on. If the number of
games to be played is unknown to the prisoners, then there will be equilibrium pairs that result in
the (N,N) strategy being played all of the time. Interestingly, many experiments have been
performed to see what happens in practice when these games are played. A. Rapoport and A. M.
Chammah (1965) and M. Guyer and B. Perkel (1972) reported that players do keep silent in
hopes that their opponents will follow suit. Once such a pattern is established they do sometimes
"punish" opponents who confessed last time by confessing themselves, but the amount that this
happens varies considerably depending upon gender, nationality, temperament, inducements, and
the way in which the experiment is conducted (Thomas, p. 56).

VI. Games with Incomplete Information

The games that have been considered so far have all been games where the players know
the exact payoffs that their opponents can obtain. This information can be key in helping players
decide which strategy is their best strategy. Thus, complete payoff information is needed to
determine a Nash equilibrium.

When this assumption is relaxed, however, players can no longer predict what would be a
best response for the other players; thus, they cannot determine what constitutes optimal behavior
for themselves (Eichberger, p. 125). The following example illustrates the type of problem that
arises when players have incomplete information about payoffs:

Consider a potential entrant to a monopolist’s market. Without entry, the
monopolist earns three units of profit and the entrant no profit. Should the
potential entrant enter the market of the monopolist, two reactions of the
incumbent have to be considered: (1) The monopolist may accommodate the
entrant, in which case its profit will be reduced to one unit and the entrant will
also make one unit of profit; or (2) The monopolist may fight entry, which will cause the entrant a loss of one unit (Eichberger, p. 124).

The critical issue for the entrant concerns the likelihood that the monopolist will fight the entry. This likelihood depends on the monopolist’s outcome if he fights the entrant. Figure 11 captures the situation.

<table>
<thead>
<tr>
<th>Entrant</th>
<th>Monopolist</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
</tr>
<tr>
<td>(e)</td>
<td>1, 1</td>
</tr>
<tr>
<td>(ne)</td>
<td>0, 3</td>
</tr>
</tbody>
</table>

**Figure 11: Competing Firms’ Payoff Matrix**
(Courtesy: Eichberger, p. 124)

Here \(e\) denotes the decision to enter, \(ne\) the decision to refrain from entering, \(a\) denotes the decision of the monopolist to accommodate the entrant, and \(f\) the decision to fight the entry. \(k\) represents the crucial payoff parameter. Without information as to whether \(k\) is larger or smaller than one, it is impossible for the entrant to predict the monopolist’s reaction to entry (Eichberger, p. 124). In this example the incomplete information is one-sided, but it is worth mentioning that each firm could be incompletely informed about the other firm’s payoff.

**VII. Closing Comments**

Game theory identifies players’ optimal strategies in the face of interdependence and uncertainty. Games can be divided into two-person and n-person games, zero-sum and nonzero-
sum games, and cooperative and non-cooperative games. Although this paper does not touch upon all of these aspects of game theory, it does offer a nice introduction of game theory and its applications for those who are not mathematical specialists.
References


