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Einstein metrics on piecewise-linear three-spheres

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EINSTEIN METRICS ON PIECEWISE-LINEAR THREE-SPHERES

An Abstract of a Thesis

Submitted

in Partial Fulfillment

of the Requirement for the Degree

Master of Arts

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ABSTRACT

Einstein metrics on manifolds are in some ways the "best" or most symmetric metrics those manifolds will allow. There has been much work on these metrics in the realm of smooth manifolds, and many results have been published. These results are very difficult to compute directly, however, and so it is helpful to consider piecewise-linear approximations to those manifolds in order to more quickly compute and describe what these metrics actually look like. We will use discrete analogues to powerful preexisting tools to do analysis on two particular triangulations of the three dimensional sphere with the intent of finding Einstein metrics on those triangulations. We find that, in one case, the intuitive solution we would expect from the literature holds, and in the other case it does not. We will discuss the differences between these two objects and will suggest possible avenues of research in the future.

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Has been approved as meeting the thesis requirement for the

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Date	Dr. Theron J. Hitchman, Chair, Thesis Committee
Date	Dr. Bill Wood, Thesis Committee Member
Date	Dr. Michael Prophet, Thesis Committee Member
Date	Dr. Michael J. Licari, Dean, Graduate College

This work is dedicated to my parents, Keith A. Pitzen, and Julie C. Pitzen.
Without their guidance, I would never have made it this far.

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CHAPTER 1

INTRODUCTION

There are many classical problems relating what is known as the curvature of a manifold to certain special metrics on that manifold. We generally consider what are known as closed Riemannian manifolds. These are manifolds which are closed (that is, they are compact and have no boundary), and Riemannian (they have a particular structure, which we will discuss later). We know that a metric is Ricci-flat on manifolds like this (provided they are dimension at least three) if they are are critical points of what is known as the *Einstein-Hilbert* functional

$$\mathcal{EH}(M,g) = \int_M R_g dV_g, \tag{1.0.1}$$

where here, R_g is the scalar curvature, and dV_g is the infinitesimal volume form for the manifold (M,g). For more information on this, see [1]. Since an arbitrary manifold can be rather nasty, it is common to constrain our manifolds to the class of manifolds with volume equal to 1. Our functionals tend to squeeze manifolds down to degenerate manifolds in the limit, so by doing this we are able to work with relatively nice spaces. This way, we can simply find critical points of the functionals and extract Einstein metrics from them. These metrics are those where the equality

$$R_g = nk$$

holds, where n is the dimension of our manifold, and k is some constant. However, instead of restricting our manifolds as above, we can instead consider a modified version of our functional, which takes the volume restriction into account. The functional we end up with is related to the Einstein-Hilbert functional above, from [1]:

$$\mathcal{NEH}(M^n,g) = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{(n-2)/n}}$$

Some simple examples of Einstein manifolds are Euclidean space, which is Ricci-flat, and so is a trivial Einstein manifold ($k = 0$). Another example is hyperbolic space, with the standard metric, where k is usually -1 (up to some scaling). It's simple to prove that any space with constant sectional curvature is Einstein [5], as $R_g = p$ for some $p \in \mathbb{R}$. Hence, since n is constant, we see $k = \frac{n}{p}$ gives us our constant, k.

The Einstein-Hilbert functional is a powerful tool when studying manifolds, since it is able to in some ways capture the "best" geometries on that manifold, with much less work than would be required through other means. The tools we use are essentially variational calculus tools and derivatives, which are very robust, and fairly simple to use. It is the power of these functionals which motivates us to use them to tackle the problem we would like to solve.

Rather than work with the smooth Riemannian manifolds, we can instead move to piecewise-flat manifolds. These are useful because we are able to collect important information about our manifolds along edges and at vertices, which reduce integrals to sums and simplify many of our calculations. Along with this fact, we have from T. Regge, an analogue to the $\mathcal{EH}$ functional on piecewise-flat manifolds [4]. The functional is named for Regge, and will be referred to as the Einstein-Hilbert-Regge functional, denoted $\mathcal{EHR}$. In mathematics, when we study the trait on a complicated object, call it A, and we approximate this object with simpler objects (specifically ones that look more and more like A). We wish the traits we study on these approximations to get closer to that trait on A. It has been shown that as the piecewise-flat manifolds are refined, $\mathcal{EHR}$ gets closer to $\mathcal{EH}$ [4]. In fact, as the PL manifolds converge, $\mathcal{EHR}$ converges to $\mathcal{EH}$. Since the computation of $\mathcal{EH}$ is potentially very difficult, this shows just how useful these functionals can be.

In this paper we wish to carefully construct two examples of piecewise-linear manifolds, and to use $\mathcal{EHR}$ and its normalizations to find Einstein metrics on those manifolds. These triangulations are interesting to us because in one case, we have some much important symmetry, and in the other we lack that same symmetry. We will discuss the importance of this symmetry and how it relates to the Einstein metrics on those manifolds.

In Chapter 2, we will introduce simplicial complexes in general. We will also introduce and discuss the object we wish to triangulate, S^3, and build the triangulations we want, which we will call $\mathcal{T}_8$, and $\mathcal{T}_5$. In Chapter 3, we will discuss the idea of a metric on simplices, and then on the triangulations as a whole. We will note that the set of all metrics is in fact a manifold itself. Within that space we will look for certain special metrics which we will point out. Following that, in Chapter 4, we will reintroduce the functionals which will be our tools of analysis on the metrics on our triangulations. We will show how they are discrete versions of the $\mathcal{EH}$ family of functionals, and will discuss the normalized versions of these. Finally, in Chapter 5, we will show which metrics from Chapter 3 are in fact critical points of our functionals, and hence that they are the Einstein metrics which we wish to find. Finally, in Chapter 6, we will discuss a few of the remaining problems, as well as some new ones. We will posit some potential avenues of inquiry for their solution in some cases, and leave them open in others.

CHAPTER 2

TRIANGULATIONS OF S^3

In this section, we will introduce the three-sphere, S^3, discuss some of its properties, and relate it to the two-dimensional sphere, S^2. Following this, we will discuss the concept of a simplex, and how that relates to our study. Finally, we will use these concepts to build two piecewise-flat approximations to S^3, which we will call $\mathcal{T}_8$, and $\mathcal{T}_5$, and will show, very carefully, how this is done.

There are many ways to think about what a three-dimensional sphere should be. The natural thing to do would be to somehow relate it back to our familiar two-sphere, and then generalize that definition so it can work in any dimension. In two dimensions, we generally think of a sphere as a sort of level set of a distance function. That is, it's the set of points equidistant from a given point. This is a nice definition, as it's extremely easy to generalize. However, since we will be working with piecewise-flat manifolds, we would prefer a more topological definition. Going back to S^2, we can think of the sphere as two disks glued together along their boundary circles.

It is somewhat difficult at first to have an intuitive sense of what it means to live in a three-dimensional sphere. Since we're working with spheres, it's often useful to refer to the more familiar S^2 when we wish to gain intuition regarding those of higher dimensions. When we think of the interesting properties of the two-sphere, often the first to come to mind is that if one were to stand in any place on a sphere and choose a direction, as long as the direction is not changed, eventually a walk in that direction would return to where they started. Since we live on an approximation to a topological sphere, this concept is not altogether hard for us to grasp. Not only do we return to where we started on such a walk, but we end up facing the same direction we started as well. For those familiar with the Klein bottle, we see a marked difference here. Another property of the two-sphere is that of contractible loops. If we have an infinitely bendy and stretchy loop, and we arrange it however we like on a sphere, we can always shrink the loop down

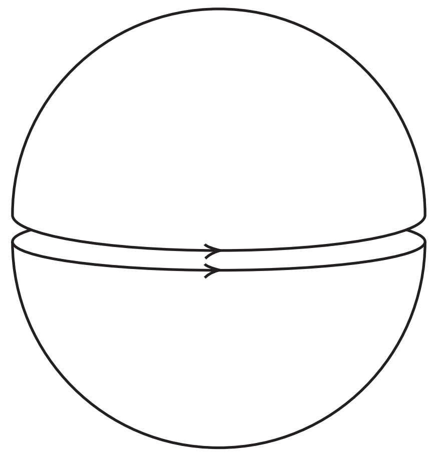

Figure 2.1: How two disks glue to make a sphere

without removing it from the surface until the loop is as small as we like.

In a three-sphere, then, we should at least have these two properties. That is, any walk in one direction (recalling that we now have three dimensions in which to walk) should lead back to the beginning of the walk without altering orientation, and any loop in a three-sphere should be contractible to a point. There have been other approaches to understanding this type of object. Some are very beautiful, like the Hopf fibration, which describes the three-sphere as a sort of bundle of fibers, each of which is a circle, on the normal two-sphere. In building a three-sphere, we approach it very much the same way we would the two-sphere. Where we would originally glue two disks together, we will instead consider the three-dimensional equivalent, the three-ball. That is, the three-ball is the interior of the two-sphere in $\mathbb{R}^3$. We will take two of these, and glue them together along their boundary spheres. The unfortunate fact is, however, that there are few, if any, elegant ways to describe this process.

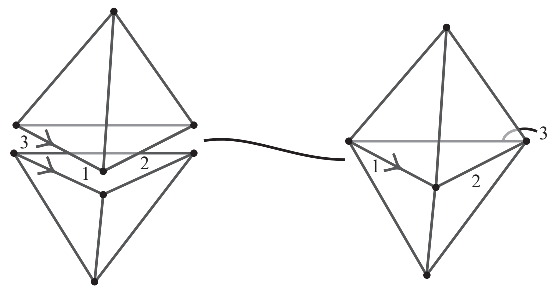

Figure 2.2: Disks made from triangles glue together to form a sphere

So far, we've introduced two-dimensional analogues to the three-dimensional objects we wish to study. This makes it easier to see the objects we're working with, as three-dimensional objects are much more difficult to visualize. Since what we want to do is approximate these three-dimensional objects, it makes sense to begin by introducing approximations to our two-dimensional analogues. In exactly the same sort of process as our gluing of disks before, we will glue two approximated disks made of triangles together to build a two-sphere. See Figure 2.2.

The triangle is the standard example of what is known as a two-dimensional simplex. We can generalize the idea of a simplex with a definition.

Definition 1 (Simplex)**.** *We define a* k-simplex *to be the* k *dimensional polytope, which is the convex hull of its* $k + 1$ *vertices.*

When we say *convex hull*, what we mean is the smallest convex set containing the vertices of the simplex. Convex is intuitively exactly what it sounds like, in that a set is *convex* if the line connecting any two points of the set does not leave the set itself. The easiest way to see this in two dimensions is to place some vertices on the plane, and stretch a rubber band around them. The shape contained within that rubber band is the convex hull of our vertices. In three dimensions, we do the same thing, but with a sheet rather than a rubber band. We have examples of simplices in dimensions other than two, such as the line in dimension one, and our simplex of interest, the tetrahedron in dimension three. In fact, for any simplex we have, we call the convex hull of any non-empty subset of the vertices of that simplex a *face* of that simplex. We say that the 1-faces are *edges*, and the $(k-1)$-faces are called facets. Though this is the case, when we work with tetrahedra, we will call the facets of the tetrahedra *faces*.

The object we will be building is a special case of what is known as a *simplicial complex*. To discuss this further, we will need a definition.

Definition 2 (Simplicial Complex)**.** *We define a* simplicial complex *to be a set of simplicies with the following conditions:*

1. *Any face of a simplex in the simplicial complex is also a face of the simplicial complex.*

2. *Any two simplices meet only along a single face.*

An example of one of these is given in Figure 2.2, where we see triangles taking the place of our simplices, and the complex being the resulting two-sphere. There are other examples, as well. Figure 2.1 shows triangles taking the place of our simplices, and a disk being our final complex. In fact, any surface made in this way is a simplicial complex, like the icosahedron, which can be realized as twenty triangles glued together in such a way that six of them meet at each vertex. The dodecahedron is a little more complicated, as we first need to build pentagons from triangles, but that can be seen to be five triangles glued along their edges so that they all meet at one central vertex. We then glue twelve such arrangements together in such a way that we get a dodecahedron. Since we can break any polygon into a finite number of triangles glued along edges, we see that every platonic solid is in fact a simplicial complex. Similarly, we can realize things like the icosahedron together with its interior as a gluing together of tetrahedra along faces. This construction is slightly more complicated, as the interior of the icosahedron becomes quickly complicated.

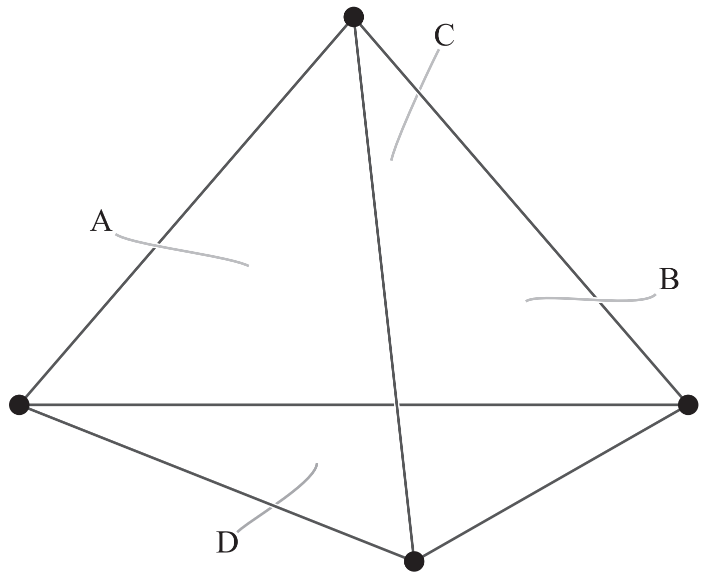

Figure 2.3: A Tetrahedron

What we will do, then, is begin to construct our own simplicial complex, which we will call the *octuple tetrahedron*, $\mathcal{T}_8$. We will start with a group of four tetrahedra, as in 2.3. We glue in such a way that what we're left with is also a tetrahedron, but with added simplicial structure, see Figure 2.4. The object we're left with after this gluing is our piecewise-flat three-ball. Similar to how we approached the gluing together of two disks above into a two-sphere, we will glue two of these constructed three-balls along their boundaries by identifying corresponding exterior faces of one with those on the other. See Figure 2.5 for an illustration of this process. We glue faces so that those sharing a letter are glued together.

This triangulation is an excellent one to begin with, as it is a simplicial complex, and often that is desirable. However, we immediately see that it lacks symmetry. While constructing this triangulation, we have inadvertently created two classes of edges. One class is what we will call the exterior class of edges, or the edges on the exterior of our

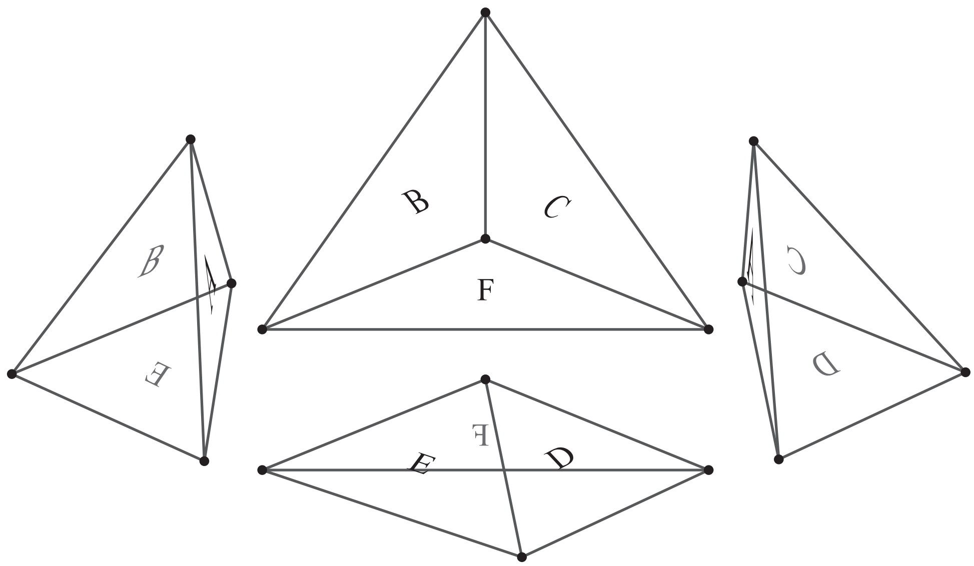

Figure 2.4: The gluing on four tetrahedra

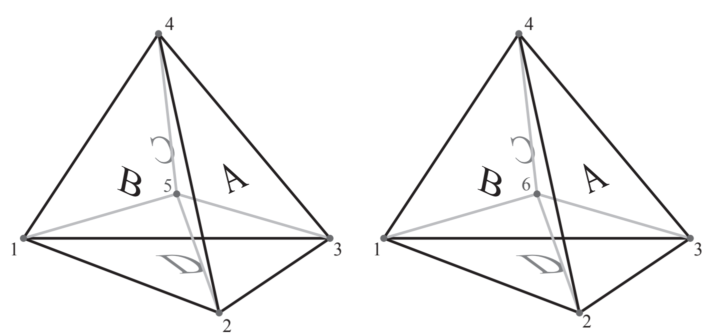

Figure 2.5: The gluing of three-balls

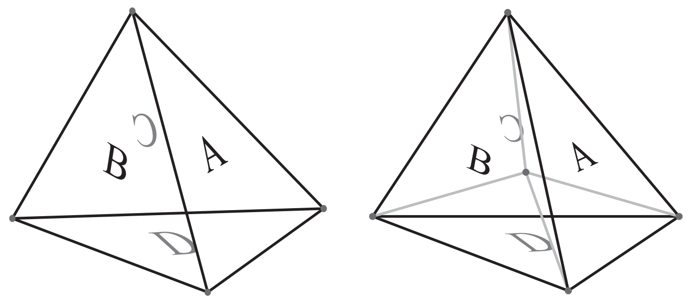

Figure 2.6: A more symmetric triangulation

original three-balls, before they were glued together. These edges are each contained in four tetrahedra, while the interior edges, or the edges in the interiors of our three-balls, are only contained in three tetrahedra. This fact will come back later. For now, we will introduce a second triangulation.

We begin our construction with two simplicial complexes. The first is just a single tetrahedron, while the second complex is a triangulated three-ball, similar to half of $\mathcal{T}_8$ introduced above. We identify the faces of the single tetrahedron with the corresponding exterior faces of the three-ball from above. That is, a single outside face from each of the four tetrahedra which comprise our three-ball is glued to each face of a fifth tetrahedron. We see each of those objects together in Figure 2.6. It is labeled in such a way that the faces to be identified share a letter. We see, again, that this is a simplicial complex. However, it has added symmetry in that every edge belongs to exactly three tetrahedra. We will see the relevance of this fact later. We will call this new triangulation the *quintuple tetrahedron* $\mathcal{T}_5$.

CHAPTER 3

METRICS ON OUR TRIANGULATIONS

Since the primary object of our study will be metrics on our simplicial complexes. We will use this section to become familiar with what such an object is. We will begin by describing what a metric looks like on just a single simplex, and after a few examples, will expand our horizon to metrics on the entire complex. Following this, we will discuss what the "space of metrics" looks like, and will describe what we believe to be the special metrics in that space.

Notation regarding simplices can get rather confusing, so to remain consistent with work done before, we will follow the model of [1] in the following ways. We will denote the *vertices* of any simplex by numbers, $\{1, 2, 3, 4, \dots\}$. The *edge between vertices i and j* will be denoted by a pair of numbers, ij. The *length of the edge, ij* will be denoted ℓ_{ij}. We will denote the *dihedral angle along edge ij* by β_{ij}, and the quantity $2\pi - \beta_{ij}$ the *angle defect along ij*. The *dihedral angle*, β_{ij}, is the interior angle along edge ij, as measured between the two faces of the simplex which meet at edge ij. There are a number of ways to measure this quantity relating either to the angle between the two faces normal vectors, or any number of other ways. We will introduce our method later. The *angle defect* along the edge is essentially how 'not flat' the simplex is at that edge. The farther from 2π the angles become, the more curved the space is, so the larger the angle defect will be. There will be more quantities that we will define as we move along, but for now, this is enough to get us started.

To begin, we will consider a triangle. As we've said before, this is a 2-simplex, and is the easiest to relate to the tetrahedron, since the one-simplex is just a line segment, and the four-simplex is much more complicated. We know, thanks to Euclid, that a triangle is uniquely defined by the lengths of its edges, and so we call such a collection of edge lengths a *metric* on the triangle. For some examples, see Figure 3.1. For those familiar with surfaces and Riemannian manifolds in general, we remember that each surface can be

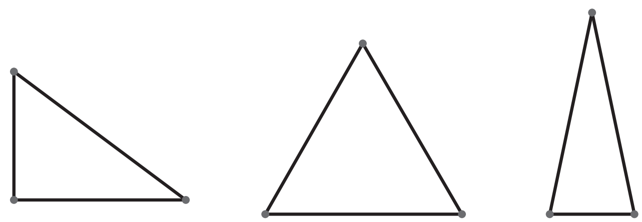

Figure 3.1: Metrics on the triangle

thought of as a sort of change of coordinates on the n-dimensional Euclidean space it represents. That is, a manifold is uniquely defined (up to isometry) by the metric on that manifold. The same holds true in our piecewise-flat spaces. So, to generalize this idea, we turn to the tetrahedron. Since a tetrahedron is determined by the triangles that compose it, and each of those triangles are determined by their edge lengths, we also have that a tetrahedron is determined up to some isometry by its six edge lengths. This gives us the motivation we need for a definition. For our definitions, we will be using generalized notation, where $\mathcal{M}$ is the manifold we wish to triangulate, a $\mathcal{T}$ with no subscript is the triangulation of that manifold, and ℓ is the metric on that triangulation. For clarity, we will only refer to our triangulations either by name or by our notation, $\mathcal{T}_5$ or $\mathcal{T}_8$.

Definition 3 (Metric)**.** *A vector $\ell \in \mathbb{R}^{|E|}$ such that each simplex can be realized as a Euclidean simplex with edge lengths determined by ℓ is called a* metric *for the triangulated manifold $(M, \mathcal{T})$, and $(M, \mathcal{T}, \ell)$ is called a* triangulated piecewise flat manifold*. The space of all metrics will be denoted $\mathfrak{met}(M, \mathcal{T})$.*

We see an interesting property of these spaces of metrics. First, we see that $\mathfrak{met}(M, \mathcal{T}, \ell) \subseteq \mathbb{R}^{|E|}$. Also, though they are used to uniquely determine triangulated manifolds, they are themselves manifolds. In fact, we will use the following definition to succinctly describe the space of metrics on a tetrahedron.

Definition 4. *Let* $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ *be the set of edge lengths on a tetrahedron,* A*. Then we let*

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ell_1^2 & \ell_2^2 & \ell_3^2 \\ 1 & \ell_1^2 & 0 & \ell_4^2 & \ell_5^2 \\ 1 & \ell_2^2 & \ell_3^2 & 0 & \ell_6^2 \\ 1 & \ell_4^2 & \ell_5^2 & \ell_6^2 & 0 \end{bmatrix} \tag{3.0.1}$$

Then let the Cayley-Menger determinant*,* $CM_3 = \det(M)$*.*

We have, thanks to Colins [2], the volume of A:

$$Vol(A) = \sqrt{\frac{\mathrm{CM}_3}{288}}.$$

We see, then, that on a single nondegenerate tetrahedron, A, that the space of all metrics, $\mathfrak{met}(A) = \left\{ \vec{\ell} \in \mathbb{R}^6 : \mathrm{CM}_3 > 0 \right\}$. Though it's unclear what the boundary of this space is, we can see that the interior of this space is a subset of $\mathbb{R}^6$.

We remember that the sphere with constant curvature is somehow special. That is, this sphere has the "best" metric on it, since other metrics would in some way cause a loss of symmetry on the sphere. We see this by comparing Figure 3.2 to Figure 3.3. We can think of these other metrics as parametrizations of the sphere in ways so that it is embedded in $\mathbb{R}^3$ differently.

A similar situation could be expected of our triangulations. We expect metrics where all edge lengths are equal to be somehow special. We call these metrics *equal length metrics*. In the paper [1], we saw results along this line, and so we will check to see if our added structure changes this at all.

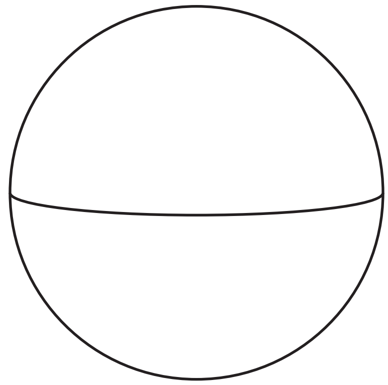

Figure 3.2: The standard metric on S^2.

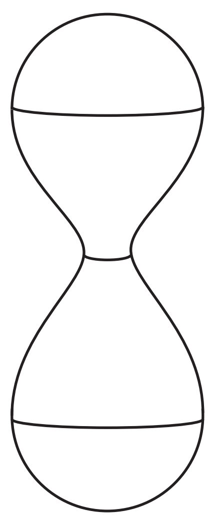

Figure 3.3: Another metric on S^2.

CHAPTER 4

THE NORMALIZED EINSTEIN-HILBERT-REGGE FUNCTIONALS

In the introduction, we talked briefly about the Einstein-Hilbert functional, $\mathcal{EH}$, and a little about what it is used for on closed Riemannian manifolds. In this section, we will introduce a discrete version of this functional, which will work very similarly to $\mathcal{EH}$, but on our simplicial complex instead. This functional comes about thanks to Regge [4], and we call it the Einstein-Hilbert-Regge functional. Similar to how we introduced normalized versions of $\mathcal{EH}$, we will discuss the normalized versions of this discrete functional, $\mathcal{EHR}$.

Following Regge's model, and our previously established notation, from Chapter 2, we will begin to build the functionals which will help us in our analysis. First, we must define what we mean by curvature in our setting.

Definition 5 (Edge Curvature)**.** *For a triangulation with metric ℓ, $(M, \mathcal{T}, \ell)$, and an edge ij, we define the* curvature along edge ij *to be the quantity*

$$K_{ij} = (2\pi - \sum_{t \in \mathcal{T}} \beta_{ij \in t})\ell_{ij},$$

where t is any simplex containing edge ij.

Where the original functional, $\mathcal{EH}$, was an integral over the surface of the sectional curvatures. Since, in our triangulations, the curvatures are collected along edges in the form of edge curvatures, we will sum the edge curvatures as an analogue to that integral. This leads us to our definition.

Definition 6 (The Einstein-Hilbert-Regge Functional)**.** *For a triangulated manifold, $\mathcal{M}$, we define the* Einstein-Hilbert-Regge Functional *to be*

$$\mathcal{EHR}(\mathcal{M}, \mathcal{T}, \ell) = \sum_{ij \in \mathcal{T}} K_{ij}.$$

Along with this, we will need a couple more definitions. These quantities will nearly always be used in the context of other formulas, and so will be defined together.

Definition 7. *We define the* total length *of* $(M, \mathcal{T}, \ell)$ *to be*

$$\mathcal{L}(M, \mathcal{T}, \ell) = \sum_{ij \in \mathcal{T}} \ell_{ij} \tag{4.0.1}$$

Let V_t *be the volume of simplex* t *in our triangulation. Then the* total volume *of* $(M, \mathcal{T}, \ell)$ *is*

$$\mathcal{V}(M, \mathcal{T}, \ell) = \sum_{t \in \mathcal{T}} V_t. \tag{4.0.2}$$

With these, we can now consider a pair of normalizations of our Einstein-Hilbert-Regge functional, rather than restricting ourselves to manifolds which behave nicely. That is, we need not restrict our triangulations to those of unit volume, or unit total length. Instead, we introduce the following.

Definition 8 (The Length Normalized Einstein-Hilbert-Regge Functional)**.** *We define the* Length Normalized $\mathcal{EHR}$ *to be*

$$\mathcal{LEHR}(M, \mathcal{T}, \ell) = \frac{\mathcal{EHR}(M, \mathcal{T}, \ell)}{\mathcal{L}(M, \mathcal{T}, \ell)}.$$

Additionally, we will introduce the Volume Normalized functional. Though we will not work with it very much, our approach to it would be similar, and it will be discussed briefly later.

Definition 9 (The Volume Normalized Einstein-Hilbert-Regge Functional)**.** *We define the* Volume Normalized $\mathcal{EHR}$ *to be*

$$\mathcal{VEHR}(M, \mathcal{T}, \ell) = \frac{\mathcal{EHR}(M, \mathcal{T}, \ell)}{\mathcal{V}^{\frac{1}{3}}(M, \mathcal{T}, \ell)}.$$

The nice thing about both of these normalizations is that if we scale the edge lengths by a non-zero constant, the value of the functionals does not change.

We will now refer to [4], and define the special metrics we are interested in.

Definition 10 (Einstein Metric)**.** *We say a metric, ℓ on $(M, \mathcal{T}, \ell)$ is* $\mathcal{L}$-Einstein *if it is a critical point of $\mathcal{LEHR}$, and is* $\mathcal{V}$-Einstein *if it is a critical point of $\mathcal{VEHR}$.*

Alternatively, we can define an *Einstein Metric* to be a metric such that for every edge, $ij \in \mathcal{T}$, $K_{ij} = \lambda_{\mathcal{L}} \ell_{ij}$ where $\lambda_{\mathcal{L}}$ does not depend on ij. That is, the total curvature along that edge is a constant multiple of the length of that edge.

These metrics are of interest to us primarily because they are in some ways the "best" metrics on a triangulation. Essentially, they are the most natural metrics to put on a manifold, as they have fairly nice curvatures. We see, at the very least, that along each edge, the curvature of that edge is some constant multiple of the length of that edge length. Since these functionals are discrete analogues to the original Einstein-Hilbert functionals, and since manifolds with constant curvature are Einstein metrics in that context, we hope that any metric which induces constant curvature on our triangulations are Einstein, as well.

CHAPTER 5

EINSTEIN METRICS ON PIECEWISE-LINEAR THREE-SPHERES

Over the course of this paper, we've built the triangulations, and functionals we need to prove some interesting facts about Einstein metrics on our triangulations. To begin, we will require some results which follow directly from the Schläfli formula, which can be found in [3]. We will also state it here, for clarity.

Remark 1 (Schläffli Formula)**.** *For a tetrahedron,*

$$\sum_{ij} \ell_{ij} \partial \beta_{ij} = 0.$$

Theorem 1 (First Derivatives)**.**

$$\frac{\partial \mathcal{EHR}(M, \mathcal{T}, \ell)}{\partial \ell_{ij}} = 2\pi - \sum_{kl} \beta_{ij,kl} \tag{5.0.1a}$$

$$\frac{\partial \mathcal{LEHR}(M, \mathcal{T}, \ell)}{\partial \ell_{ij}} = \mathcal{L}^{-1} \left(\frac{K_{ij}}{\ell_{ij}} - \frac{\mathcal{EHR}(M, \mathcal{T}, \ell)}{\mathcal{L}(M, \mathcal{T}, \ell)} \right). \tag{5.0.1b}$$

Proof. We will begin by proving (5.0.1a). We recall that $\mathcal{EHR} = \sum_{ij} (2\pi - \sum_{kl} \beta_{ij,kl}) \ell_{ij}$ where the sum is taken over all kl such that $ijkl$ is a tetrahedron in our triangulation. So, by a simple product-rule argument, and since the derivative can move inside our sum, we see that

$$\frac{\partial \mathcal{EHR}}{\partial \ell_{ij}} = \sum_{i^* j^*} \left(\ell_{i^* j^*} \frac{\partial}{\partial \ell_{ij}} (2\pi - \beta_{i^* j^*}) + (2\pi - \beta_{i^* j^*}) \frac{\partial}{\partial \ell_{ij}} (\ell_{i^* j^*}) \right).$$

From the Schläfli formula, we get that $\sum_{i^* j^*} \ell_{i^* j^*} \partial \beta_{i^* j^*} = 0$, and since $\frac{\partial}{\partial \ell_{ij}} (\ell_{i^* j^*}) = 0$ for all $i^* j^* \neq ij$, we have $\frac{\partial \mathcal{EHR}}{\partial \ell_{ij}} = 2\pi - \beta_{ij}$.

Next, we will prove (5.0.1b). We use the quotient rule, and see that

$$\frac{\partial \mathcal{LEHR}}{\partial \ell_{ij}} = \frac{\mathcal{L}(M, \mathcal{T}, \ell) \frac{\partial \mathcal{EHR}}{\partial \ell_{ij}} - \mathcal{EHR} \frac{\partial \mathcal{L}(M, \mathcal{T}, \ell)}{\partial \ell_{ij}}}{\mathcal{L}(M, \mathcal{T}, \ell)^2}. \tag{5.0.2}$$

By a similar argument as above, and by simplifying, we have that

$$\frac{\partial \mathcal{LEHR}}{\partial \ell_{ij}} = \mathcal{L}^{-1}\left(\frac{K_{ij}}{\ell_{ij}} - \frac{\mathcal{EHR}(M,\mathcal{T},\ell)}{\mathcal{L}(M,\mathcal{T},\ell)}\right).$$

□

As in [1], we check to see if equal-length metrics are $\mathcal{L}$-Einstein. That is, whether they are critical points of $\mathcal{LEHR}$. To do this, we must compute $\mathcal{LEHR}$ on each of our triangulations. We will begin with $\mathcal{T}_8$. On equal length metrics, we see that each tetrahedron is equilateral. Because of this, and the spherical cosine law, we know that the dihedral angle along each edge is exactly $\arccos(\frac{1}{3})$. So, setting every edge length to the same constant, a, we have that

$$\mathcal{EHR}(\mathcal{T}_8) = 6a\left(2\pi - 4\arccos\left(\frac{1}{3}\right)\right) + 8a\left(2\pi - 3\arccos\left(\frac{1}{3}\right)\right) \tag{5.0.3a}$$

$$\mathcal{LEHR}(\mathcal{T}_8) = \frac{6a\left(2\pi - 4\arccos\left(\frac{1}{3}\right)\right) + 8a\left(2\pi - 3\arccos\left(\frac{1}{3}\right)\right)}{14a}. \tag{5.0.3b}$$

Theorem 2 (Einstein Metrics on $\mathcal{T}_8$)**.** *Equal length metrics on $\mathcal{T}_8$ are not Einstein metrics.*

Proof. From equation (5.0.2), we see that a metric is Einstein here only if

$$\frac{K_{ij}}{\ell_{ij}} = \mathcal{LEHR}(M,\mathcal{T},\ell) \tag{5.0.4}$$

for all ij. For this triangulation, we see that $\frac{K_{ij}}{\ell_{ij}} = 2\pi - 4\arccos\left(\frac{1}{3}\right)$ for the class of exterior edges, and for the interior edges, we have $\frac{K_{ij}}{\ell_{ij}} = 2\pi - 3\arccos\left(\frac{1}{3}\right)$. So, not only do neither of these equal $\mathcal{LEHR}(\mathcal{T}_8)$, as in 5.0.3a, but they are not equal themselves. Hence, an equal length metric on this triangulation is not Einstein, contrary to what might be expected. It is exactly the lack of symmetry mentioned in §2 that causes this failure, as the curvature along each edge is not the same. □

We then turn to our second triangulation, $\mathcal{T}_5$, and check the same quantities. In this example, we see that since each edge belongs to the same number of tetrahedra, our computation of $\mathcal{LEHR}$ is much less complicated. We will, again, let each edge be of length a, where a is some positive constant. We then see the following equalities:

$$\mathcal{EHR}(\mathcal{T}_5) = 10a\left(2\pi - 3\arccos\left(\frac{1}{3}\right)\right) \tag{5.0.5a}$$

$$\mathcal{LEHR}(\mathcal{T}_5) = \frac{10a\left(2\pi - 3\arccos\left(\frac{1}{3}\right)\right)}{10a} \tag{5.0.5b}$$

$$= \left(2\pi - 3\arccos\left(\frac{1}{3}\right)\right). \tag{5.0.5c}$$

We see, almost immediately, then, that equal length metrics satisfy our equality 5.0.4, since in $\mathcal{T}_5$, we have $\frac{K_{ij}}{\ell_{ij}} = 2\pi - 3\arccos(\frac{1}{3})$ for all ij.

Theorem 3 (Einstein Metrics on $\mathcal{T}_5$)**.** *1. Equal length metrics are Einstein metrics on $\mathcal{T}_5$.*

2. The eigenspaces and eigenvalues for the Hessian matrices of $\mathcal{LEHR}$ at equal length metrics (where the edge lengths are all equal to a) are the following:

Table 5.1: Eigenanalysis of $\mathcal{T}_5$

eigenspace	spanning vectors	eigenvalues
V_{λ_1}	$(1,1,1,1,1,1,1,1,1,1)$	$\lambda_1 = \frac{19\sqrt{6}-12\sqrt{2}}{30}a^{-2} \approx 0.9856a^{-2}$
V_{λ_2}	$(1,0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{2}{3},-\frac{2}{3},-\frac{2}{3})$ $(0,1,0,0,\frac{1}{3},-\frac{2}{3},-\frac{2}{3},\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ $(0,0,1,0,-\frac{2}{3},\frac{1}{3},-\frac{2}{3},\frac{1}{3},-\frac{2}{3},\frac{1}{3})$ $(0,0,0,1,-\frac{2}{3},-\frac{2}{3},\frac{1}{3},-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$\lambda_2 = \frac{4\sqrt{6}-2\sqrt{2}}{30}a^{-2} \approx 0.2323a^{-2}$
V_{λ_3}	$(1,0,0,-1,0,0,-1,-1,1,1)$ $(0,1,0,-1,0,0,0,-1,0,1)$ $(0,0,1,-1,0,0,0,-1,1,0)$ $(0,0,0,0,1,0,-1,-1,0,1)$ $(0,0,0,0,0,1,-1,-1,1,0)$	$\lambda_3 = \frac{4\sqrt{6}(\frac{3}{3}-2)-3\sqrt{6}}{30}a^{-2} \approx -0.7095a^{-2}$

Proof. For equal length metrics on $\mathcal{T}_5$, where each edge length is equal to some constant, a, we have that $\frac{K_{ij}}{\ell_{ij}} = 2\pi - 3\arccos\left(\frac{1}{3}\right)$. From (5.0.5b), we see that, for equal length metrics on $\mathcal{T}_5$, $\frac{K_{ij}}{\ell_{ij}} = \mathcal{LEHR}(\mathcal{T}_5)$ for all ij. Hence, equal length metrics are Einstein on $\mathcal{T}_5$.

We will now prove the second statement in our theorem. To compute the Hessian of $\mathcal{LEHR}$, we will begin by computing the general second derivatives of $\mathcal{LEHR}$. Using a general product-rule argument, we see

$$\frac{\partial^2 \mathcal{LEHR}}{\partial \ell_{ij}^2} = -\mathcal{L}^{-2}\left[\frac{K_{ij}}{\ell_{ij}} - \mathcal{LEHR}\right] + \mathcal{L}^{-1}\left[\frac{\partial}{\partial \ell_{ij}}\left(\frac{K_{ij}}{\ell_{ij}}\right) - \mathcal{L}^{-1}\left(\frac{K_{ij}}{\ell_{ij}} - \mathcal{LEHR}\right)\right].$$

Since we are interested in evaluating this only on equal length metrics, we see the first term on the right hand side disappears, as will the second term in the second set of parenthesis. So we are left with the following

$$\begin{aligned}
\frac{\partial^2 \mathcal{LEHR}}{\partial \ell_{ij}^2} &= \mathcal{L}^{-1}\left(\frac{\partial}{\partial \ell_{ij}}\frac{K_{ij}}{\ell_{ij}}\right) \\
&= \mathcal{L}^{-1}\left[\frac{\partial}{\partial \ell_{ij}}\left(2\pi - \sum_{kl}\beta_{ij,kl}\right)\right] \\
&= -\mathcal{L}^{-1}\left(\sum_{kl}\frac{\partial}{\partial \ell_{ij}}\beta_{ij,kl}\right) \\
&= -\mathcal{L}^{-1}\left(\frac{\partial}{\partial \ell_{ij}}\beta_{ij}\right).
\end{aligned}$$

We must also calculate the general mixed partial derivative, when the two edges which we vary are different. The argument is similar, and we see the following:

$$\frac{\partial^2 \mathcal{LEHR}}{\partial \ell_{ij}\partial \ell_{i^*k^*}} = -\mathcal{L}^{-2}\left[\frac{K_{ij}}{\ell_{ij}} - \mathcal{LEHR}\right] + \mathcal{L}^{-1}\left[\frac{\partial}{\partial \ell_{i^*j^*}}\left(\frac{K_{ij}}{\ell_{ij}}\right) - \mathcal{L}^{-1}\left(\frac{K_{i^*j^*}}{\ell_{i^*j^*}} - \mathcal{LEHR}\right)\right].$$

By a similar argument as above, we get:

$$\frac{\partial^2 \mathcal{LEHR}}{\partial \ell_{ij}\partial \ell_{i^*k^*}} = -\mathcal{L}^{-1}\left(\frac{\partial}{\partial \ell_{i^*j^*}}\beta_{ij}\right).$$

Since we need only find the derivative of each dihedral angle with respect to each edge, we will use the spherical cosine law again to do this. There are three classes of derivatives we need to find. The first is when the dihedral angle we measure and edge we vary are the same. We begin by denoting the area of the triangle between vertices ijk by

$$A_{ijk} = \frac{1}{4}\sqrt{(\ell_{ij}^2 + \ell_{ij}^2 + \ell_{jk}^2) - 2(\ell_{ij}^4 + \ell_{ik}^4 + \ell_{jk}^4)}.$$

On equal length metrics, we see that $\frac{\partial A_{ijk}}{\partial \ell_{i^*j^*}}$ is the same, regardless of our choice of $\ell_{i^*j^*}$, provided they are two of ijk. Otherwise, the partial derivative would be zero.

We then consider the following

$$\frac{\partial A_{ijk}}{\partial \ell_{ij}} = \frac{1}{4}\left[\left(2\ell_{ij}(2(\ell_{ij}^2 + \ell_{ik}^2 + \ell_{jk}^2)\right) - 8\ell_{ij}^3\right]\left[\frac{1}{8A_{ijk}}\right]$$

Since we're considering equal length metrics, $\ell_{ij} = \ell_{ik} = \ell_{jk} = a$, and we have

$$\begin{aligned} \left.\frac{\partial A_{ijk}}{\partial \ell_{ij}}\right|_{\ell_{ij}=\ell_{ik}=\ell_{jk}=a} &= \frac{1}{4}(2a(2(3a^2)) - 8a^3) \cdot \frac{1}{2\sqrt{3}a^2} \\ &= (3a^3 - 2a^3) \cdot \frac{1}{2\sqrt{3}a^2} \\ &= \frac{a}{2\sqrt{3}} \qquad \forall i, j, k. \end{aligned}$$

We notice that for the case when either or both $l, m \notin \{i, j, k\}$, that $\frac{\partial A_{ijk}}{\partial \ell_{lm}} = 0$.

We introduce a new quantity

$$B_{ij,kl}^2 = \frac{1}{16}\left(4\ell_{ij}^2\ell_{kl}^2 - \left((\ell_{ik}^2 + \ell_{jl}^2) - (\ell_{il}^2 + \ell_{jk}^2)\right)^2\right).$$

Thanks to the symmetry we have, we get the following equalities:

$$\begin{aligned}\frac{\partial B_{ij,kl}}{\partial \ell_{ij}} &= \frac{\partial B_{ij,kl}}{\partial \ell_{kl}} \\ \frac{\partial B_{ij,kl}}{\partial \ell_{ik}} &= \frac{\partial B_{ij,kl}}{\partial \ell_{jl}} \\ \frac{\partial B_{ij,kl}}{\partial \ell_{il}} &= \frac{\partial B_{ij,kl}}{\partial \ell_{jk}}.\end{aligned}$$

From this, we see that

$$\begin{aligned}\left.\frac{\partial B_{ij,kl}}{\partial \ell_{ij}}\right|_a &= \frac{a}{4} \\ \left.\frac{\partial B_{ij,kl}}{\partial \ell_{il}}\right|_a &= \left.\frac{\partial B_{ij,kl}}{\partial \ell_{ik}}\right|_a = 0.\end{aligned}$$

So, using the spherical cosine law, we get that the dihedral angle along edge ij, in tetrahedron $ijkl$ is given by

$$\beta_{ij,ijkl} = \arccos\left(\frac{A_{ijk}^2 + A_{ijl}^2 - B_{ij,kl}^2}{2A_{ijk}A_{ijl}}\right).$$

We then take a derivative and have

$$\frac{\partial \beta_{ij,ijkl}}{\partial \ell_{i^*j^*}} = \frac{-\left(2A_{ijk}A_{ijl}\left(2A_{ijk}\frac{\partial A_{ijk}}{\partial \ell_{i^*j^*}}+2A_{ijk}\frac{\partial A_{ijl}}{\partial \ell_{i^*j^*}}-2B_{ij,kl}\frac{\partial B_{ij,kl}}{\partial \ell_{i^*j^*}}\right)-2\left(A_{ijk}^2+A_{ijl}^2-B_{ij,kl}^2\right)\left(A_{ijl}\frac{\partial A_{ijk}}{\partial \ell_{i^*j^*}}+A_{ijk}\frac{\partial A_{ijl}}{\partial \ell_{i^*j^*}}\right)\right)}{4A_{ijk}^2A_{ijl}^2\sqrt{1-\cos^2(\beta_{ij,ijkl})}}. \tag{5.0.6}$$

By the above, we have on equal length metrics that $A_{ijk} = \frac{1}{4}\sqrt{9a^4 - 6a^4} = \frac{\sqrt{3}}{4}a^2$ $\forall i, j, k$. Since $B_{ij,kl}^2 = \frac{a^4}{4}$ $\forall ij, kl$, we have $B_{ij,kl} = \frac{a^2}{2}$.

We already have that $\left.\frac{\partial A_{ijk}}{\partial \ell_{ij}}\right|_{\ell_{ij}=\ell_{ik}=\ell_{jk}=a} = \frac{a}{2\sqrt{3}}$. Since these quantities are evaluated at equal length metrics, we get that

$$\begin{aligned}
\left.\frac{\partial\beta_{12}}{\partial\ell_{12}}\right|_a &= \left.\frac{\partial\beta_{12,1234}}{\partial\ell_{12}}\right|_a + \left.\frac{\partial\beta_{12,1235}}{\partial\ell_{12}}\right|_a + \left.\frac{\partial\beta_{12,1245}}{\partial\ell_{12}}\right|_a \\
&= \left.\frac{3\partial\beta_{12,1234}}{\partial\ell_{12}}\right|_a \\
&= \frac{-3}{\sqrt{\frac{2}{3}}}\left(\frac{\left(\frac{3}{8}a^4\right)\left(\frac{1}{4}a^3\right)-\left(\frac{7}{8}a^4\right)\left(\frac{1}{2}a^3\right)}{\frac{9}{64}a^8}\right) \\
&= \frac{2\sqrt{6}}{3}a^{-1}.
\end{aligned}$$

Because of our symmetry, and the fact that these are evaluated at equal-length metrics, we have $\frac{\partial\beta_{ij}}{\partial\ell_{ij}} = \frac{\partial\beta_{12}}{\partial\ell_{12}}$ for all $ij \in \mathcal{T}_5$.

We now look at derivatives where the dihedral angle is taken along an edge which shares exactly one vertex with the length we vary. We see that these derivatives are of the form $\frac{\partial\beta_{ij}}{\partial\ell_{ik}}$. We know, from symmetry and since these are evaluated at equal length metrics the following for all i, j, k:

$$\begin{aligned}
\frac{\partial\beta_{ij}}{\partial\ell_{ik}} &= \frac{\partial\beta_{12}}{\partial\ell_{13}} \\
&= \left.\frac{\partial\beta_{12,1234}}{\partial\ell_{13}}\right|_a + \left.\frac{\partial\beta_{12,1235}}{\partial\ell_{13}}\right|_a + \left.\frac{\partial\beta_{12,1245}}{\partial\ell_{13}}\right|_a = (\star)
\end{aligned}$$

Since $\left.\frac{\partial\beta_{12,1245}}{\partial\ell_{13}}\right|_a = 0$, because edge 13 is not in tetrahedron 1245, we have:

$$(\star) = \left.\frac{2\partial\beta_{12,1234}}{\partial\ell_{13}}\right|_a.$$

By (5.0.6), we get that for all i, j, k,

$$\left.\frac{2\partial\beta_{12,1234}}{\partial\ell_{13}}\right|_a = \frac{8}{3\sqrt{2}}(1-\sqrt{3})a^{-1} = \frac{\partial\beta_{ij}}{\partial\ell_{ik}}$$

Finally, we have the derivatives in which the angle we measure shares no vertices with the edge we vary. These are of the form

$$\left.\frac{\partial \beta_{ij}}{\partial \ell_{kl}}\right|_a .$$

Again, by symmetry and since the edge lengths are all equal, we have the following:

$$\left.\frac{\partial \beta_{ij}}{\partial \ell_{kl}}\right|_a = \left.\frac{\partial \beta_{12}}{\partial \ell_{34}}\right|_a = (\dagger)$$

Since the edge we vary only affects the angle we measure in one tetrahedron, we get

$$(\dagger) = \left.\frac{\partial \beta_{12,1234}}{\partial \ell_{34}}\right|_a = \frac{\sqrt{6}}{3} a^{-1}$$

We recall that our derivatives are each multiplied by $\mathcal{L}^{-1}$, and we have our Hessian matrix. We first let the quantity $b = \frac{2\sqrt{3}}{3} - 4$, and our matrix is the following:

$$A = -\frac{\sqrt{6}}{30} a^{-2} \begin{bmatrix} 2 & b & b & b & b & b & b & 1 & 1 & 1 \\ b & 2 & b & b & b & 1 & 1 & b & b & 1 \\ b & b & 2 & b & 1 & b & 1 & b & 1 & b \\ b & b & b & 2 & 1 & 1 & b & 1 & b & b \\ b & b & 1 & 1 & 2 & b & b & b & b & 1 \\ b & 1 & b & 1 & b & 2 & b & b & 1 & b \\ b & 1 & 1 & b & b & b & 2 & 1 & b & b \\ 1 & b & b & 1 & b & b & 1 & 2 & b & b \\ 1 & b & 1 & b & b & 1 & b & b & 2 & b \\ 1 & 1 & b & b & 1 & b & b & b & b & 2 \end{bmatrix}$$

By solving the system of equations $Av = \lambda v$, we get the following values of λ:

$$\lambda_1 = \frac{-\sqrt{6}}{30}a^{-2} \cdot (4\sqrt{3} - 19) = \frac{19\sqrt{6} - 12\sqrt{2}}{30}a^{-2}$$
$$\lambda_2 = \frac{-\sqrt{6}}{30}a^{-2} \cdot \left(\frac{2\sqrt{3}}{3} - 4\right) = \frac{4\sqrt{6} - 2\sqrt{2}}{30}a^{-2}$$
$$\lambda_3 = \frac{-\sqrt{6}}{30}a^{-2} \cdot \left(3 - 4\left(\frac{\sqrt{3}}{3} - 2\right)\right) = \frac{4\sqrt{6}\left(\frac{\sqrt{3}}{3} - 2\right) - 3\sqrt{6}}{30}a^{-2}.$$

And that each of these has the following corresponding eigenspaces:

$$Av_{\lambda_1} = \lambda_1 v_{\lambda_1}$$

where

$$v_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and we have

$$Av_{\lambda_2} = \lambda_2 v_{\lambda_2}$$

where

$$v_{\lambda_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \\ \frac{-2}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \frac{1}{3} \\ \frac{-2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \frac{-2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \frac{-2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

and finally,

$$Av_{\lambda_3} = \lambda_3 v_{\lambda_3}$$

where

$$v_{\lambda_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

These results, we see, agree with those in Table 5.1. Hence, our results are proved.

□

We have now shown that equal length metrics on $\mathcal{T}_5$ are indeed Einstein, and are also saddle points of the length normalized Einstein-Hilbert-Regge functional.

CHAPTER 6
FURTHER QUESTIONS AND POTENTIAL WORK

Throughout this paper, we have introduced and discussed what the object we call a three-sphere is, and what it looks like. We have talked about what it means to triangulate such an object, and what it means to have a metric on such a triangulation. Our main results centered around a certain special class of metrics on two such triangulations, the first showing that with added structure comes added complexity. We saw this complexity break our intuition about what types of metrics should be called the 'best' metrics on this triangulation. However, when we simplified the triangulation only slightly, we saw that added symmetry in our triangulation brought that intuition back into focus in the form of equal-length metrics on $\mathcal{T}_5$. However, this still leaves many questions unanswered.

First, the obvious question to ask would be regarding Einstein metrics on $\mathcal{T}_8$. This was investigated for some time while working on this problem, and it seems to be the case that since we have two different classes of edges, (recall that we have one class of edges on the 'exterior' of the triangulation, belonging to four tetrahedra each, and a class on the 'interior', which each belong to three tetrahedra) the curvature along each edge on equal length metrics is different between the two classes. Hence, since each edge length is the same, we clearly do not meet the Einstein condition from Definition 10. Further work could be done by using the Einstein metrics as a restriction on our metrics to see what possible metrics could be considered Einstein. When we do this, we see the following equality as a necessary condition, for each edge ij:

$$\frac{K_{ij}}{\ell_{ij}} = \left(2\pi - \sum_{kl} \beta_{ij,kl}\right) \ell_{ij}.$$

While this is a succinct restriction, it is also quite complicated. Not only must this be satisfied, but so must a large number of other conditions. For a study in the edge-length restrictions on a tetrahedron, see e.g. [7]. There are also possible avenues of inquiry regarding the solid angles at each vertex, but the path forward here seems unclear.

The second question we must ask is concerning $\mathcal{V}$-Einstein metrics. Though we introduce the volume normalized functional in this paper, we limit its discussion, as the computations of derivatives of volume with respect to edge length become quickly very complicated. Some headway was made in this area, and will be discussed now.

We begin the discussion by recalling our quantity CM_3, the Cayley-Menger determinant, which is related to the volume of a 3-simplex, A, in the following way:

$$Vol(A) = \sqrt{\frac{CM_3}{288}}.$$

We wish to find the quantity $\frac{\partial Vol(A)}{\partial \ell_{ij}}$, since we recall $\mathcal{VEHR}(M, \mathcal{T}, \ell) = \frac{\mathcal{EHR}(M,\mathcal{T},\ell)}{\mathcal{V}(M,\mathcal{T},\ell)^{\frac{1}{3}}}$. To do this, we will use the chain rule, and check $\frac{\partial CM_3}{\partial \ell_{ij}}$. Since $CM_3 = \det(M)$, where M is the Cayley-Menger matrix for our simplex, A, what we need is a tool for relating the derivative of a determinant to something potentially more tractable. Fortunately, we have precisely that tool in Jacobi's formula [6]. We first define, since all valid Cayley-Menger matrices are invertible, the *adjugate* matrix of A to be $\mathrm{adj}(A) = \det(A)A^{-1}$. We see that

$$\begin{aligned}\frac{\partial CM_3}{\partial \ell_{ij}} &= \mathrm{tr}\left(\mathrm{adj}(A)\frac{\partial A}{\partial \ell_{ij}}\right) \\ &= \mathrm{tr}\left(\det(A)A^{-1}\frac{\partial A}{\partial \ell_{ij}}\right) \\ &= \mathrm{tr}\left(CM_3 A^{-1}\frac{\partial A}{\partial \ell_{ij}}\right).\end{aligned}$$

Here, the problem arises, as the quantity CM_3A^{-1} is rather complicated. See Figure 6. This is only a small part of the final computation for the derivative of volume. It's an important part, but the computations became far too intractable at this point, so inquiry into the volume normalizations was halted. Further investigation should yield results, however, and it looks fairly optimistic once this hurdle can be overcome.

Let $\{a, b, c, d, e, f\}$ be the set of edge lengths of the tetrahedron A. This is inconsistent with our previous notation, but we omit the character ℓ for brevity.
Then $CM_3M^{-1} =$

$$\begin{bmatrix} c^2d^2-2bced-2acfd+b^2e^2a^2f^2-2abef & -cd^2+bcd+ced+afd+cfd-2efd-be^2-af^2+aef+bef & -eb^2+cdb+ceb+afb-2cfb+efb-af^2-c^2d+acf+cdf & -fa^2+cda+bea-2cea+cfa+efa-be^2-c^2d+bce+cde & -fa^2-2bda+cda+bea+bfa+dfa-cd^2+bcd-b^2e+bde \\ -cd^2+bed+ced+afd+cfd-2efd-be^2-af^2+acf+bef & d^2-2ed-2fd+e^2+f^2-2ef & -f^2-2af+bf+cf+df+ef-bd+cd+be-ce & -e^2+ae-2be+ce+de+fe-ad+cd+af-cf & -d^2+ad+bd-2cd+ed+fd-ae+be+af-bf \\ -eb^2+cdb+ceb+afb-2cfb+efb-af^2-c^2d+aef+cdf & -f^2-2af+bf+cf+df+ef-bd+cd+be-ce & b^2-2cb-2fb+c^2+f^2-2cf & -c^2+ac+bc-2dc+ec+fc-ab+be+af-ef & -b^2+ab+cb+db-2eb+fb-ac+cd+af-df \\ -fa^2+cda+bea-2cea+cfa+efa-be^2-c^2d+bce+cde & -e^2+ae-2be+ce+de+fe-ad+cd+af-ef & -c^2+ac+bc-2dc+ec+fc-ab+be+af-ef & a^2-2ca-2ea+c^2+e^2-2ce & -a^2+ba+ca+da+ea-2fa-bc+cd+be-de \\ -fa^2-2bda+cda+bea+bfa+dfa-cd^2+bcd-b^2e+bde & -d^2+ad+bd-2cd+ed+fd-ae+be+af-bf & -b^2+ab+cb+db-2eb+fb-ac+cd+af-df & -a^2+ba+ca+da+ea-2fa-bc+cd+be-de & a^2-2ba-2da+b^2+d^2-2bd \end{bmatrix}.$$

Figure 6.1: A Very Large Matrix

BIBLIOGRAPHY

[1] Daniel Champion and David Glickenstein and Andrea Young *Regge's Einstein-Hilbert Functional on the Double Tetrahedron*. Oxford arXiv:1007.0048v1[math.DG], 2010.

[2] Colins, K. *Cayley-Menger Determinant*. `http://mathworld.wolfram.com/Cayley-MengerDeterminant.html`, [Online; accessed 04-June-2013], 2013.

[3] David Glickenstein, *Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds*. arXiv:0906.1560v1[math.DG], 2009.

[4] Regge, T., *General Relativity without Coordinates.*. Il Nuovo Cimento, 1960.

[5] Wikipedia, *Einstein Manifold*. `http://en.wikipedia.org/wiki/Einstein_manifold`, [Online; accessed 11-June-2013], 2013.

[6] Wikipedia, *Jacobi's formula*. `http://en.wikipedia.org/wiki/Jacobi's_formula`, [Online; accessed 04-April-2013], 2013.

[7] Wirth,K. and Dreiding,A., *Edge lengths determining tetrahedrons*. Elem. Math. 64 (2009) 160.