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Introduction to fractal geometry: Definition, concept, and applications

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INTRODUCTION TO FRACTAL GEOMETRY: DEFINITION, CONCEPT, AND APPLICATIONS

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". . . clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line, . . ."

Benoit B. Mandelbrot

For centuries, geometers have utilized Euclidean geometry to describe, measure, and study the world around them. The quote from Mandelbrot poses an interesting problem when one tries to apply Euclidean geometry to the natural world.

In 1975, Mandelbrot coined the term "fractal." He had been studying several individual "mathematical monsters."

At a young age, Mandelbrot had read famous papers by G. Julia and P. Fatou concerning some examples of these monsters. When Mandelbrot was a student at Polytechnique, Julia happened to be one of his teachers. Mandelbrot became interested in iteration and, using iteration and some mathematical monsters, took off in a new dirction. Up until Mandelbrot's find, these monsters were thought to be separate entities, with no correlation between them. It was Mandelbrot who found the unity in these constructions. In doing so, he introduced a new language to the world of mathematics, providing a new image for mathematics and revolutionizing the application of non-Euclidean geometric constructs to science.

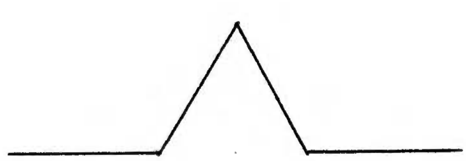
When Mandelbrot first coined the term "fractal", he did not give a precise mathematical definition. It is truly difficult to pin down an exact definition of the word "fractal." Mandelbrot would describe the shapes he was studying (and calling fractals) as sharing the property of being "rough but self-similar." He said that each shape conforms to the loose sense of "alike." This quality is probably the most definitive fractal characteristic.

Looking at a particular fractal structure, the triadic Koch curve, will help to demonstrate this characteristic of self-similarity. The Koch curve is generated as follows:

> Begin with a straight line. Let n stand for generation or time. As you can see, a straight line is the 0th generation.

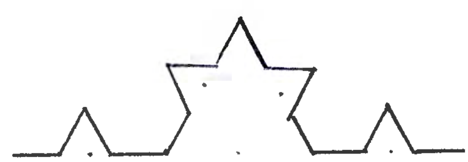
$n=0$

> To produce the 1st generation Koch curve, remove the middle third of the line segment and replace it with two segments that are each the same in length as the segment third which was just removed.



$n=1$

Repeating this process for each line segment in the n-th generation will produce the n+1-th generation. From the 1st generation, the 2nd generation is produced.



n=2

This procedure demonstrates another interesting phenomena. Notice that each stage of the Koch curve increases in total length in a ratio of 4/3. Therefore, the limit curve is of infinite length.

The Koch curve is a fine example of self-similarity. Each small segment is similar to the 1st generation. The 1st generation is underlying in all successive generations. A fractal is a shape which is made up of parts similar to the whole in some way.

A second definitive characteristic of fractals is the dimensional realm in which they exist. Fractals have dimensions in between 1 and 2, and in between 2 and 3.

In 1982, Mandelbrot finally gave a tentative definition of a fractal using the idea of dimension. A fractal is, by definition, a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension. Now, introducing an important theorem, if K is a curve and is self-similar, then the Hausdorff-Besicovitch dimension of K is the same as its self-similarity dimension.

To find the self-similarity dimension of the Koch curve, begin by looking at its length function. The length of the n-th generation fractal is given by

$$L(\delta) = (4/3)^n$$

The length of each of the small line segments is

$$\delta = 3^{-n}$$

By noting that the generation number n may be written in the form

$$n = -\ln\delta / \ln 3$$

it becomes obvious that the length may be expressed as follows:

$$L(\delta) = (4/3)^n = \exp\left(-\frac{\ln \delta[\ln 4 - \ln 3]}{\ln 3}\right) = \delta^{1-D}$$

This results in the equation

$$D = \ln 4 / \ln 3 \doteq 1.2618$$

This is the self-similarity dimension of the Koch curve. By the previously stated theorem, since the Koch curve is self-similar, this dimension is equivalent to the Hausdorff-Besicovitch dimension. Now, each generation of the construction of the Koch curve may be stretched to form a straight line, and therefore it is concluded that the topological dimension of the Koch curve is 1. Since the Hausdorff-Besicovitch dimension ($\sim$1.2618) strictly exceeds the topological dimension (1), this is a fractal curve according to Mandelbrot's definition. Furthermore, the fractal dimension equals the self-similarity dimension equals the Hausdorff-Besicovitch dimension, which is approximately 1.2618.

Another important characteristic of fractals is their artistic quality. The development of computer generated graphics provided a rich soil from which fractals were harvested. Computer graphics, adding visualization to these mathematical monsters, were a necessary factor for growth in the field of fractal geometry. Interestingly enough, when Mandelbrot was a Visiting Professor of Mathematics at Harvard in 1979-80, Harvard's Science Center had its first Vax computer (brand new), a Tektronix cathode ray tube was used to view the pictures generated, and a Versatec device was used to print hard copies of the first computer generated fractal images. There has been much improvement in computer generated fractals, from a time when Mandelbrot said ". . . no one knew how to set properly [the computer devices]. . ." to more recent computer generated fractal forgeries, sometimes used in the filming industry, of everything from plants to planet rises. These computer generated fractal images are finding their way into galleries, and artists are even hand painting huge portions of the fractal named the "Mandelbrot set." Much of fractal art resembles natural phenomena such as mountain ranges, valleys, clouds, and the list goes on. Returning to the notion of Euclidean geometry being difficult to apply to nature, it seems as though fractal geometry provides a mathematical language which is perfect for application to the world around us. As a matter of fact, there are many items which we often see which have fractal characteristics such as clouds, ferns, and cauliflower (all have the quality of self-similarity), and lightning and dust (whose movement is of fractal nature).

Yet another characteristic of fractals is their numerous applications. In metallurgy, which is the study of the properties of metals, a type of metal powder grain is produced by the disintegration of a jet of liquid metal with subsequent cooling of the droplets as they fall down a tall cooling tower. This is called a shotting process. The structure of irregularly-structured aluminum shot is such that it can be described by a fractal dimension at coarse resolution. Also, it appears that chemical crusts on eroded surfaces look like fractals or have fractal characteristics, although no research was found that has been conducted to this extent. Fractals have been implemented in the mining industry, also, since it has been proven that there is a fractal structure to a body failing under impact stress produced by a high-velocity projectile. Special events combining music and fractals have occurred in New York at the Guggenheim Museum in April 1990 and at Lincoln Center in March 1991. There is fractal structure apparent in cosmic fineparticles, that is, cosmic dust. There is also fractal structure in some types of sand grains. As one last example of fractal application, when coal is burned to provide energy, some of the ash present in the original piece of coal escapes from the stack of the power station as a very fine material which is known as flyash. Many flyash fine particles have a fractal structure. The utility and/or health hazard of any particulate flyash may depend on that fractal structure.

It has become evident that fractals are not to be tied down to one compact, Webster-style, paragraph definition. The foremost qualities of fractals include self-similarity and dimensionality. One cannot help but appreciate the aesthetic beauty of computer generated fractal art. Beyond these characteristics, when trying to grasp the idea of fractal geometry, it is helpful to learn about its many applications. Fractal geometry is opening new doors for study and understanding in diverse areas such as science, art, and music. All of these facets of fractal geometry unite to provide an intriguing, and alluring, wardrobe for mathematics to wear, so that mathematical study can now be enticing for the artist, the scientist, the musician, etc., as well as the mathematician.

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