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Fred Behroozi
University of Northern Iowa

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In Praise of the Catenary

F. Behroozi, University of Northern Iowa, Cedar Falls, IA

When a chain hangs loosely from its end points, it takes the familiar form known as the catenary. Power lines, clothes lines, and chain links are familiar examples of the catenary in everyday life. Nevertheless, the subject is conspicuously absent from current introductory physics and calculus courses. Even in upper-level physics and math courses, the catenary equation is usually introduced as an example of hyperbolic functions or discussed as an application of the calculus of variations. We present a new derivation of the catenary equation that is suitable for introductory physics and mathematics courses.

Introduction

Due to its historical and pedagogical significance, treatment of the catenary was standard fare in physics and mathematics textbooks through the first half of the 20th century. In 1891 E. J. Routh introduced a novel derivation of the catenary equation without resorting to the calculus of variations, and a decade later an alternative derivation, due to H. Lamb, gained widespread acceptance. Lamb’s derivation is still used in some intermediate mechanics textbooks.

The Routh/Lamb derivation of the catenary equation is rather convoluted and arduous, which explains why it has gradually disappeared from contemporary textbooks. For instance, the early editions of the well-known calculus text by Thomas included a derivation of the catenary equation covering several pages of the text. But in later editions the catenary was relegated to exercises for hyperbolic functions. This practice is now the norm in contemporary calculus textbooks.

A more extensive historical background and several alternative derivations of the catenary equation appear elsewhere. The appendix in Ref. 8 also gives a streamlined version of the modern derivation of the catenary equation based on the calculus of variations.

Here we present a new and simpler route to the catenary equation suitable for an introductory course in physics or mathematics. In the next section, we consider the equilibrium conditions for a section of the chain under the effect of gravity and tension. This analysis leads directly to a pair of equations, one of which gives the tension $T(\theta)$ and the other gives the arc length $s(\theta)$ as functions of the angle between the chain and the horizontal.

In the section “Catenary Equation in Rectangular Coordinates,” by judicious use of elementary calculus, we derive the familiar catenary equation in rectangular coordinates, $y = a \cosh x/a$, where $a$ is the scale factor. In the “Discussion” section, we discuss the results and note that a hanging chain is characterized most conveniently by two physical parameters: the chain length and the distance between its support points. We describe the procedure for finding the scale factor for a hanging chain of known length and end point separation.

Finally, we discuss the fact that all catenary curves are similar to one another in the same sense that all circles are similar to one another. Furthermore, since many of the catenary features can easily be demonstrated in class, we suggest a few simple demonstrations.

Catenary equation in angular coordinates

Figure 1 shows a hanging chain of length $2l$, linear mass density $\lambda$, and end point separation $2b$. The $Y$-axis passes through the vertex of the chain and divides the chain into two symmetric halves. For ease of analysis, it is convenient to focus on the right half of the chain where $s(\theta)$ stands for the arc length from the vertex to a point along the chain where the angle between the chain direction and the horizontal is $\theta$.

![Figure 1. A hanging chain of length 2l, linear mass density $\lambda$, and end separation 2b is shown schematically. The Y-axis passes through the vertex of the chain and forms an axis of symmetry. The chain section s (shown in red) is in equilibrium under the action of three forces: the tensions $T$ and $T_0$ and its weight $\lambda gs$.](image)

In Fig. 1, $T_0$ is the tension at the vertex; the tension increases along the chain to its maximum, $T_m$, at the end point of the chain. The angle between the chain and the horizontal, $\theta$, is zero at the vertex and reaches its maximum, $\theta_{max}$, at the top. The chain length $s(\theta)$, shown in red and extending from the vertex up to a point along the chain at angle $\theta$, is in equilibrium under the action of three forces: gravitational force $\lambda gs$ and the two tension forces, $T_0$ and $T$. Therefore we have,

$$T \cos \theta = T_0$$  \hspace{1cm} (1)

$$T \sin \theta = \lambda gs.$$  \hspace{1cm} (2)

We note here that the shape assumed by a hanging chain is determined by its two physical parameters $l$ and $b$. However, as we shall show below, for chains with the same values of $l$ and $b$, the tension scales linearly with the weight density $\lambda g$ for each chain. Consequently, the ratio of $T_0/\lambda g$ is a constant.
for all catenaries sharing the same \( l \) and \( b \). For this reason and for later convenience we define a new constant \( a \), where

\[
a = \frac{T_0}{\lambda g}. \tag{3}
\]

The parameter \( a \) has the dimension of length; its explicit dependence on the catenary parameters \( l \) and \( b \) will be given later. Furthermore, as shown later, the catenary equation takes its simplest form in a Cartesian coordinate system in which the \( Y \)-axis is the symmetry axis and the \( y \)-intercept is chosen to be equal to \( a \), as shown in Fig. 1.

Using the new parameter \( a \), Eq. (1) may be written as

\[
T(\theta) = \lambda ga \cos \theta. \tag{4}
\]

Dividing Eq. (2) by Eq. (1) leads to

\[
s(\theta) = a \tan \theta. \tag{5}
\]

Equation (5), known as the Whewell equation,\(^9\) gives the arc length of the chain from its vertex to any point of the chain above the vertex as a function of the angle between the chain and the horizontal. Similarly Eq. (4) gives the tension in the chain at a given point as a function of the angle between the chain and the horizontal. Note that as stated before, once the parameter \( a \) is fixed by the choice of \( l \) and \( b \), according to Eq. (4) the tension \( T(\theta) \) scales up or down with the chain weight density \( \lambda g \).

Equations (4) and (5) give the tension and arc length at any point along the chain once we know the parameter \( a \) in terms of the more readily available physical parameters. We can take a step toward this goal by noting that at the end point of the chain, \( s = l \) and \( \theta = \theta_m \). Therefore, by Eq. (5), we have

\[
a = \frac{T_0}{\lambda g} = l / \tan \theta_m. \tag{6}
\]

Equation (6) gives \( a \) in terms of \( l \) and \( \theta_m \). Therefore, \( a \) can be determined for any catenary by measuring \( \theta_m \) when the half-length \( l \) is known. Later we will describe how \( a \) may be obtained in terms of the more readily accessible parameters \( l \) and \( b \).

**Catenary equation in rectangular coordinates**

In rectangular coordinates the chain element \( ds \) is given by

\[
ds = [(dx)^2 + (dy)^2]^{1/2} = dx[1 + (dy/dx)^2]^{1/2}. \tag{7}
\]

In light of Eq. (5), we further have

\[
dy/dx = \tan \theta = s/a. \tag{8}
\]

Therefore, Eq. (7) may now be cast into

\[
ds/dx = [1 + (s/a)^2]^{1/2}, \tag{9}
\]

which in turn leads to

\[
\int dx = a \int ds (a^2 + s^2)^{-1/2}. \tag{10}
\]

Integration of Eq. (10) results in

\[
x + c_1 = a \sinh^{-1} (s/a). \tag{11}
\]

Therefore,

\[
s = a \sinh [(x + c_1)/a]. \tag{12}
\]

Referring to Fig. 1, we note that \( s = 0 \) when \( x = 0 \), which gives \( c_1 = 0 \). Therefore,

\[
s = a \sinh (x/a). \tag{13}
\]

In light of Eq. (8) we now have

\[
s/a = \sinh (x/a) = dy/dx. \tag{14}
\]

Integration of Eq. (14) immediately yields

\[
y + c_2 = a \cosh (x/a). \tag{15}
\]

Again, referring to the coordinates of Fig. 1, we note that \( y(0) = a \), and thus \( c_2 = 0 \), and hence

\[
y = a \cosh (x/a). \tag{16}
\]

Equation (16) is the familiar form of the catenary equation in rectangular coordinates where the \( y \)-axis is the axis of symmetry and the parameter \( a \) is the \( y \)-intercept.

**Discussion**

A little known but intriguing fact is that all catenaries are similar to one another in the same sense that all circles are similar to one another. Though this fact is rather obvious for circles, it is not so in the case of catenaries. The idea becomes clear when we compare the equation for a circle to that for a catenary. In rectangular coordinates, the equation for a circle of radius \( a \) that is centered on the origin is given by

\[
(x^2/a^2) + (y^2/a^2) = 1. \tag{17}
\]

When the radius \( a = 1 \), we have the unit circle

\[
x^2 + y^2 = 1. \tag{18}
\]

Clearly all circles represented by Eq. (17) are similar to the unit circle with \( a \) providing the scale factor. Similarly the catenary equation in rectangular coordinates (Eq. [16]) may be recast in the form

\[
y/a = \cosh (x/a). \tag{19}
\]
In this form, the parameter \(a\) is clearly seen to act as a length scale. Therefore, all catenaries are similar to a unit catenary for which the scale factor \(a = 1\) with the equation

\[
y = \cosh x.
\]  

(20)

Figure 2 shows the catenary graphs when the scale factor is chosen to be 0.5, 1, 2, and 4. The dashed graph for which \(a = 1\) represents Eq. (20), the unit catenary. The upper most graph results when the unit catenary expands by a factor of 4 (\(a = 4\)). The lowest graph results when the unit catenary shrinks by a factor of 2 (\(a = 0.5\)).

Equation (21) gives the relation between the length scale factor \(a\) and the physical parameters of the chain, \(l\) and \(b\). In practice, it is more useful to ask this question differently: for a chain of half-length \(l = 1\), what is the value of the parameter \(a\) for a given \(b\). By choosing \(l\) to be our unit of length we, in effect, divide \(a\) and \(b\) by \(l\), thus constructing a normalized plot of \(b\) vs. \(a\), which would be applicable to all chains.

![Graphs of the catenary equation, \(y = \cosh \left( \frac{x}{a} \right)\), with four values for the \(y\)-intercept \(a\) ranging from 0.5 to 4.0. The dashed graph represents the unit catenary for which the scale factor \(a = 1\). The intercept of each graph gives the value of the scale factor \(a\) for that graph. For example, the upper most graph results when the unit catenary expands by a factor of 4 (\(a = 4\)). The lowest graph results when the dashed graph shrinks by a factor of 2 (\(a = 0.5\)).](image1)

![Right half of a hanging chain with length \(l\), linear density \(\lambda\), and end point distance \(b\) from the \(y\)-axis. The intercept \(a\) is related to \(l\) and \(b\) through the relation \(l = a \sinh \left( \frac{b}{a} \right)\).](image2)

Figure 4 shows the normalized graph of \(b\) vs. \(a\). Since \(b\) is always smaller than \(l\), the fraction \(b/l\) is always between 1 and 0 (see Fig. 1).

Note that when \(b\) is zero, the chain hangs vertically along the \(y\)-axis with \(a = 0\). On the other hand, when \(b/l = 1\), the chain is horizontal with \(a\) tending to infinity. These features are evident in Fig. 4 where there is an asymptote for \(a\) at \(b/l = 1\).

Finally, we note that the scale factor \(a\) is the only parameter that determines the shape of a catenary through the equation \(y = a \cosh \left( \frac{x}{a} \right)\). But according to Eq. (21), \(a\) depends only on the two physical parameters \(l\) and \(b\). Since \(a = \frac{T_0}{\lambda g}\), we conclude that \(T_0/\lambda g\) must remain constant for any hanging chain with fixed \(l\) and \(b\). In other words for a catenary with fixed \(l\) and \(b\), the tension at the vertex changes linearly with \(\lambda g\). Furthermore, since the chain tension is given by \(T = T_0/\cos \theta\), along the chain the tension also changes linearly with \(\lambda g\).

On Earth where \(g\) is a constant, the catenary tension increases proportionately with the chain mass density \(\lambda\), but the shape of the chain does not change if \(l\) and \(b\) are not changed. On the other hand, when the same chain is taken to the Moon, the shape remains the same if \(b\) is unchanged, but the...
tension everywhere along the chain is reduced by about a factor of six since the gravitational acceleration on Moon is 1/6 of that on Earth.

Many of the catenary features discussed above can easily be demonstrated in class. Here are a few specific suggestions:

1) **Two chains of the same length**, one light (plastic) and one heavy (metallic), assume identical shapes when they are held up together with their end points coinciding. Changing the end point separation alters the catenary shape, but the two chains remain identical in form.

2) **A hanging chain** can be made to assume any of the shapes in Fig. 2 by adjusting the end point separation. When the separation is reduced to zero, the chain hangs straight down in a double strand. In this state the scale factor is zero and the vertex point marks the origin of the Cartesian coordinates. By increasing the end point separation of the chain along the horizontal, the vertex rises as the scale factor increases and the shape goes through the stages shown in Fig. 2.

3) **When a chain is held asymmetrically**, each side of the chain is represented by the same catenary equation since the tension at the vertex \( T_0 = a \rho g \) assures that the scale factor \( a \) is identical for both sides. This feature can be demonstrated by starting with a chain held symmetrically (as in Fig. 1) and noting that the right-hand side keeps its shape when the left side of the chain is held at any other point along its length.

References
5. George B. Thomas Jr., *Calculus and Analytic Geometry*, 3rd ed. (Addison-Wesley, 1960), pp. 516–519 and 529–531. In this edition first the catenary problem is broached and a solution is pulled “out of the hat” (pp. 516–519); next the differential equation is solved by separation of variables (pp. 529–531).
6. G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 9th ed. (Addison-Wesley, 1996), pp. 528–529. In this edition the catenary equation is mentioned in “exercises on hyperbolic functions” and the reader is asked to verify the catenary equation as the solution to a given differential equation.