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What Happened When a Mathematician Found Himself Funded by the National Endowment for the Humanities

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ABSTRACT

Exploring the relationships between mathematics and the humanities can be stimulating for both faculty and students. That there are connections should surprise no one in interdisciplinary studies, but finding them might prove to be more difficult; humanists and mathematicians have not been involved in extensive dialogue in recent years. This study provides a field guide to the kinds of connections that one might find with examples of each.

Exploring the relationships between mathematics and the humanities can be stimulating for both faculty and students. That there are connections should surprise no one in interdisciplinary studies, but finding them might prove to be more difficult; humanists and mathematicians have not been involved in extensive dialogue in recent years. Here I will provide a field guide to the kinds of connections that one might find.[1]

First, one can use humanistic tools and ideas to understand mathematics, and mathematicians as well. In mathematics, the best theorems are those of beauty and elegance that contain a surprise. Mathematics research that proves only what is believed to be true is, in general, of less interest than research that proves something unexpected. Mathematicians such as Paul Erdös speak of proofs that are "in the Book"; those proofs that, because of their elegance, simplicity, and clarity, seem to be the most fitting, the most apt proofs of certain results.[2] Mathematicians argue about which basic concepts are the "right" ones. Almost all mathematicians originally chose their field of study because of the joy and beauty they find in the patterns of mathematics.

Almost all mathematics was developed in response to particular problems that arose within an historical context. Topics and questions in this category include: Why did the ancient Greek mathematicians emphasize the need for proofs for mathematical statements? Why didn't Archimedes develop the calculus? One idea in calculus is the infinite; what difficulties did the Greeks have with the notion of infinity? What effect did the mathematics of China and India have on the development of number theory? Why were Arabic numerals introduced in Western Europe in the tenth through thirteenth centuries, and what led to their adoption? What contributed to the development of probability and its acceptance as a tool for making decisions, especially in the twentieth century under such guises as the game theoretic decisions made during World War II?

Recent television shows, plays, and movies have highlighted characteristic and stereotypical traits of mathematicians. *NUMB3RS*, a popular television show currently airing on CBS, depicts a mathematician who consults for the FBI, solving in a few hours (for example, in the time required to rescue a kidnapped child), the kind of applied problems that might be expected to take a mathematician several months to solve. Both he and his female graduate student are portrayed as friendly but socially inept.[3] The dramas, *Arcadia*, by Tom Stoppard,[4] and the Pulitzer prize-winning *Proof*, by David Auburn,[5] depict female mathematicians struggling with societal expectations of the ability of women in mathematics.[6] The 2001 Academy award-winning best picture, *A Beautiful Mind*, is a biography of the Nobel prize-winning mathematician John Nash that graphically depicts his mental illness.[7] Another Academy-award winner, the 1997 film *Good Will Hunting*, depicts a troubled young janitor with surprising mathematical ability.[8] The 1998 thriller *Pi* shows a paranoid mathematician who seeks to use cabalistic mathematics to discover the secret of the world.[9]

A second kind of connection is that mathematics provides a tool for the study of the arts and humanities. Of course, mathematical considerations of proportion and symmetry occur throughout the study of art, with da Vinci's works and Islamic art simply some of the more obvious examples. Steven Brams and Douglas Muzzio have explored game theoretic analyses in religion (considering theological terms and stories from the Old Testament)[10] and political science (presidential elections and decisions made during the Watergate scandal).[11] Some of the conclusions can be counter-intuitive; for example, Brams shows that omniscience can work to God's disadvantage in dealing with reasoning humans.[12] Recently, the 2005 Nobel Prize in Economics was given to Robert Aumann and Thomas Schelling for applications of Game Theory to situations from personal decisions through arms control. Aumann (although primarily a theoretical mathematician) wrote a game-theoretic analysis of a bankruptcy problem in the Talmud.[13]

Examples of mathematics in music include the analysis of musical forms and discussions of systems of tunings both within western music and in ethnomusicology. A statistical analysis of the use of major thirds in the various keys of Bach's *Well-Tempered Clavier* suggests that the tuning of the instrument was likely not equal-tempered.[14]

Many mathematical theories and apparent successes have inspired humanistic and artistic ideas and expressions as well. A number of different mathematical structures have served as guides for other humanistic creations. For example, Euclid's model of truth and certainty as presented in the *Elements* has been significant in philosophy throughout the centuries. Simply one example is Spinoza's presentation of his *Ethics*.

Artists have explored theories of perspective for centuries, using them to organize their material.

Models of non-Euclidean geometries inspired several of M. C. Escher's prints.[15] In a letter to his son Arthur of 9 November 1958, Escher writes, "I've barely finished my sphere spirals (a coloured woodcut) before I'm engrossed again in the study of an illustration which I came across in a publication of the Canadian professor H. S. M. Coxeter, of Ottawa (whom I met in Amsterdam some time ago), *A Symposium on Symmetry*."[16] At the time, Escher was striving to represent infinity; previously he had objects becoming smaller toward the middle. Compare for example Escher's October 1956 wood engraving and woodcut, "Smaller and Smaller," with the November 1958 woodcut, "Circle Limit I," and the December 1959 woodcut, "Circle Limit III."[17] The pattern of "Smaller and Smaller" must continue outwards forever. Escher therefore has failed to capture infinity in the pattern. By contrast, in the "Circle Limit" series, inspired by Coxeter's illustration, the limiting circle around the design itself is the edge of the world. There is nothing beyond.[18] The entire infinite world is captured in the print itself. Escher himself was dissatisfied with "Circle Limit I," for the birds depicted there ran into each other nose to nose. In "Circle Limit III," the creatures show a motion across the plane, each following the other, or as if each track showed one fish caught in stop action at various points in its journey.[19]

Explicit use of golden rectangles provides an example of the use of another kind of geometry by artists. Golden rectangles are distinguished by a version of self-similarity. If a square is removed from a golden rectangle, the remaining rectangle is similar, i.e., has the same shape as the original rectangle; see Figure 1. An important property of any golden rectangle is that the remaining rectangle is itself a golden rectangle.

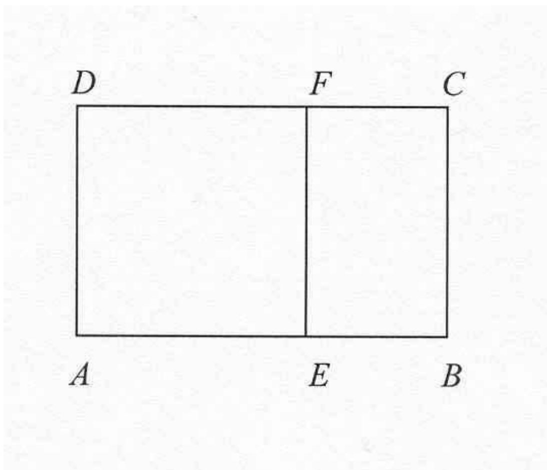

Figure 1. Rectangle ABCD is a *golden rectangle*; if square AEFD is removed, the remaining rectangle EBCF is similar to the original rectangle ABCD. Rectangle EBCF is then itself a golden rectangle.

While experts still argue about which classical artists used golden rectangles to lay out their canvases, the UNI emeritus faculty member John Page (b. 1923, became emeritus professor in 1987) used them explicitly in many of his prints. The first example is *Threshold VI*, a screen print created in 1976, presently on display in the south lobby of the Strayer-Wood Theatre on the UNI campus;[20] see Figure 2. The artist speaks of the "whirling squares" that are provided by the series of golden rectangles that appear in the work.[21]

Figure 2. *Threshold VI*.[22]

A second example is *Chickenhouse*, a color intaglio from 1984, also in the UNI collection; see Figure 3. Geometric objects found in this work include a square, a semicircle, a logarithmic spiral (on top of the vent), and a golden rectangle framing a window of the barn. Page also created an outdoor wood sculpture using golden rectangles for the Cedar Falls home of the Unitarian Universalist Society of Black Hawk County.[23]

Figure 3. *Chickenhouse*.[24]

Examples of mathematical ideas and objects helping composers, such as Arnold Schoenberg and Alban Berg, structure their music are found in twentieth-century serial music. After selecting a tone row of the twelve pitch classes, composers apply temporal and pitch-related versions of geometric transformations to the tone row to obtain the related forms: reflections (giving the retrograde, inversion, and retrograde inversion) as well as translations (pitch transpositions). Bela Bartók made explicit use of the terms of the Fibonacci series in some of his work.[25] Iannis Xenakis used set theory/symbolic logic and stochastic processes to create some very intense, powerful music.[26] Another example of random processes in music is provided by a version of *A Musical Dice Game*, K. 516f, included in Philips' 1991 set of the complete works of Wolfgang Amadeus Mozart. In this recording, Sir Neville Mariner and Erik Smit enact a scene in which they roll the dice, consult a chart, and create a brief minuet for harpsichord.[27]

One more example of the use of mathematics in structuring art is found in the textile patterns created by designer Jhane Barnes (b. 1954).[28] She has made use of several fractal designs, including the Koch curve[29] and the Mandelbrot set,[30] in such designs as "Koch snowflake curve" and "Mandelbrot cloud."

Having seen a number of examples of the use of mathematics to structure humanistic and artistic expression, we now turn to mathematics as content of such expression. Some twentieth-century artists have chosen to depict mathematical ideas explicitly. For example, Max Bill (Swiss, 1908-94), has provided a colorful depiction of the most famous Pythagorean triple, 3-4-5, in his 1980 oil-on-linen work *Konstruktion um das Thema 3:4:5*; see Figure 4.

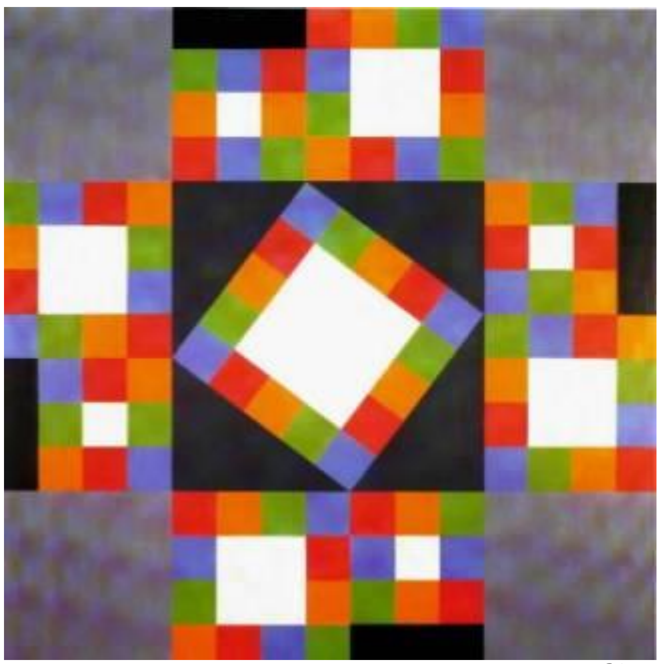

Figure 4. *Konstruktion um das Thema 3:4:5*.[31]

Two other interesting examples of the depiction of mathematical ideas come from Rune Mields (German, b. 1935). His 1979 aquatec-on-linen piece, *Das Lo-shu und seine Spiegelungen*, depicts a 3x3 magic square; see Figure 5. *Lo-shu* is the

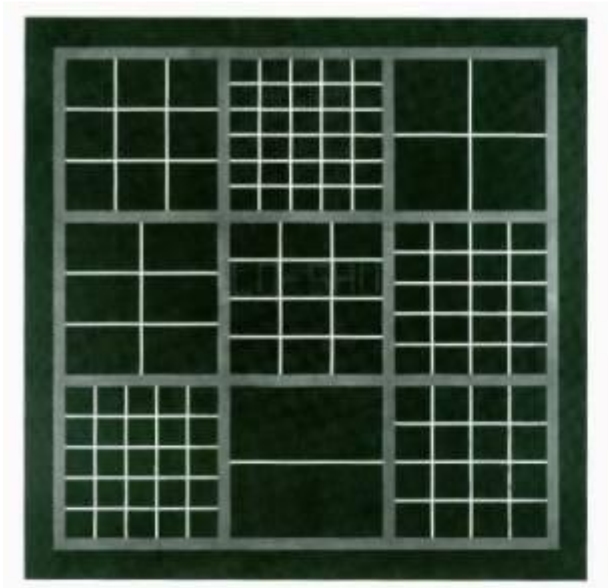

Figure 5. *Das Lo-shu und seine Spiegelungen*.[32]

name given to a legendary Chinese magic square, with stories told that the diagram was found on the back of a turtle by the mythological Emperor Yü around 2100 B.C.E., though it can be traced back to only 400-200 B.C.E by historians.[33] In Mields' piece, count the number of lines in each of the nine cells; the result is shown in Figure 6. In this square, the sum of the numbers in each

4	9	2
3	5	7
8	1	6

Figure 6. 3x3 Magic Square

row, each column, or the two main diagonals is the same, hence the name *magic square*.

A second work of Mields' is *Das Sieb des Eratosthenes III*, from 1977, in India ink on linen; see Figure 7. In this work, Mields has placed a white square in the

Figure 7. *Das Sieb des Eratosthenes III*.[34]

position occupied by a prime number, one whose only whole number factors are itself and one. The three lower panels in Figure 7 all begin with a square representing the number 1 in the upper left hand corner. Contrary to common practice, Mields has chosen to regard 1 as prime, so that you will see that the first three squares are white, corresponding to the primes 1, 2, and 3. There is then a black square for the non-prime number 4, followed by a white square for 5, etc. The difference between the three lower panels is that the first has 89 columns of squares, the second 90, and the third 91. The vertical striping in the center panel is the easiest to explain: the only even prime number is 2, so that white squares occur only in odd numbered columns; because 90 is a multiple of 2, the even numbered squares fall in columns. Because 5 is also a divisor of 90, all multiples of 5 which must themselves be composite, also line up in vertical columns, and hence are colored black. The diagonal striping in the outside panels occurs because 89 and 91 are odd. That the stripes run from upper right to lower left in the third panel, but from upper left to lower right in the first, is a consequence of 7 dividing 91 evenly, but failing to divide 89. The less prominent vertical stripes in the third panel correspond to the necessarily composite multiples of 7, which line up because 7 divides 91 evenly. Finally, the top three panels, instead of beginning with 1, start with much larger, different numbers, in the upper left corner.

The title of the work, "Sieve of Eratosthenes III," refers to a method for compiling a list of prime numbers that is often taught in schools yet today, named for the 3 rd century B.C.E. Alexandrian scholar. Write down the whole numbers beginning with 2. Circle 2, then cross out every second number after 2. Circle 3, then cross out every third number after 3. Circle 5, then cross out every fifth number after 5. Continuing in this way, one will find every prime number circled and every composite number crossed out; see Figure 8. The relationship of this process to Mields' work is clear.

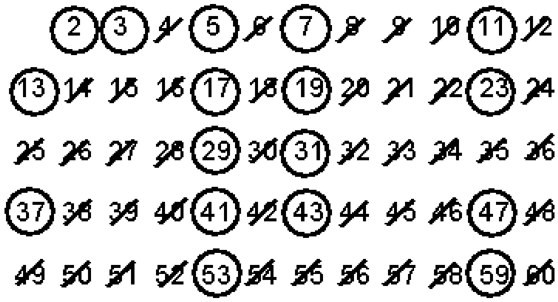

Figure 8. Sieve of Eratosthenes.

In other art forms, sculptures have depicted ideas of continuity and infinity; of course, these concepts are not unique to mathematics. In music, an example of mathematics as content is provided in by Lars Erickson's *Pi Symphony*, based on the digits in the decimal expansion of π.[35]

Finally, mathematical ideas have been used metaphorically, in a broad sense of this term. For example, the deterministic view of the world presented by early eighteenth-century physics and the calculus affected certain social and political views which were to emerge in Europe later in the century. The loss of certainty with the discovery of non-Euclidean geometry, Russell's paradoxes of set theory and logic, and Gödel's results on the incompleteness and undecidability of mathematical theories, all have parallels in our non-mathematical culture.[36] The idea of a fourth spatial dimension and non-Euclidean models of geometry proposed as models of the universe had a great impact on artists in the early twentieth century.[37] For millennia, the famous construction problems of Greek antiquity, to use compass and straightedge to double a cube, trisect an angle, or square a circle, have been icons of impossibility, inspiring, for example, the title of a Tom Stoppard play, *Squaring the Circle*, about a believed-to-be impossible political situation in Poland.[38]

For some lesser known, more specific examples, consider first a work of d'Arcy Hayman (1924-1994). Hayman attended a beginning calculus class taught by Louis Leithold (1924-2005). Though she learned nothing of calculus, Hayman created artistic, m ythological interpretations of terms and phrases that are used in this subject, later collected in *The Calculus Virgin*.[39] As one example, see Figure 9, showing Hayman's interpretation of x *Gets Closer and Closer to One*, which to a mathematician would mean that the difference between x and 1 became negligible. In her own words, Hayman's interpretation depicts "the interrelationship of all things and the direct connections between macro and micro environments."[40]

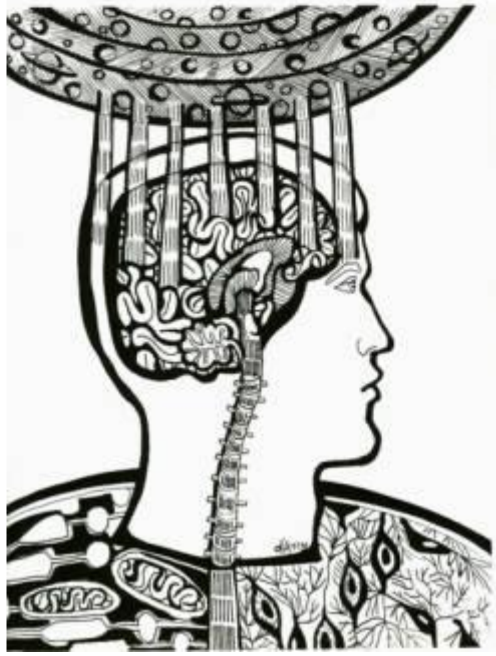

Figure 9. x *Gets Closer and Closer to One*.[41]

Less professional, but more personal, is the anonymous modern algebra student who offered the following sketch after learning of Hayman's work; see Figure 10. To fully appreciate it, one must know a definition from abstract

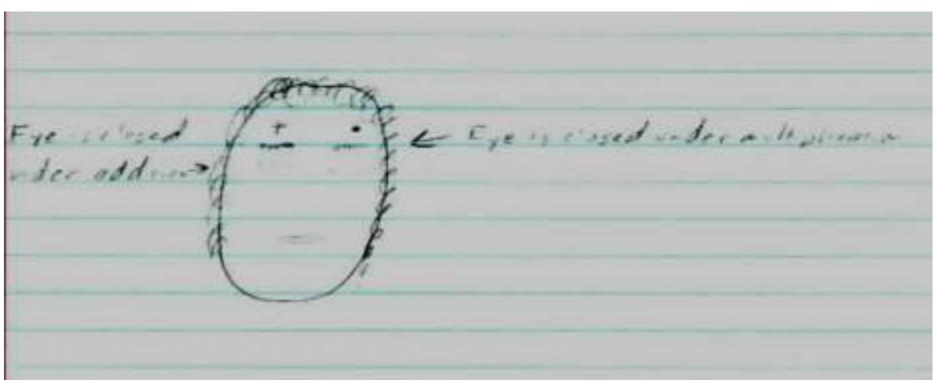

Figure 10. "Eye is closed under addition; eye is closed under multiplication."

algebra: a subset I of a ring R is a *subring* if I is closed under the operations of addition and multiplication. A mathematical object that has become part of our popular culture is the Möbius strip, named for August Möbius (1790-1868), though, as is typical, this object was neither first found nor first published by him; that honor goes to Johann Listing (1808-1882), whose work preceded, but was nearly contemporaneous, with that of Möbius. A *Möbius strip* is a one-edged, one-sided surface, easily constructed by giving a strip of paper a half-twist before taping the ends together. In this form, it provides the structure of a short story, "Frame-Tale," by John Barth (b. 1930), who asks the reader to cut out a strip from the edge of the page which read, " ONCE UPON A TIME THERE " on one side and " WAS A STORY THAT BEGAN " on the other. When the Möbius strip is formed, the story becomes an endlessly recursive "… once upon a time there was a story that began once upon a time there was a story

… "[42]The Möbius strip provides less physical structure to the short story "Moebius Strip" by Julio Cortázar (Argentine, 1914-1984), a tale of rape, murder, and suicide.[43] The qualities of endlessness and eternity suggested by the Möbius strip have also made popular its use in wedding paraphernalia and in jewelry. For example, a wedding invitation designed by a self-proclaimed "geek's geek" shows the Möbius strip; see Figure 11.

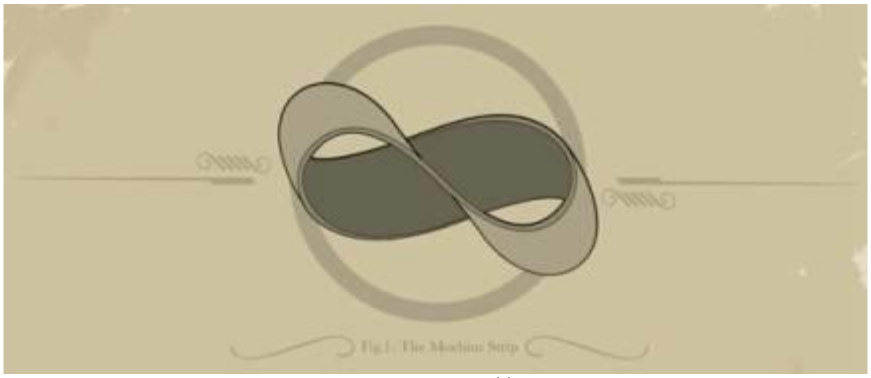

Figure 11.[44]

The jewelry in Figure 12 is an infinity love pendant from Jerusalem; the inscription, a verse from the Song of Songs, "'Ani Ledodi Vedodi Li' - I am to my beloved and he is mine, expresses the infinite love between a man and a woman or between human being and God."[45] Möbius strips also appear inscribed with religious sentiments. An example is Deuteronomy 6:4, " Sh'ma Yisrael Adonai Elohaynu Adonai Echa" ("Hear O Israel the Lord your God is One").[46]

Figure 12.

The last connection I wish to discuss is the impact of the arts and humanities on mathematics. While most people are acquainted with developments in mathematics that arose in response to questions from the sciences, few are aware that some of the problems that stimulated mathematics arose in the arts and humanities. For example, the discovery of perspective in painting in the fifteenth and sixteenth centuries inspired the development of projective geometry; one of the first texts in this area was written by the

German artist Albrecht Dürer (1471-1528). Studies of mathematical symmetry were motivated by repetitive patterns in both crystallography and the art of various cultures. Questions arising in ancient African and Indian religions led to developments in both counting and geometry.[47] As a personal case study, I was led to questions in combinatorics and modern algebra by the rhythmic pattern of *Clapping Music* (1972) by Steve Reich (b. 1936).[48] This piece, for two musicians clapping, is the last of Reich's gradual process pieces, and was written "out of a desire to create a piece of music needing no instruments at all beyond the human body."[49] He wanted it to be similar to the gradual phasing process, i.e., shifting a part slightly in time against another, that he had used in the tape-piece *It's Gonna Rain* (1965),[50] but felt that this was overly complex for such simple instrumentation as clapping. Reich instead introduced a discrete version of phasing: shifting the contents of an entire unit of time, one beat of the piece, at a time. *Clapping Music* is based on a rhythmic pattern of 8 claps within a 12-beat measure, represented in Figure 13. One of the musicians will clap this pattern

Figure 13. Original pattern of *Clapping Music*.

throughout the piece. The second musician begins clapping this pattern, then at some point changes to a second pattern, the first variant, shown in Figure 14,

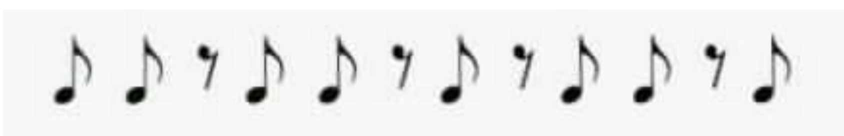

Figure 14. Second pattern (first variant) of *Clapping Music*.

which is derived from the original pattern by moving the content of the first beat to the twelfth position. After some number of repetitions of this pattern, the second musician will move on to the next, second variant, shown in Figure 15,

Figure 15. Third pattern (second variant) of *Clapping Music*.

obtained once again by moving the content of the first beat to the twelfth position. The piece comes to an end after the second musician has preformed repetitions of all twelve patterns (the original and eleven variants) that arise in this manner, followed by the two musicians again clapping in unison.

A mathematical description of the piece would discuss the permutation of the content of the twelve beats of the original pattern that moves the content of the first position to the twelfth position. Applying this action twelve times ultimately returns the pattern to its original form, passing through all the variations. This permutation is an element of the group of all possible permutations of twelve objects, called *S* 12, the *symmetric group on 12 symbols*. A permutation that mathematicians single out of this group is the *identity* permutation, which moves nothing. The *order* of the permutation that Reich has employed is 12, as it takes 12 repetitions of the permutation before one would return to the identity permutation. Could there be a more complicated piece of music created from Reich's initial pattern, using a permutation of twelve objects that had a higher order than 12? Certainly. For example, if one were to cyclically permute the content of the first five beats and at the same time the remaining seven beats, the content of the first five beats would repeat after 5, 10, 15, or any multiple of 5 repetitions, while the content of the remaining seven beats would repeat after 7, 14, 21, or any multiple of 7 repetitions. Hence, it would take 35 repetitions before the original pattern reappeared. Thus, we have located an element of order 35 in *S* 12. Is this the best we can do, i.e., is 35 the largest order of an element in *S* 12? No. The largest order in fact is 60, obtained by breaking the pattern into three parts, say the first four beats, the next three beats, and the final five beats, allowing each of these parts to cycle among itself. The resulting permutation has as its order the least common multiple of 3, 4, and 5, namely 60. The action of this order-60 permutation on Reich's original pattern is presented below in Figure 16, which shows the original pattern, the first variant, and the second variant. Peculiarly, the first application of the order-60 permutation leads to the same first variant as did

Figure 16. Original pattern and first and second variants of *Variation on "Clapping Music."*

Reich's order-12 permutation. The thirtieth variant and the original pattern, simply for comparison purposes, appear in Figure 17. Notice how closely the thirtieth variant matches the original pattern. Of course, because 30 is divisible by both 3 and 5, the content of beats 5 through 12 of the thirtieth variant matches

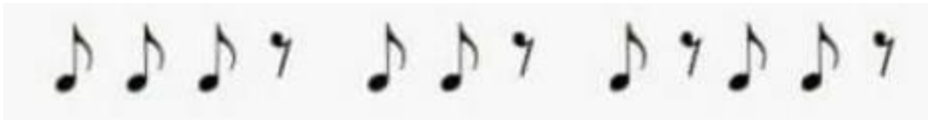

Figure 17. Thirtieth variant and original pattern of *Variation on "Clapping Music."*

those of the original pattern. The fifty-ninth variant together with the original pattern appear in Figure 18. Notice that one application of the order-60 permutation to the fifty-ninth variant will return the original pattern.

Figure 18. Fifty-ninth variant and original pattern of *Variation on "Clapping Music."*

The resulting piece of music, this *Variation on "Clapping Music,"* may be heard at http://www.math.uni.edu/~haack/. To be able to hear the difference between the content of the first four beats, the next three beats, and the final five beats, instrumentation other than clapping is required. Here, piano sounds generated on a Roland W-30 Music Workstation were used, with the role of the first musician clapping being played on middle C, and the second musician's role played by pitches a fourth, a third, and a fifth above middle C, respectively. A comment from a student who heard the piece at Clarke College was, "Easy to dance to, but not much melody," an accurate summary of its musical value. But hearing the mathematics is interesting nonetheless.

Another mathematical question then arises from our work. What is the maximal order of an element in S_n, the symmetric group on n symbols, where n is a whole number? We would expect that this maximal order would increase as n increases, but, in each of S_{12} and S_{13}, the maximal order is 60, a surprise. (To see this, note that, for example, 12 = 4+3+5 and the least common multiple of 4, 3, and 5 is 60, while 13 = 4+3+5+1 and the least common multiple of 4, 3, 5, and 1 is also 60. Some experimentation will show that no other decomposition of 13 into summands will yield a larger least common multiple.) Why should it be a surprise that the maximal order did not increase? One reason is that S_{12} has 479,001,600 elements while S_{13} has 6,227,020,800 elements, so that we would expect one of those in S_{13} to have larger order. The result of a search of the mathematical literature leads to the following information about this problem. The maximal order of an element of S_n is asymptotic to $e^{\sqrt{n \log n}}$, so that it does increase without bound.[51] But the behavior we saw with S_{12} and S_{13} occurs infinitely often, in fact, there are arbitrarily long intervals of n on which the maximal order of an element of S_n is constant.[52]

This ends our field guide to connections between mathematics and the arts and humanities. This model could also effectively be used to explore connections between other disciplines as well, of course. These connections provide a rewarding way to begin to bridge the gap between C. P. Snow's "two cultures," the culture of science and the culture of the arts and humanities, discussed in his pivotal 1959 lecture.[53] Finally, I must explain the title of the article. My explorations in this area began with two grants to Oklahoma State University by the National Endowment for the Humanities that funded two-week summer camps for gifted and talented high school students to study the medieval world and ancient Greece. I had the privilege of developing and delivering course content in mathematics that fit these eras and that had connection to the arts and humanities. Reflecting on how I could do this provided the basis for this field guide for exploring the connections between mathematics and the arts and humanities.

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[1] Acknowledgment. I thank Syed Kirmani and Eugene Wallingford of the Graduate Faculty, and Dean Susan Koch and Associate Dean Sue Joslyn of the Graduate College, at the University of Northern Iowa for the invitation to present this material as part of the Brown Bag Lecture Series.

[2] See, for example, Martin Aigner, et al.'s book *Proofs from THE BOOK*, (New York, 2003).

[3] Mathematicians have created several web sites containing explanations of the mathematical content from the episodes of *NUMB3RS*; examples include http://www.cbs.com/primetime/numb3rs/ti/activities.shtml and http://www.atsweb.neu.edu/math/cp/blog/. Both sites Accessed 25 January 2006.

[4] Tom Stoppard, *Arcadia* (London, 1993). The University of Iowa Theatre Arts Department mounted an excellent production of *Arcadia* in 1997.

[5] David Auburn, *Proof* (New York, 2001). Local productions include those by the Riverside Theatre in Iowa City (2002) and by the Waterloo Community Playhouse (2004). Of course, in 2005 it was made into a motion picture; see http://www.imdb.com/title/tt0377107/. Accessed 25 January 2006.

[6] A video containing an explanation of some of the mathematics in *Arcadia* and an interview with Tom Stoppard is available from the Mathematical Sciences Research Institute: *Mathematics in Arcadia: Tom Stoppard in Conversation with Robert Osserman, February 19, 1999*. A web site discussing the mathematics of *Arcadia* has been prepared by one of the leading expositors of chaos theory, Robert L. Devaney, available at http://math.bu.edu/DYSYS/arcadia/. Accessed 25 January 2006.

[7] For details, see http://www.imdb.com/title/tt0268978/. Accessed 25 January 2006.

[8] For details, see http://www.imdb.com/title/tt0119217/. Accessed 25 January 2006.

[9] For details, see http://www.imdb.com/title/tt0138704/. Accessed 25 January 2006.

[10] Steven J. Brams, *Superior Beings: If They Exist, How Would We Know: Game-Theoretic Implications on Omniscience, Omnipotence, Immortality and Comprehensibility* (New York, 1983) and Steven J. Brams, *Biblical Games: Game Theory and the Hebrew Bible* (Cambridge, MA, 1980).

[11] Steven J. Brams, *The Presidential Election Game* (New Haven, 1978) and Douglas Muzzio, *Watergate Games: Strategies, Choices, Outcomes* (New York, 1982).

[12] Brams, *Superior Beings*, 69-71.

[13] Robert Aumann and Michael Maschler, "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud," *Journal of Economic Theory* 36 (August 1985), 195-213.

[14] John Barnes, "Bach's Keyboard Temperament: Internal Evidence from the *Well-Tempered Clavier*," *Early Music* 7 (1979), 236-249.

[15] The non-Euclidean geometry that Escher found inspirational is the Poincaré disk model of hyperbolic geometry, in which there are infinitely many lines parallel to a given line through any point not on the line. A brief explanation of the model can be found in George E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane* (New York, 1975), 59, where it is called the Poincaré Incidence Plane.

[16] As translated in F. H. Bool et al., *M. C. Escher: His Life and Complete Graphic Work* (New York, 1981), 91. A similar illustration can be found in H. S. M. Coxeter, *Introduction to Geometry*, 2 nd ed. (New York, 1989), 285.

[17] For "Smaller and Smaller," see http://www.tessellations.org/eschergallery22.htm. For "Circle Limit I," see http://www.dartmouth.edu/~matc/math5.pattern/lesson7math.html. For "Circle Limit III," see http://www.tessellations.org/eschergallery24.htm. All sites accessed 25 January 2006.

[18] In another letter to his son Arthur, Escher writes, "My woodcut ["Circle Limit I"] inspired by the Coxeter system is finished, and I consider it to be the finest of the 'reductions' that I have made. The circular limitation of infinitely small motifs, surrounding the whole thing logically and inevitably, is something I cannot take my eyes off." [From Bool et al., 92.]

[19] M. C. Escher, *Escher on Escher: Exploring the Infinite* (New York, 1989), 125-127.

[20] Viewed 30 January 2006.

[21] *John Page: A Retrospective Exhibition in Three Parts*, exhibition catalogue (Cedar Falls and Waterloo, 1992), Plate 15.

[22] *John Page: A Retrospective Exhibition in Three Parts*, Plate 15.

[23] Thomas Thompson, "Fifty Years of Art," *John Page: A Retrospective Exhibition in Three Parts*.

[24] *John Page: A Retrospective Exhibition in Three Parts*, Plate 25.

25 Erno Lendvai, *Bela Bartók, An Analysis of His Music* (London, 1971). The Fibonacci series, which appears first as a solution to a problem in Leonardo Pisano's (Fibonacci's) *Liber Abaci* of 1202, is defined recursively with each term the sum of the preceding two, beginning with 1, 1, and hence continuing 2, 3, 5, 8, 13, etc. A translation by Lawrence Sigler of the *Liber Abaci* was published in 2002 by Springer-Verlag.

26 A powerful example is Xenakis's 1960-61 solo piano piece *Herma*, recorded on *Xenakis: Complete Works for Solo Piano*, Aki Takahashi (piano), compact disc, mode, 1999.

27 Wolfgang Amadeus Mozart, "A Musical Dice Game," *Rarities & Surprises, Complete Mozart Edition*, compact disc, Phillips, 1991.

28 Her most recent designs can be found at http://www.jhanebarnes.com/1/index.php. Accessed 25 January 2006.

29 The Koch curve is described well at http://www.jimloy.com/fractals/koch.htm. Accessed 25 January 2006. Helge von Koch (1870-1924) first described this object in 1904 in order to give a geometrically intuitive example of a nowhere differentiable curve; see his "On a continuous curve without tangents constructible from elementary geometry," *Classics on Fractals*, ed. Gerald A. Edgar (Boulder, CO, 2004), 25-45.

30 For pictures of the Mandelbrot set, see http://aleph0.clarku.edu/~djoyce/julia/explorer.html. Accessed 25 January 2006. Benoit Mandelbrot first saw a computer-generated drawing of the set which came to be called the Mandelbrot set in 1980 [Benoit Mandelbrot, *Fractals and Chaos: The Mandelbrot Set and Beyond* (New York, 2004), 13].

31 Dietmar Guderian, *Mathematik in der Kunst der letzten dreißig Jahre: Von der magischen Zahl über das endlose Band zum Computerprogramm* (Paris, 1990), 117.

32 Guderian, 65.

33 Frank Swetz, "The Evolution of Mathematics in Ancient China," Frank Swetz, ed., *From Five Fingers to Infinity: A Journey through the History of Mathematics* (Chicago, 1994), 319.

34 Guderian, 47.

35 See http://www.pisymphony.com/gpage.html. Accessed 25 January 2006.

36 A good discussion of these mathematical concepts in this context is given by Morris Kline in *Mathematics: The Loss of Certainty* (New York, 1980).

37 Linda Dalrymple Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* (Princeton, 1983).

38 Tom Stoppard, *Squaring the Circle* (London, 1985).

[39] D'Arcy Hayman and Louis Leithold, *The Calculus Virgin: An Artist's View of the Language of Calculus* (Santa Monica, CA, 1992).

[40] Hayman and Leithold, 24.

[41] Hayman and Leithold, 25.

[42] John Barth, "Frame-Tale," *Lost in the Funhouse: Fiction for Print, Tape, Live Voice* (New York, 1968), 1-2.

[43] Julio Cortázar, "Moebius Strip," *We Love Glenda So Much and A Change of Light* (New York, 1984), 129-145.

[44] From http://www.baseboard.net/forums/showthread/t-11265.html. Accessed 25 January 2006.

[45] From http://www.fine-jewelry-from-jerusalem.com/store/merchant.mv?Screen=PROD&Store_Code=P&Product_Code=IMP-1&Category_Code=13. Accessed 25 January 2006.

[46] See, for example, http://www.christianforums.com/u12572. Accessed 14 October 2005.

[47] See, for example, Abraham Seidenberg, "The Ritual Origin of Counting," *Archive for History of Exact Sciences* 2(1962), 1-40 and George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics* (London, 1992), 225-233.

[48] One recording is Steve Reich, "Clapping Music," *An American Collection*, The Sixteen under Harry Christophers, compact disc, Collins, 1992.

[49] Steve Reich, *Writings about Music* (Halifax, NS, 1974), 64.

[50] Steve Reich, *Early Music*, compact disc, Nonesuch, 1992.

[51] Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, (Leipzig and Berlin, 1909).

[52] Jean-Louis Nicolas, "Ordre maximal d'un élément du groupe S_n des permutations et <<highly composite numbers>>," *Bulletin de la Société Mathématique de France* 97 (1969), 129-191.

[53] C. P. Snow, *The Two Cultures and the Scientific Revolution* (Cambridge, 1959).