Selected problems involving the probability of ruin for an insurance company

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Abstract

In the actuarial science literature, an insurance company is said to be ruined if, at some time $t > 0$, the aggregate claims up to time t exceed the sum of the initial surplus and the total premium collected up to time t. The calculation and/or estimation of the probability of ruin is of fundamental importance. In this paper we use computer simulation to estimate the probabilities of ruin over a finite horizon of time when the aggregate claims process is Compound Poisson and the distribution of the claim sizes is: (i) Weibull, and (ii) exponential with a random parameter. The obtained estimates are compared with the case of exponential claim sizes and it is found that, in all but the increasing failure rate Weibull case, the probability of ruin, assuming exponential claims, overestimates the actual probability of ruin. Thus, one should be extremely careful in using the exponential claim formula uncritically.

Key Words and Phrases: Adjustment coefficient, Classical model of risk theory, Compound Poisson process, Gamma distribution, Pareto distribution, Probability of ruin, Relative security loading, Weibull distribution

1. Introduction

1.1 Definition of the probability of ruin

The probability of ruin is a concept used to describe the risk an insurance company faces. The risk consists of the possibility that the company's initial surplus and premium income may prove insufficient to meet the cumulative claims. In order to understand the concept more fully, the following definitions and notations are necessary (See Bowers, et al., 1986, 345-347). Let

t = point in time

c = rate at which premiums are received (hitherto, assumed constant over time)

u = initial surplus of the insurer

S(t) = aggregate claims up to time t

U(t) = surplus of the insurer at time t

U(t) is defined as:

$$U(t) = u + ct - S(t) \qquad , t \geq 0$$

Figure 1 gives a graphical depiction of U(t) when the successive claims arrive at time points $T_1, T_2, \ldots$ The graph of U(t) is increasing up until T_1. At this point a claim is filed by a client, and some of the surplus is removed to account for this. Similarly, claims are also filed at times T_2 through T_4. Note what occurs at T_4. At this point, the surplus drops below zero. It is at this point that we say the insurer has been ruined, since adequate funds no longer exist with which to pay premiums.

We refer to this time of ruin as T, and are interested in the probability of the event '$T < \infty$'. More precisely,

$$T = \begin{cases} 0 < t < \infty & \text{, if } U(t) < 0 \text{ for some } t > 0 \\ +\infty & \text{, if } U(t) \geq 0 \text{ for all } t > 0 \end{cases}$$

The probability of ruin is then defined as:

$$\psi(u) = \Pr(T < \infty)$$

The notation of $\psi(u)$ emphasizes that the probability of ruin depends on the initial surplus u, as well.

1.2 Classical model of risk theory

The probability of ruin depends on the process of aggregate claims as well as on u and c. The classical model of risk theory assumes that the process $\{S(t), t \geq 0\}$ is a

Compound Poisson process with $S(t) = X_1 + \ldots + X_{N(t)}$, where $\{N(t), t \geq 0\}$ is a Poisson Process with rate λ and $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables. N(t) denotes the number of claims made by time t, while X_i denotes the size of the ith claim. We refer to Bowers, et al. (1986) for the necessary definitions and other details.

It is easy to see that $E[S(t)] = \lambda t E(X)$ when X denotes the magnitude of a random claim against the insurer. For later use, we note that the arrival of claims follows a Poisson process of rate λ if and only if the time between claims follows an exponential distribution with mean $1/\lambda$. This relationship between the Poisson distribution and the exponential distribution is useful for simulating the times at which claims arrive.

1.3 Probability of ruin for the classical model

It is well known (See Bowers, et al., 1986, 352-353) that, for the classical model described above; the probability of ruin is given by

$$\psi(u) = \frac{\exp\{-Ru\}}{E[\exp\{-RU(T)\} \mid T < \infty]}.$$

where the "adjustment coefficient" R satisfies the equation $\lambda + cR = \lambda M_X(R)$. Here $M_X(R)$ denotes $E[\exp(RX)]$. The denominator in the above expression for $\psi(u)$ is calculated with respect to the distribution of negative surplus, given that ruin occurs. However, such an analysis of $\psi(u)$ is not practical since $E[\exp\{-RU(T)\} \mid T < \infty]$ cannot be evaluated explicitly (except for the case of exponential claims). Since the latter must exceed 1, we find that $\psi(u) < \exp\{-Ru\}$. This relationship is commonly referred to as Lundberg's inequality.

1.4 Relative security loading and rate of premium collection

The relative security loading (See Bowers, et al., 1986, 37) specifies the extra amount of premium that an insurer must collect from policy holders in order to make a profit while still paying the numerous claims filed against them. For obvious reasons, simply collecting premiums that are equal to the expected amount of claims will not yield a profit or generate surplus funds for the insurer. Thus, the premium amount collected must exceed the expected aggregate claims. Recall that the premium amount collected up to time t is ct. Expected aggregate claims in a Poisson process may be stated as $\lambda tE(X)$. We then define the relative security loading, θ, as follows:

$$\theta = \frac{\text{premium collected - expected aggregate claims}}{\text{expected aggregate claims}}$$

Or symbolically as:

$$\theta = \frac{ct - \lambda tE(X)}{\lambda tE(X)}$$

This expression can then be rewritten as follows:

$$c = \lambda E(X)(1 + \theta)$$

1.5 Probability of ruin for exponential claim sizes

The probability of ruin is mathematically tractable when X, the magnitude of a randomly selected claim, has the exponential distribution. The probability density function for the exponential distribution is as follows:

$$f(x) = \beta \exp\{-\beta x\} \quad , x > 0 \; ; \beta > 0$$

In this case the expression:

$$E[\exp\{-RU(T)\} \mid T < \infty]$$

reduces to:

$$\frac{\beta}{\beta - R}$$

and the probability of ruin is found to be:

$$\psi(u) = \frac{(\beta - R)\exp\{-Ru\}}{\beta}$$

Further analysis of the exponential distribution shows that R may be written in terms of β and θ (the relative security loading) as:

$$R = \frac{\theta\beta}{1+\theta}$$

Substituting this value of R into the previous equation for probability of ruin yields:

$$\psi(u) = \frac{\exp\{-\theta u / (1+\theta)E(X)\}}{1+\theta}$$

This equation can now directly be used to compute the probability of ruin when claims follow an exponential distribution with mean $1/\beta$. It should be noted, however, that the above is the probability of ruin over an infinite time horizon.

1.6 Simulation of finite time horizon probability of ruin for exponential claim sizes

Probability of ruin is the probability that ruin will occur sometime in the interval $(0, \infty)$. However, one may be interested in the probability that ruin will occur sometime in the finite interval $(0, t]$. The objective of this subsection is to indicate how simulation may be used to estimate:

$\psi(u, t)$ = the probability that ruin will occur sometime in the interval $(0, t]$

The simulation entails generating values from an exponential distribution with a given mean, and then using these values as data for determining an insurer's surplus. As was mentioned earlier, when claim times follow a Poisson process, the time in between claims follows an exponential distribution with parameter λ. Thus, we can simulate these claim times by generating a set of these numbers with an application such as MINITAB. In a similar fashion, the claim amounts can also be simulated since they follow an exponential distribution with parameter β.

In addition to simulating times and claims with the MINITAB software, we have written a short C program to analyze the data (See Appendix A). Observe that the program reads in the data generated by MINITAB, and then uses the values to simulate the probability of ruin. This is accomplished by computing the insurers' surplus as time progresses, and continually checking to see if ruin has occurred or not. Each simulation involved running 1000 separate trials to test for ruin occurring. These results were continually printed to the screen for analysis as the program ran.

For the purpose of simulation, the following assumptions were made:

u = initial surplus of insurer = 100

c = rate of premium collection = 1.1

$1/\lambda$ = mean time between successive claims = 5

Initially, $\theta = 0.15$, $1/\beta = 4.78$, and the time horizon (0, t] = (0, 500] was considered. The computer read the data and analyzed whether or not ruin occurred in 1000 separate trials (Note: 1000 trials were used for each separate simulation). In 28 of these trials, ruin occurred. Thus the simulated probability of ruin was 28/1000 = 0.028. This value differed from the theoretical value of 0.057 by quite a large margin.

The simulation was repeated for various choices of θ, β, and (0, t] = (0, 700]. Appendix B shows the results of the eight simulations run, and the various values of θ that were used. The last three simulations, where $\theta = 0.2$ and $1/\beta = 4.58$ yielded a probability of ruin very close to the infinite time horizon theoretical value

The objective of this paper is to estimate $\psi(u, t)$, which denotes the probability of ruin over the finite time horizon 0 to t, for selected t when (i) the claim size distribution is Weibull and (ii) the claim size distribution is exponential with parameter β, but β itself is the observed value of a Gamma random variable.

2. Probability of ruin for Weibull claim sizes

2.1 The Weibull distribution

The Weibull distribution's probability density function (See Leemis, 1995, 88-89) may be stated as follows:

$$f(x) = k(1/\alpha)^k x^{k-1} \exp\{-(x/\alpha)^k\}$$

Here, k and α are the parameters of the Weibull distribution that will adjust the form of the distribution depending on their values. The quantity $k(1/\alpha)^k x^{k-1}$ in the above distribution is called the failure rate of the Weibull distribution. If $k > 1$, it is increasing and if $0 < k < 1$, it is decreasing. The expected value of the Weibull distribution is

$$\mu = (\alpha/k)[\Gamma(1/k)]$$

where $\Gamma(x)$ is the well known gamma function. The values of $\Gamma(x)$ may be found with the aid of several readily available mathematical tables (See Jeffrey, 1995, 222-224).

2.2 Choosing the Weibull Distribution

Modeling the probability of ruin for the Weibull distribution was chosen for two specific reasons. First, the theoretical probability of ruin when claim sizes follow a Weibull distribution is not tractable mathematically. Recall that the relationship between claim size and claim time is:

$$\lambda + cR = \lambda M_X(R)$$

In the case when claims follow a Weibull distribution, $M_X(R)$ cannot be calculated. Thus, simulation is the only tool available for computing the probability of ruin.

The second reason for examining the Weibull distribution is its close relationship to the exponential distribution. Indeed, the exponential distribution is simply a special case of the Weibull distribution with k=1. If insurers are assuming that their claim sizes follow an exponential distribution, then they are also assuming that claim sizes follow a Weibull distribution with k=1 since these are equivalent. The interest here is in overestimating or underestimating the probability of ruin. What happens if an insurer assumes exponential claim sizes, but claims actually follow a Weibull distribution with k=0.5 or k=5? Would the insurer be making the grave error of underestimating probability of ruin and not have enough surplus on hand? The opposite could also be true if the insurer is overestimating the probability of ruin. In that case, an excess amount of surplus would be accumulated that is not altogether necessary.

2.3 Simulating $\psi(u)$ when claim sizes follow a Weibull distribution

The next step in modeling the probability of ruin for the Weibull case is to take the mean used for our exponential claim size (4.58), and set this value equal to the mean of the Weibull distribution ($(\alpha/k)[\Gamma(1/k)]$) over the time horizon (0, t] = (0, 700]. Here the values of k and α are unknown. Since our primary interest lies in evaluating the probability of ruin as k varies, we have selected various values of k to use, and then computed α. These k values range from 0.2 to 5.

Now that α and k have been computed, we can use MINITAB to once again generate random values, this time from the Weibull distribution. For each separate k value, three separate simulations were run. The results are displayed in Appendix C. For these simulations, we are still assuming that the time between individual claims follows an exponential distribution with a mean of 5. The only change is in the claim size distribution.

2.4 Conclusions

As can be seen in Appendix C, our simulations reveal that when claims follow a Weibull distribution and k is less than 1, the probability of ruin is greatly inflated. As k increases in value, the probability of ruin rapidly decreases in value. When k reaches a value of 3 or greater, the probability of ruin is virtually zero.

By averaging the simulated probability of ruin at each value of k, we now have a better estimate of the probability of ruin for each k value. These values have been graphed in Figure 2 to show the decreasing trend of the probability of ruin as k increases.

So what does this mean to insurers? Since the probability of ruin is so much greater when k is less than 1, we conclude that an insurer who assumes that claims are following an exponential distribution (k=1) would be grossly underestimating the probability of ruin if claims actually did follow a Weibull distribution and k was actually less than 1. Conversely, the insurer is greatly overestimating the probability of ruin if the claims follow a Weibull distribution and k is greater than 1. The danger for the insurer here is if k is less than 1. Our results suggest that the insurer should take great caution in representing claims with an exponential distribution.

3. Probability of ruin for exponential claims with a random parameter

We will now assume that the claim sizes $X_1, X_2, \ldots, X_n$ are i.i.d. exponential with probability density function:

$$f(x) = \beta\exp\{-\beta x\} \quad , \ x > 0$$

where the parameter β is the observed value of a random variable B having the Gamma density function:

$$f_B(\beta) = (\lambda^{\alpha}/\gamma(\alpha))(\beta^{\alpha-1})\exp\{-\lambda\beta\} \quad , \beta > 0$$

This scenario is realistic when the value of β is not completely known, but $f_B(\beta)$ represents our prior partial knowledge about β.

3.1 Unconditional distribution of claim sizes

The final section of our analysis lies in once again simulating the probability of ruin with the previous computer program. This time we are again assuming that the time between claims follows an exponential distribution with a mean of 5. In addition, our claim sizes are once again following an exponential distribution, but now some information is known about the parameter (β) of the exponential claim distribution.

Rather than having this parameter remain constant, we are now interested in observing what occurs when β is constantly changing. In particular, we are assuming that β now follows a Gamma distribution with parameters α and λ. This sort of information about β is referred to as the conjugate prior.

The probability density function of the Gamma distribution is as follows:

$(\lambda^{\alpha}/\Gamma(\alpha))(\beta^{\alpha-1})\exp\{-\lambda\beta\} \quad , \beta > 0$

Further, this distribution has a mean of α/λ. We have chosen to model β with a Gamma distribution because this particular distribution is very flexible. What this means is that by modifying α and λ appropriately, it is relatively simple to model a variety of shapes.

It can b shown that if claim sizes are following an exponential distribution with mean $1/\beta$ and β is following a Gamma distribution with parameters α, λ then the claim size has the probability density function:

$$f(x) = \frac{\alpha(\lambda^\wedge\alpha)}{(\lambda+x)^{\alpha+1}} \quad , x > 0$$

This effectively means that our claims are now following a Pareto distribution with a mean of $\lambda/(\alpha-1)$ for $\alpha>1$.

3.2 Simulating claim size from the Pareto distribution

We now go about simulating the probability of ruin as before. In this case we are interested in setting our new mean of $\lambda/(\alpha-1)$ equal to 4.58. Ideally a wide range of values for α would be considered here. We have, however, limited our analysis to the case when $\alpha=2$ and hence $\lambda=4.58$. These are the parameters for our Pareto claim size distribution.

The next step is to simulate claim sizes from the Pareto distribution using MINITAB. Since MINITAB does not have a way of simulating values from the Pareto distribution, values were generated from a Uniform(0,1) distribution and then a simple transformation was performed on them so that they followed a Pareto distribution with $\alpha=2$ and $\lambda=4.58$ (See Appendix D for an explanation of this transformation). Our claim times were then generated from, once again, an exponential distribution with a mean of 5.

Finally, three separate simulations were run for these claim sizes and times. The following probabilities of ruin were observed:

$\psi(u) = 0.187 , 0.184 , 0.186$

3.3 Conclusions

These probabilities of ruin are once again extremely high and hence extremely risky for this particular case. As α and λ are modified in the Pareto claim distribution, we would expect these values to be altered slightly. However, we conclude that the conjugate prior information may have the effect of once again inflating the probability of ruin. Insurers need to be cautious of this particular situation and take steps to reduce the high risk of ruin that was observed in our simulations.

References

Bowers, N. L., et al. (1986) *Actuarial Mathematics.* The Society of Actuaries. Itasca, Illinois.

Jeffrey, A. (1995) *Handbook of Mathematical Formulas and Integrals.* Academic Press. San Diego, California.

Klugman, S. A. (1992) *Bayesian Statistics in Actuarial Science.* Kluwer Academic Publishers. Norwell, Massachusetts.

Leemis, L. M. (1995) *Reliability.* Prentice-Hall. Englewood Cliffs, New Jersey.

Minitab, Inc. (1992) *Minitab Reference Manual, Release 9.* Sowers Printing. Lebanon, Pennsylvania.

Press, S. J. (1989) *Bayesian Statistics: Principles, Models, and Applications.* John Wiley & Sons. New York, New York.

Figure 1

Graphical Depiction of U(t)

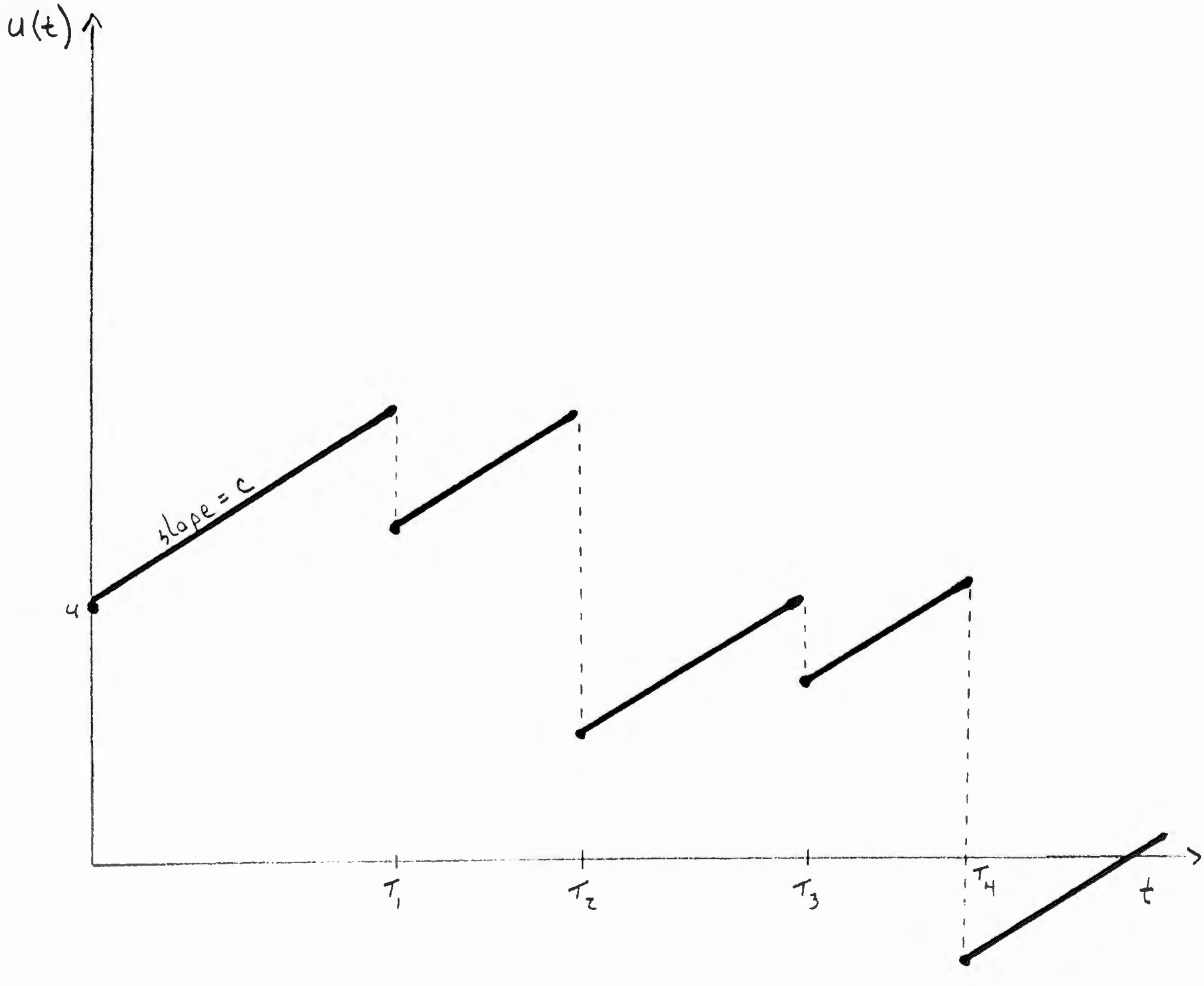
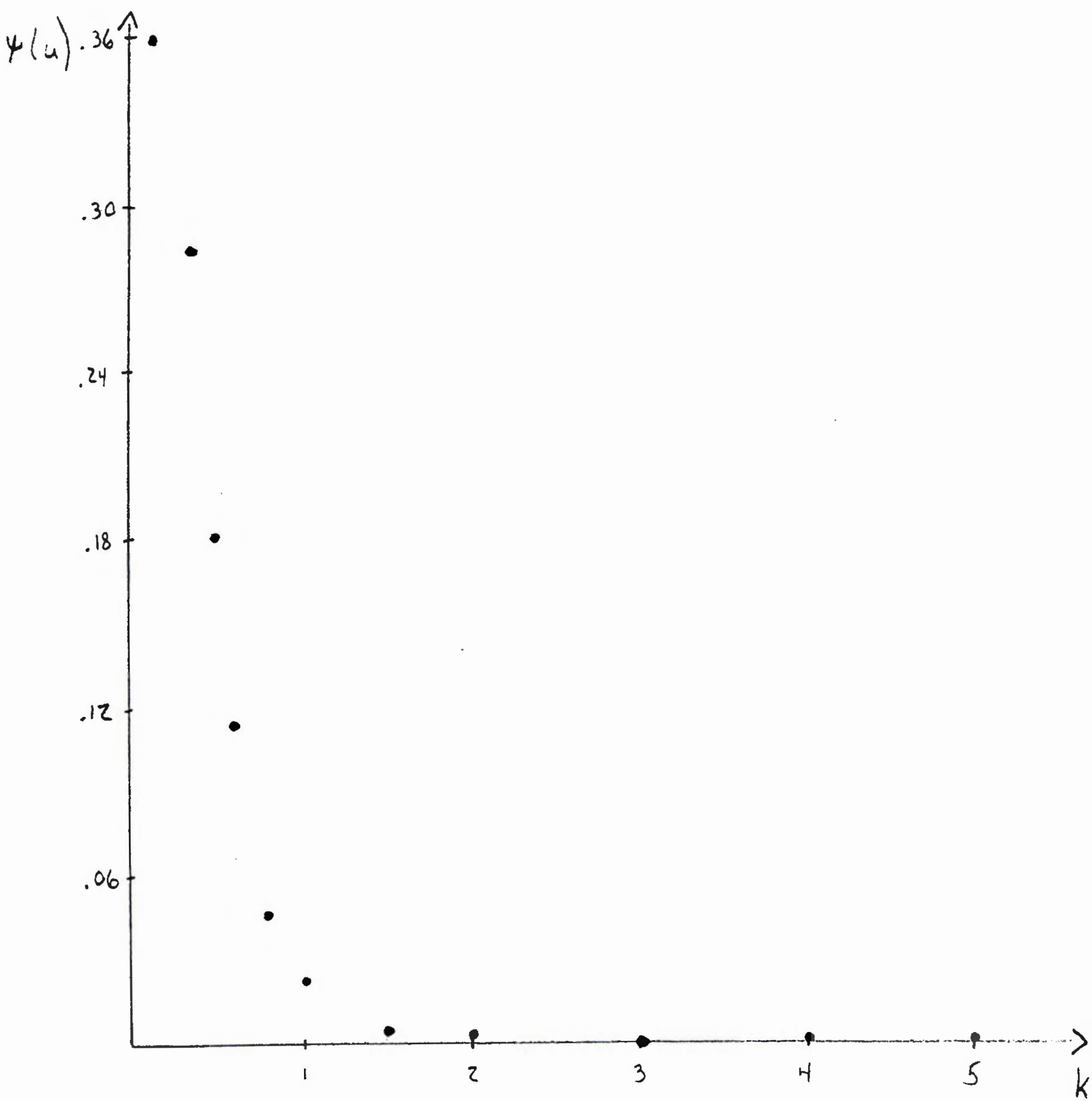


Figure 2

Graphical Representation of ψ(u) when claims follow a Weibull distribution



Appendix A

```
#include <stdio.h>
#include <stdlib.h>

main()
{
        FILE *inpf;
        FILE *infp;
        int j, counter, ruin_count;
        float time, claim, total_time, surplus, premium;

        inpf = fopen("H:\TIMES.DAT", "r");
        infp = fopen("H:\CLAIMS.DAT", "r");

        ruin_count = 0;

        for (j = 1; j <= 1000; j++)
        {
        surplus = 100;
        total_time = 0;
        do {
           fscanf(inpf, "%f", &time);
           fscanf(infp, "%f", &claim);
           surplus = (surplus+(1.1 * time)-(claim));
           total_time = total_time + time;
           } while ((surplus >= 0) && (total_time <= 700));

        if (surplus < 0)
           ruin_count = ruin_count++;

        printf("Number of ruins = %3d", ruin_count);
        printf(" Total time = %f", total_time);
        printf(" Surplus at end time = %f\n", surplus);
        }

        fclose(inpf);
        fclose(infp);
        return 0;
}
```

Appendix B

Simulating probability of ruin during the time interval [0, t] when the claim sizes are exponential with mean $1/\beta$

t	θ	$1/\lambda$	$1/\beta$	Theoretical $\psi(u)$	Simulated $\psi(u)$
500	0.15	5	4.78	0.057	0.028
700	0.15	5	4.78	0.057	0.045
700	0.10	5	5.00	0.148	0.081
700	0.19	5	4.62	0.027	0.015
700	0.21	5	4.55	0.018	0.016
700	0.20	5	4.58	0.022	0.023
700	0.20	5	4.58	0.022	0.016
700	0.20	5	4.58	0.022	0.023

Appendix C

Simulating probability of ruin during the time interval [0, t] when claim sizes are Weibull with mean $(\alpha/k)\Gamma(1/k) = 4.58$

k	α	$\psi(u)$
0.2	0.038	0.364
0.2	0.038	0.358
0.2	0.038	0.358
0.4	1.378	0.276
0.4	1.378	0.294
0.4	1.378	0.277
0.5	2.29	0.182
0.5	2.29	0.169
0.5	2.29	0.189
0.6	3.057	0.108
0.6	3.057	0.119
0.6	3.057	0.113
0.8	4.042	0.042
0.8	4.042	0.050
0.8	4.042	0.045
1.0	4.58	0.023
1.0	4.58	0.016
1.0	4.58	0.023
1.2	4.854	0.009
1.2	4.854	0.006
1.2	4.854	0.004
1.4	4.999	0.002
1.4	4.999	0.008
1.4	4.999	0.003
1.5	5.096	0.002
1.5	5.096	0.004
1.5	5.096	0.004
1.6	5.145	0.004
1.6	5.145	0.006
1.6	5.145	0.002
2	5.168	0.001
2	5.168	0.004
2	5.168	0.004
3	5.075	0.000

Appendix C (continued)

Simulating probability of ruin during the time interval [0, t] when claim sizes are Weibull with mean $(\alpha/k)\Gamma(1/k) = 4.58$

k	α	$\psi(u)$
3	5.075	0.000
3	5.075	0.000
4	5.053	0.001
4	5.053	0.000
4	5.053	0.000
5	4.988	0.001
5	4.988	0.000
5	4.988	0.000

Appendix D

The Pareto distribution has probability density function:

$$f(x) = \frac{\alpha(\lambda^\wedge\alpha)}{(\lambda+x)^{\alpha+1}} \quad , \; x > 0$$

And hence has cumulative density function:

$$F(x) = 1 - [\lambda^{\alpha}/(\lambda+x)^{\alpha}]$$

If we generate a value, U, from a Uniform(0,1) distribution, then $F^{-1}(U)$ will have a distribution from F(x) since:

$$P(F^{-1}(U) \leq x) \;=\; P(U \leq F(x)) \;=\; F(x)$$

Let y = F(x). Then:

$$y = 1 - [\lambda^{\alpha}/(\lambda+x)^{\alpha}]$$

Which may be reduced to:

$$x = \lambda[(1-y)^{-1/\alpha} - 1]$$

Therefore:

$$F^{-1}(U) = \lambda[(1-U)^{-1/\alpha} - 1]$$

For our case, λ=4.58 and α=2.

$$F^{-1}(U) = (4.58)[(1\text{-}U)^{-1/2} - 1]$$

where U is a random value from a Uniform(0,1) distribution.