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Meaningful Distributed Instruction— Developing Number Sense

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What Is Number Sense?

The primary goal for elementary and middle school mathematics is to help students learn to use numbers meaningfully, reasonably and flexibly in everyday life. This means that students must develop a deep understanding of

- numbers and operations,
- when operations can appropriately be used to solve problems, and
- judging the reasonableness of their solutions to problems.

They also need to develop attitudes so they

- believe they can make sense of mathematics, and
- habitually try to make sense of mathematics.

In other words, students need to develop number sense.

Understandings Needed for Number Sense

To develop number sense related to a specific topic students need to have a deep understanding about

1. when to use that topic in everyday situations,
2. how to represent that topic in multiple ways, and
3. how to think and reason with that topic in multiple ways.

Fraction Example

1. **Know When Fractions Can Be Used To Solve Problems**
   Grade six students who deeply understand fractions will know that fractions can be used to solve a variety of problems when a whole is split into equal-sized parts. Problem contexts might be linear, involve area or volume, or include wholes that are comprised of individual items. Students need to learn that fractions can be applied in a variety of different contexts with a variety of different problem structures.

2. **Flexibly And Fluently Use Multiple Representations Of Fractions And Their Relationships To Other Topics,**
   When students first begin learning fractions, they often directly model the structure of the problem. This leads to different representations of fractions. Students must make sense of these different representations.

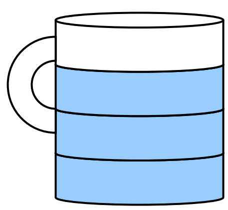
[1] Adapted from a manuscript that will be published by the National Council of Teachers of Mathematics.

Halves, fourths, sixths, and eighths of a pie or pizza are familiar situations to many students. Experiences like this lead many of them to associate fractions with area models, and more specifically splitting circles into equal parts.

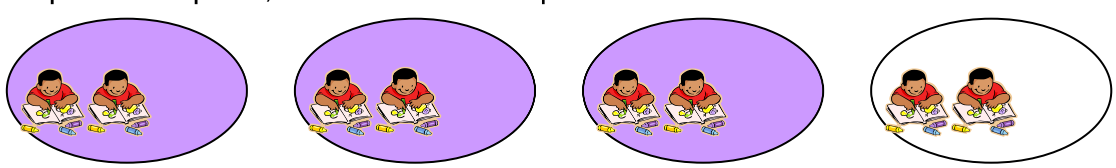
Area models can also involve any other shapes that are split into equal parts. Each of the following illustrates the fraction 3/4.

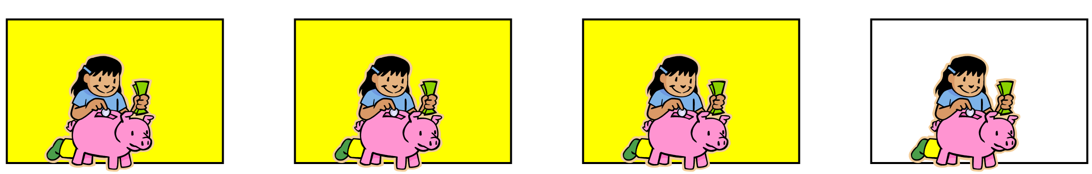
Linear measurement and the use of rulers lead to a number line representation of fractions. In this case a unit length is split into equal-sized parts.

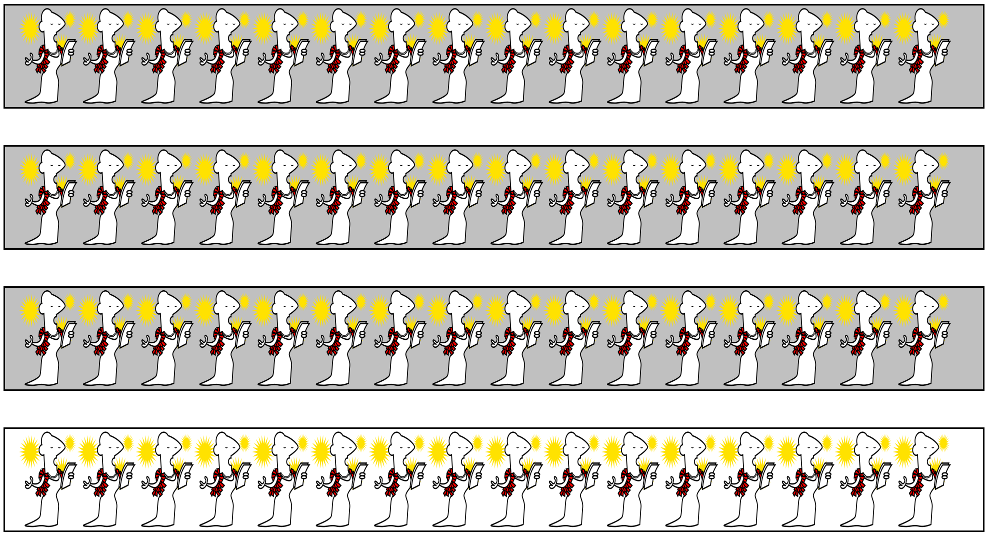
Measuring volume leads to a three-dimensional representation of fractions. Measuring cups used in cooking are typical examples.



All of the representations above are connected or continuous. Fractions are also used in discrete situations. Groups of items, which are split into equal-sized parts, lead to different representations.

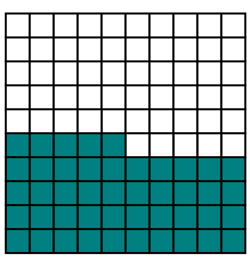






In contrast, students who only represent fractions in symbolic form often have difficulty using fractions in all of the different types of situations shown above. This limits their ability to know when to use fractions in everyday situations.

These representations also help students make connections between fractions and other topics. For example, using a unit square split into 100 equal-sized parts by forming 10 rows and 10 columns provides a natural way to connect fractions to decimals and percents. Forty-five of 100 equal parts illustrates 45/100; forty-five hundredths illustrates 0.45; and forty-five per one hundred illustrates 45%.

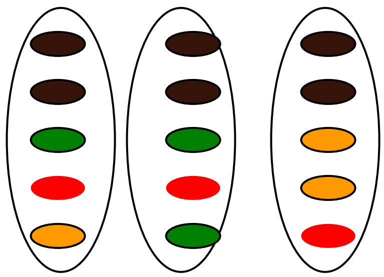


45/100

0.45

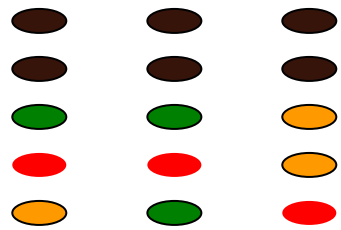
45%

Ratios, which are comparisons of a part to the whole, can be represented by a model that also illustrates a related fraction. If the ratio of the number of brown candies to the total number of candies is 2 to 5, that can be represented by a model to illustrate 2/5. Two-fifths of the candies are brown.



ratio

2 browns to
5 candies



fraction

2 rows of brown
out of 5 rows

2//5

3. Flexibly And Fluently Use Multiple Reasoning Strategies With Fractions And Their Relationships With Other Topics

In order to make sense of fractions and use them effectively to solve everyday problems, students need to think about the following just to understand the meanings of the fractions.

- What is the whole (unit)?
- Are there equal-sized parts?
- How many parts are you considering?
- How many parts are there in the whole (unit)?

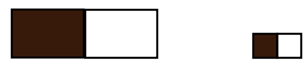
Students with number sense understand the “size” of fractions. Just like they understand that 58 is close to 60, they understand that 9/10 is close to 1 and that 4/7 is a little more than 1/2. When solving problems, it is often much easier to use a nice number that is close to a fraction than to use the fraction itself. One-half is often much easier to use than 4/7.

Size of Fractions

fraction	benchmarks	thinking
2/13	0	Two parts is very small compared to 13 parts in the unit. So 2/13 is not that much more than 0.
1/35	0	One part is very small compared to 35 parts in the unit. So 1/35 is not that much more than 0.
4/7	1/2	Four is little more than half of 7, so 4/7 is a little more than 1/2.
11/25	1/2	Eleven is a little less than half of 25. So 11/25 is a little less than 1/2.
8/9	1	Eight parts is almost the same as 9 parts in the unit, so 8/9 is almost 1.
14/12	1	Fourteen parts is a little more than 12 parts in the unit. So 14/12 is a little more than 1.

Reasoning that is used to compare fractions is much more complex than reasoning to compare whole numbers. One thing that confuses many students is that fractions with larger numbers aren't necessarily greater than fractions with smaller numbers. For example, 5/8 is not as great as 3/4. But, 7/9 is greater than 2/3. Sometimes the fraction with larger numbers is greater and sometimes it is not. If you cannot decide which of two fractions is greater by comparing the size of the numbers involved, how can you decide?

Comparing Fractions

Which is greater?	strategy	thinking
1/2 or 1/2	compare the size of the units	If the unit for one fraction is greater, 1/2 of that unit is greater. 
3/8 or 5/8	common denominators, then compare the size of the numerators	If the fractions have the same size unit and they have the same number of parts per unit (common denominators), then you can compare the number of parts you are considering (numerators). Three parts is less than 5 parts, so 3/8 is less than 5/8. 
3/4 or 3/8	common numerators, then compare the size of the denominators	If fractions have the same size unit and they have the same number of parts being considered (common numerators), then you can compare the number of parts per unit (denominators). Fourths are bigger parts than eighths, so 3/4 is greater than 3/8. 
3/8 or 4/7	compare to benchmark fractions	Three is less than half of 8, so 3/8 is less than 1/2. Four is greater than half of 7, so 4/7 is greater than 1/2. So, $3/8 < 1/2 < 4/7$
6/7 or 8/9	compare the "missing" parts to make a unit or benchmark fraction	Six-sevenths is only 1/7 away from 1. Eight-ninths is only 1/9 away from 1. Since sevenths are greater than ninths, 8/9 is closer to 1 than 6/7. That means 8/9 is greater.

Fractions are different from whole numbers in many ways. There is no whole number between 23 and 24. But there are an infinite number of fractions between any two given fractions, no matter how close together. Fractions are dense.

fractions between	Strategy	thinking
2/3 and 3/4	change both to a common denominator (much larger than the least common denominator), then choose a numerator between	$\frac{2}{3} x \frac{20}{20} = \frac{40}{60}$ $\frac{3}{4} x \frac{15}{15} = \frac{45}{60}$ 41/60; 42/60; 43/60; and 44/60 are all fractions between 2/3 and 3/4
2/3 and 3/4	average the two fractions; that is, add the fractions and divide by two	$\frac{8}{12} + \frac{9}{12} = \frac{17}{12}$ (17/ 12) ÷ 2 = 17/24 So, 17/24 is a fraction between 2/3 and 3/4. In fact, it is “halfway” between 2/3 and 3/4.

Using the standard symbols there is only one way to write each whole number. With fractions greater than one, they can be written as a mixed number or as an improper fraction, with a numerator greater than the denominator. Changing a mixed number to an improper fraction or back relies on making sense of 3/3, 5/5, 16/16, etc. Each of those fractions is just 1.

Changing a Mixed Number to an Improper Fraction

mixed number	thinking	improper fraction
$3\frac{1}{4}$	4/4 is 1, so three times that or 12/4 is 3. There is an extra 1/4, so it is 13/4.	13/4

Changing an Improper Fraction to a Mixed Number

improper fraction	thinking	mixed number
11/3	3/3 is 1, so 9/3 is three times as much or 3. That leaves 2/3.	$3\frac{2}{3}$

With fractions there are an infinite number of ways to write the same number. For example, 3/5 is the same number as 6/10 or 15/25 or 24/40 or (3 x n)/(5 x n) for any non-zero number.

Equivalent Fractions

equivalent fractions	strategy	thinking
fraction equivalent to 2/3	multiply both numerator and denominator by the same non-zero number	When you double the number of parts in the unit and double the number of parts you are considering, the amount is the same. The same is true if you multiply both numerator and denominator by the same non-zero number. For example, $\frac{2}{3} \xrightarrow[\text{x } 4]{\text{x } 4} \frac{8}{12}$
Is 6/8 equivalent to 9/12?	cross multiply and compare products	Since 6 x 12 = 8 x 9, the fractions are equivalent. $\frac{6}{8} x \frac{12}{12} = \frac{72}{96}$ $\frac{9}{12} x \frac{8}{8} = \frac{72}{96}$ The numerators are simply the cross products. You don't have to compare the denominators because they are the same.

To add and subtract fractions, you have to be combining or separating parts that are the same size, that is, they have a common denominator. How do you find a common denominator?

Finding Common Denominators

find common denominator	strategy	thinking
2/3 and 3/4	list multiples of each denominator	3, 6, 9, 12, 15, 18, ... 4, 8, 12, 16, ... The first common multiple is 12, so 12 is a common denominator, in fact, it is the least common denominator.
3/5 and 2/3	list multiples of the greatest denominator, check to find a multiple of the other denominator	5, 10, 15 Three does not divide 5 or 10. Three does divide 15, so 15 is a common denominator, in fact, it is the least common denominator.

5/12 and 3/21	factor the denominators into prime factors, then combine factors to insure that both denominators divide the product.	12 = 2 x 2 x 3 21 = 3 x 7 Factors of a common denominator must include at least 2 twos and 1 three so that it is a multiple of 12; it must include at least 1 three and 1 seven so that it is a multiple of 21. So, 2 x 2 x 3 x 7 = 84
7/12 and 7/18	multiply the denominators and divide by the greatest common factor	Factors of a common denominator must include at least 2 twos and 1 three so that it is a multiple of 12; it must include at least 1 two and 2 threes so that it is a multiple of 18. 12 x 18 = 2 x 2 x 3 x 2 x 3 x 3 There is one extra two and one extra three in that product, 2 x 3 is the greatest common factor of 12 and 18. So, (12 x 18) ÷ 6 = 36

Counting by fractions, just like counting by multiples of 4 can be helpful in adding and subtracting fractions. Counting by fractions eliminates finding a common denominator, converting both fractions to equivalent fractions with that common denominator, then adding or subtracting. However, counting by fractions is complicated because some multiples of fractions are whole numbers.

$$\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2, \ldots$$

Counting by fractions is even more complicated because often a multiple of a fraction is equivalent to a simplified fraction.

$$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2, \ldots$$

Counting by Fractions to Add or Subtract

computation	strategy	thinking
$2\frac{1}{2}+\frac{3}{4}$	count on by one-fourths	$2\frac{1}{2}, \ldots\ 2\frac{3}{4}, 3,\ 3\frac{1}{4}$
$3\frac{1}{4}-\frac{3}{8}$	count back by one-eighths	$3\frac{1}{4}, \ldots\ 3\frac{1}{8}, 3,\ 2\frac{7}{8}$

Often fractions can be related to decimals, money, or even percents. Some students might find it easier to think of 1/4 as 0.25, a quarter, or 25%. Similarly, 2/5 can be 0.4 or 40%. Common fraction decimal equivalents include the following.

Common Fraction Decimal Percent Equivalents

fraction	decimal	percent
1/2	0.5	50%
1/3; 2/3	about 0.33; 0.67	about 33%; 67%
1/4; 3/4	0.25; 0.75	25%; 75%
1/5; 2/5; 3/5; 4/5	0.2; 0.4; 0.6; 0.8	20%; 40%; 60%; 80%
1/10; 3/10; 7/10; 9/10	0.1; 0.3; 0.7; 0.9	10%; 30%; 70%; 90%

Other fractions can often be converted to decimals and/or percents by using the equivalents listed above.

Other Fraction Decimal Percent Equivalents

fraction	decimal	percent
1/6	1/3 is a little more than 0.33, so 1/6 is half of that or about 0.167	16.7%
3/8	1/4 or 2/8 is 0.25, so 1/8 is half of that or 0.125 3/8 is 0.25 + 0.125 or 0.375	37.5%
7/20	1/10 is 0.1, so 1/20 is half of that or 0.05 7/20 is 7 x 0.05 or 0.35	35%

Every fraction can be converted to a decimal, simply divide the numerator by the denominator. But because decimals only have denominators that are powers of ten, some fractions have terminating decimals and others have repeating decimals. If a fraction, in simplest form, has a denominator with prime factors that only include twos and/or fives, it will have a terminating decimal.

Converting a Fraction to a Decimal

fraction	strategy	thinking	decimal
3/20	multiply numerator and denominator by the same number to get a power of ten in the denominator	20 = 2 x 2 x 5 If the number of twos matches the number of fives as factors, then they can be paired up to make tens. If you multiply the denominator by 5, then 20 x 5 = 2 x 2 x 5 x 5 = (2 x 5) x (2 x 5). That is 10 x 10 or 100, a power of ten. You can multiply both numerator and denominator of 3/20 by 5 to get 15/100. Because the fraction has a denominator that is a power of ten it can be written as fifteen hundredths.	0.15
41/250	multiply numerator and denominator by the same number to get a power of ten in the denominator	250 = 2 x 5 x 5 x 5 If the number of twos matches the number of fives, then they can be paired up to make tens. If you multiply the denominator by 2 x 2, then 250 x 2 x 2 = 2 x 2 x 2 x 5 x 5 x 5 = (2 x 5) x (2 x 5) x (2 x 5). That is 10 x 10 x 10 or 1000, a power of ten. You can multiply both numerator and denominator of 41/250 by 4 to get 164/1000. The fraction can be written as one hundred sixty-four thousandths.	0.164

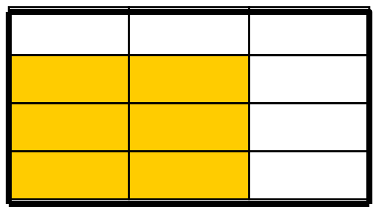
5/6	$$\begin{array}{r} 0.8333 \\ 6\overline{)5.0000} \\ \underline{48} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$	When you divide by six, there are only six possible remainders for each division, 0- 5. This division will never have a remainder of 0 because six has a factor of 3. That means it cannot be multiplied by any whole number to get a power of ten. Powers of ten only have factors of twos and fives. So this division will repeat. As soon as the same remainder appears a second time, the quotient will begin repeating. In this case when 20 is divided by 6, the remainder is 2, which becomes 20 when the next place value is added ("brought down") to that remainder. The quotient will repeat from that point on.	0.8333...

Students should learn to compute using standard fraction algorithms, but they also should be able to use mental computation and estimation strategies to solve problems, with exact answers and with estimations. Variations of many of the strategies that students use with whole numbers work with fraction computation.

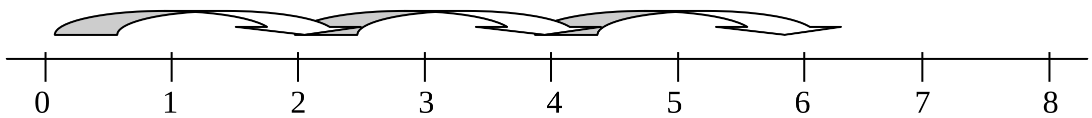
Mental Computation and Estimation With Fractions

computation	Strategy	thinking
$2\frac{3}{4}+1\frac{3}{4}$	use a unit	Start with $2\frac{3}{4}$ and add enough to make 3. That leaves $1\frac{1}{2}$. $3+1\frac{1}{2}=4\frac{1}{2}$
$5\frac{1}{4}-1\frac{3}{8}$	use a unit	Subtract 1/4 to get 5. Then subtract the remaining $1\frac{1}{8}$. $5-1\frac{1}{8}=3\frac{7}{8}$
$2\frac{1}{4}+3\frac{1}{3}$	front end, then adjust	2 + 3 is 5 1/4 + 1/3 is a little more than 1/2 so, it's a little more than 5 and 1/2
$5\frac{2}{3}-2\frac{1}{4}$	front end, then adjust	5 – 2 is 3 2/3 – 1/4 is a little less than 1/2 so, it's a little less than 3 and 1/2
$1\frac{7}{8}+3\frac{4}{7}$	use benchmark fractions	1 and 7/8 is about 2 3 and 4/7 is about 3 and 1/2 2 plus 3 and 1/2 is 5 and 1/2
$4\frac{3}{5}-1\frac{5}{16}$	use benchmark fractions	4 and 3/5 is about 4 and 1/2 1 and 5/16 is about 1 and 1/4 4 and 1/2 minus 1 and 1/4 is 3 and 1/4

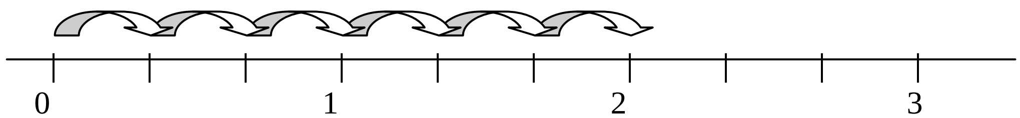
Models to represent operations with fractions are often extensions of models to represent operations with whole numbers. For whole numbers, 3 rows with 2 in each row represents 3 x 2. Below, the shaded rectangle with a height 3/4 of the unit rectangle (outlined with a heavy border) and a length of 2/3 of the unit rectangle is 3/4 x 2/3 of the unit rectangle. That is, 3 rows of 2 shaded parts out of 4 rows of 3 parts in the unit rectangle is 6 parts out of 12 parts or 6/12 of the unit rectangle.



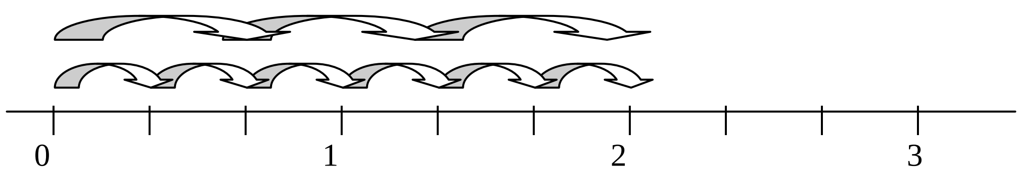
Similarly, division of fractions can be represented on a number line just like division of whole numbers. Six divided by 2 and be represented by splitting the 6 into groups of 2.



Two divided by 2/3 can be modeled in two steps. First, how many thirds are there in 2? Since 3/3 is a 1, there are three times as many thirds as the unit. 3/3 is 1, so 6/3 is 2. Multiplying the denominator by the number of units determines the number of thirds in the total number of units.



Since you need to know the number of 2/3 in 2, you can simply divide the number of thirds by 2. Dividing the total number of thirds by the numerator determines the number of 2/3 in the number of units. The number line below illustrates there are 6 thirds in 2, but only half that many groups of 2/3.



Students, who understand what is described above, will be able to

- Confidently and meaningfully use fractions,
- Flexibly use appropriate and efficient representations and reasoning strategies with fractions, and
- Perform extremely well on any fraction items in any assessment, from skills through problem solving.

What Is The Current Status?

In contrast, many sixth grade students

- Have many misconceptions about fractions,
- Do not understand the relative size of the number or which nice fractions are close, and
- Almost always have memorized symbolic procedures without understanding,

Most curriculum materials in grade six include two, three, or four units of instruction on fractions. If each unit takes about three weeks, that means students must learn

- all of the different situations when fractions can be used,
- multiple ways to represent fractions, and
- multiple reasoning strategies for solving problems with fractions.

The examples above include over 30 different reasoning strategies. And students only have about nine weeks to make sense of them. This is an impossible task for both the teacher and the students.

First, it needs to be recognized that our current programs are not successfully helping students develop a deep understanding of fractions, and they certainly do not help students develop number sense with fractions.

Second, no current curriculum materials include sufficient instructional time to help students make sense of all of the ideas shown in the examples above. Because there are so many different contexts for problems, so many different ways to represent those problems, and so many different reasoning strategies used to solve those problems, students need an extended amount of time to understand.

Third, making sense of concepts and reasoning strategies simply takes too long. With current curriculum materials students do not have the opportunity to learn and to develop a deep understanding of fractions.

How Can We Improve Achievement?

Organizing Curriculum to Develop Number Sense

One alternative is to spend more time on each of the current units of instruction. This approach fails in two ways. First, our current materials focus on only a few of the ideas shown in the example above, so they still will not provide a complete instructional program. Second, this means the focus on at least some topics other than fractions will need to be diminished, if not ignored. Developing understanding of one topic at the expense of another is a questionable practice.

Another alternative is to distribute the instructional experiences throughout the school year, a few minutes each day. Since this approach cannot be used for every topic, one or two key topics need to be selected for each grade level.

Daily five-minute lessons on that key topic will provide time for students to develop number sense and still leave time for the current curriculum. The extra twenty-five minutes of instruction each week will provide students repeated opportunities to make sense of multiple representations, thinking strategies, and contexts for using the key topic.

Research has clearly established that distributed practice is more effective than massed practice. A few minutes of practice each day is more effective than spending a similar amount of time practicing once a week.

The same principle applies to mathematical concepts and to the development of reasoning strategies. Five minutes of daily conceptual instruction throughout the school year is more effective than a three-week unit of instruction—even two, three or four units of instruction. Meaningful distributed instruction provides valuable time for students to make sense of mathematical concepts and thinking strategies.

Meaningful Distributed Instruction
Guidelines for these brief five-minute lessons include

1. Use a problem-centered approach
2. Focus on conceptual ideas and routine problem solving
3. Expect students to make sense
4. Encourage students to develop fluency and flexibility

1. Use A Problem-Centered Approach

Students should have repeated opportunities to choose their own way to represent problems and to choose their own idiosyncratic solution strategies—ones that make sense to them. With repeated experiences and with many opportunities to model and see illustrations and to hear other explanations, students will gradually enlarge their repertoire of these representations and thinking strategies. This permits them the luxury of choosing an appropriate and efficient representation or strategy for each situation. This allows them to develop flexibility in their use of the key topic.

Students should also communicate about their solutions of these problems. They should have repeated opportunities to explain their representations and solution strategies. These explanations should include both how the representations were used to solve the problem and why they were selected. Students should also be encouraged to actively listen and ask for clarification as other students explain their solutions.

2. **Focus On Conceptual Ideas And Routine Problem Solving**
   These five-minute lessons are meant to develop understanding; they are not intended to provide extra time for practicing skills. Manipulatives, diagrams, drawings, and other concrete models should be used to illustrate solutions. These visual presentations together with explanations help students understand the concepts and the reasoning. This extra time spent on helping students understand concepts is time well spent on helping students make sense of symbolic procedures that are introduced later.

   Routine word problems enable students to apply a topic in a variety of everyday contexts. These distributed experiences through the school year enable students to decide when a topic can be used. The different contexts in these routine problems also encourage a variety of representations and thinking strategies. They provide the stage for students to develop a deep understanding of a key topic.

3. **Expect Students To Make Sense**
   Students should be expected to try to solve these daily problems. They also should be expected to make sense of multiple representations and multiple thinking strategies and ask for clarification when they do not understand. Students should be actively involved.

4. **Encourage Students To Develop Fluency And Flexibility**
   The distributed curriculum is designed to promote flexibility. As students become familiar with and understand various representations and thinking strategies, they are able to make strategic choices, which are efficient.

   Ultimately, it is important for students to be reasonably fluent with skills. If students need to hesitate and rethink a procedural step in their performance of a skill, they are less likely to use it in everyday life. While practice is important in developing fluency, it is much more effective after students have a deep understanding of the underlying concepts. The daily five-minute lessons are great preparation for skill fluency, but practice needs to be delayed.

Five-Minute Lessons

These lessons are designed to provide the experiences needed to help students develop deep understanding. These lessons need to become a routine in the classroom—a routine for the teacher and a routine for the students. Everyone in the classroom needs to know what to expect.

It is important to create an appropriate classroom climate. Students need to try to make sense of these problems, try to represent them appropriately, and try to use an efficient thinking strategy to solve the problem. Students also need to feel comfortable trying new approaches, explaining their solutions, and asking for clarifications.

It should be noted that routine word problems are suggested. While they are only one kind of problem, they can be used to provide a variety of contexts, a variety of problem structures, and a variety of situations where efficient thinking strategies can be used. More complex problems are important too, but they cannot be completed in the amount of time allotted for these lessons.

Typically the lessons are structured as below.

1. Present a routine problem to the students (about 20 seconds)
2. Give students time to think and solve the problem (about 30 seconds)
3. Ask two or three students to explain their solutions (about 2 minutes)
4. Highlight and illustrate one of the explanations (about 1 minute)
5. Ask students to try using that highlighted strategy to solve a new problem; have one student explain using that thinking (about 1 minute)

Each of the five parts of these lessons are more fully described below.

1. **Present A Routine Problem To The Students (about 20 seconds)**
   Select a routine problem with nice numbers. As students gain proficiency, more complex numbers can be used

   Usually this problem will be read to the students. It might be projected on a screen so students can read the problem as well.

2. **Give Students Time To Think And Solve The Problem (about 30 seconds)**
   All of the students need to be thinking about how to solve the problem. If it helps to keep all students involved, paper and pencil may be used. This also allows the teacher to check any drawings or diagrams the students might have used and their solutions.

   Students should not be using manipulatives at this time. It's not that manipulatives aren't helpful; they should be used at other times. It simply takes too much time for students to use manipulatives during these brief lessons.

3. **Ask Two Or Three Students To Explain Their Solutions (about 2 minutes)**
   Two or three students should explain their solutions. Students should develop an expectation that their solution strategies need to be different from any previous explanations. Students should expect to actively listen and ask questions to clarify any misunderstandings. As they gain experience, students will communicate better and feel more confident in their explanations.

4. **Highlight And Illustrate One Of The Explanations (about 1 minute)**
   After the explanations, the teacher should select one of the efficient strategies and highlight it. This choice should be made to promote a strategy, which will help the class make progress in their collective thinking strategies.

   Once the choice has been made, that thinking needs to be explained once or twice more—perhaps explained by a different student and perhaps explained in conjunction with a teacher demonstrated use of models or a diagram along with a student or a teacher explanation.

5. **Ask Students To Try Using That Highlighted Strategy To Solve A New Problem; Have One Student Explain Using That Thinking (About 1 Minute)**
   The "same" problem, but probably with larger numbers, can be presented again. This time students should be directed to try to use the highlighted strategy. "Everyone try to use Marissa's thinking to solve this problem."

   After about ten seconds to think, ask one student to explain using that same thinking. Ask another student if the highlighted thinking was used. Repeat that thinking one more time.

Sample Five-Minute Fraction Lesson

Teacher "Listen carefully and solve this problem."

> Jonas ate 5/6 of a small pizza. His dad ate 7/8 of a small pizza. Which one ate more of their small pizza?

Wait about twenty to thirty seconds.

Teacher "Cam, will you explain your solution?"

Cam "His dad ate more. 7/8 is bigger because if you change them to a common denominator 7/8 is 42/48 and 5/6 is only 40/48."

Teacher "Did anyone solve it a different way? (brief pause) Yes, Maddie."

Maddie "Jonas had 1/6 of his pizza leftover. His dad had 1/8 of his pizza leftover. 1/8 is less than 1/6, so his dad had less leftover. That means he ate more."

Teacher "How do you know that 1/8 is less than 1/6?"

Maddie "If you cut 8 slices of pizza instead of 6, there are more pieces so each piece is smaller."

Teacher "Ok, today, let's focus on Maddie's way of thinking. Erin, can you explain how Maddie solved the problem?"

Erin — "She looked at how much was leftover. Jonas's dad had a smaller piece leftover, so that means he ate more."

Teacher — While Erin is explaining, the teacher is drawing a diagram on the board.

Teacher — "So, Maddie thought about the parts that were leftover. She knew that 1/8 is less than 1/6, so Jonas's dad had less pizza leftover. That means he ate more.

Sometimes you can compare fractions by checking to see how close they are to a benchmark or nice number. Let's all try that same thinking on this problem.

> Kaitlyn ate 3/8 of a large pizza. Hayden ate 4/10 of a large pizza. Who ate more pizza?

Wait about ten seconds.

Teacher — "Who can explain Maddie's thinking with this problem? (brief pause) Marissa, will you try this?"

Marissa — "You can compare these to one-half. 4/8 is one-half. Kaitlyn ate 3/8, so she ate 1/8 less than one-half. 5/10 is one-half. Hayden ate 1/10 less than one-half. Since 1/10 is less than 1/8, Haydn ate more."

Teacher — While repeating the explanation, draw a diagram showing this same thinking. Look at the diagrams. 3/8 is 1/8 less than one-half. 4/10 is 1/10 less than one-half. Which is less, 1/8 or 1/10?"

4/8 3/8 5/10 4/10

Felipe — "1/10 is smaller because it has more pieces in the pizza."

Teacher — "You can compare fractions by comparing them to benchmarks. Sometimes it is easy to look at the "missing" part like we did today."

Summary

Key number and operation topics can be identified at each grade level. To develop number sense for those key topics, students need to have a deep understanding about

- when to use that topic in everyday situations,
- how to represent that topics in multiple ways, and
- how to think and reason with that topic in multiple ways.

Given the structure of current curriculum materials, most students do not have an opportunity to develop that deep understanding. There are just too many applications, too many models to represent a topic, and too many ways to think and reason with the topic to make sense of all that in only two, three or four units of instruction.

Meaningful distributed instruction provides students a much better opportunity to develop number sense. A few minutes of instruction each day keeps the ideas in the foreground all year long. Instead two or three conceptual lessons on a topic prior to practicing related skills, students use and see representations daily. Instead of three or four lessons on a specific thinking strategy, which might be spread over five or six months, students will use and listen to that strategy being used repeatedly throughout the school year.

Routine problems in the five-minute lessons provide a variety of

- contexts for using a topic,
- problem structures to represent in different ways, and
- situations to develop more efficient ways of thinking and reasoning.

As students begin to develop understandings of these ideas, they become more flexible in their use. With a minimal amount of practice on ideas they understand, they begin to develop better fluency.

Meaningful distributed instruction, as a supplement to any regular mathematics curriculum, can help students make sense and help them learn to use their number sense in everyday life.

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