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Meaningful Distributed Instruction— Conceptual Previews For Symbolic Procedures

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Conceptual Previews For Symbolic Procedures[1]

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What Does It Mean To Understand A Symbolic Procedure?

Understanding a symbolic procedure means far more than "getting the right answer." A mathematical symbolic procedure or written skill involves step-by-step thinking that leads from a computational problem to a solution. Memorizing this step-by-step procedure may enable a student to answer the problem, even answer it correctly. Yes, that is important, but understanding means much more.

To understand a symbolic procedure or skill, students must

1. **understand concepts** underlying the skill (refer to *Meaningful Distributed Instruction: Developing Number Sense, Iowa Council of Teachers of Mathematics Journal Volume 36 Fall 2009)* and
2. **connect**
   a. the **step-by-step actions** that can be used with models to solve problems to
   b. the corresponding thinking procedures that can be used to **record those steps** with paper and pencil.

A student with this deep understanding will understand the meaning of each of the symbols that is recorded in an symbolic procedure or algorithm, be able to illustrate the step-by-step process with manipulatives or diagrams, and be able to explain how the step-by-step actions with manipulatives or diagrams is connected to the symbols that get recorded in the algorithm. This deep understanding enables students to use the computational procedure flexibly in everyday situations. For example, if the numbers involved are nice, they may recognize that they can solve the same problem mentally and not even have to write the symbolic procedure on paper.

Division Example

1. ***Understand Concepts Underlying Division***
   Grade four students who deeply understand division will know that division can be used to solve problems when the product and one of the factors is known and they want to determine the other factor. If both factors are known, then division is not appropriate.

[1] Adapted from a manuscript that will be published by the National Council of Teachers of Mathematics.

Division is far more complex than just sharing or repeated subtraction. Students need to learn that division can be applied in a variety of different contexts and problem structures including

Equal parts

- **number in each group is unknown**—Janna had 24 cookies. She wants to put an equal number on each of 3 plates. How many cookies should she put on each plate?
- **number of groups is unknown**—Dillard wants to put 4 desks in each group. He had 24 desks to arrange. How many groups of desks will he make?
- **total number or product is unknown**—Stephanie had 3 packages of gum. There were 5 sticks of gum in each package. How many sticks of gum did she have?

Arrays

- **number of rows or columns is unknown**—Lon planted 24 apple trees. He put 6 trees in each row. How many rows did he plant?
- **total number or product is unknown**—Mrs. Smith had the desks in her classroom arranged in 4 rows of 6. How many desks did she have in her classroom?

Comparison

- **number in the small set is unknown**—Marta scored 12 points in the basketball game. That was 4 times as much as LaTrice scored. How many points did LaTrica score?
- **multiple or scale is unknown**—Jonah has collected autographs of 36 major league baseball players. Micah has only collected 9 autographs. How many times greater is Jonah's collection?
- **number in the large set is unknown**—Ryan has read 11 books in the past month. Justin has read twice as many books. How many books has Justin read?

Rates

- **rate is unknown**—Rae drove 480 miles in 6 hours of driving time. On the average, about how many miles per hour did she drive?
- **number of units or factor is unknown**—Robin drove 420 miles. She drove about 70 miles per hour. How many hours of driving time did it take her?
- **total or product is unknown**—Rachel drove 60 miles per hour for 3 hours. How many miles did she drive?

Combinations

- **number in one set is unknown**—Julia had several sweaters that she could wear with 3 pairs of slacks. She could use these to create 15 outfits. How many sweaters did she have to go with those slacks?
- **number of combinations is unknown**—Nicole had 4 sweaters that she could wear with 2 pairs of slacks. How many different outfits did she have?

Students also need to make sense of multiple representations and multiple reasoning strategies related to division well enough to develop flexibility and fluency in their use. For example, students need to make sense of division models including

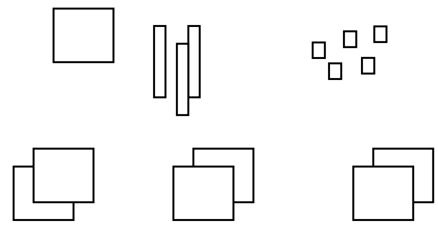
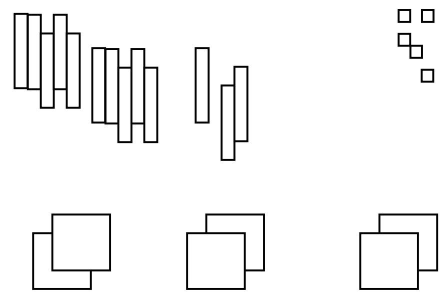
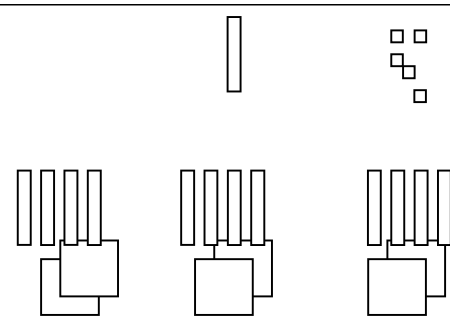
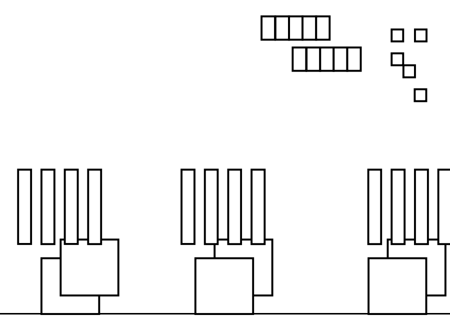
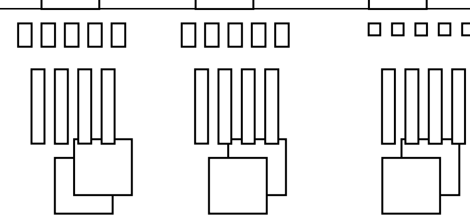
- separating a quantity into a known number of equal-sized groups
- separating a quantity into parts of a known size
- sharing
- measuring one quantity by another
- using arrays
- using equal parts on a number line
- using area and volume models
- using multiplication

Understanding requires a student to make sense in a way that enables flexible use of the skill. For example students should be able to use a standard paper-and-pencil algorithm, but they also should recognize when it is easier to solve problems mentally. Flexibility like this, that involves the strategic choice of an efficient way of thinking, requires a deep understanding of the operation. For 294 ÷ 6, students might recognize that

- 50 x 6 is 300; that's 50 sixes; since 294 is one less six, the answer is 49,
- 300 ÷ 6 is 50, so 294 ÷ 6 is one less six, or 49, or
- 294 = 240 + 54; then 240 ÷ 6 = 40 and 54 ÷ 6 = 9, so the answer is 40 + 9 or 49.

2. Connect Step-By-Step Actions On Models To Corresponding Thinking Procedures For A Standard Division Algorithm

Parents and teachers expect students to learn a standard paper-and-pencil division algorithm. Although each of the reasoning strategies described above illustrates understanding and flexible use of numbers, none of them provides evidence that the student understands the actions needed to make sense of a standard algorithm. Consider the step-by-step actions with base ten blocks that are needed to make sense of a symbolic solution to the standard long division algorithm involving 735 ÷ 3.

Actions	Actions on Base Ten Blocks
Start with 7 hundreds 3 tens and 5 ones.	Make three equal groups of blocks.
Share the hundreds. We have 7 hundreds to share. Put 2 hundreds in each pile. We used 6 hundreds and have 1 hundred left over.	
Trade the extras for tens. Trade the extra hundred for 10 tens & combine them with the 3 tens.	
Share the tens. We have 13 tens to share. Put 4 tens in each pile.	
Trade the extras for ones. Trade the extra ten for 10 ones & combine them with the 5 ones.	
Share the ones. Put 5 ones in each pile. There are no extras 735 has been split into 3 equal piles, with 245 in each pile. So, 735 ÷ 3 = 245.	

Making sense of these actions is the basis for making sense of the most common standard division algorithm. But these actions also need to be connected to recording the symbols in the algorithm. Each set of actions with the model corresponds with and can be recorded as part of a paper-and-pencil long division algorithm.

Actions	Connecting to Symbols	Algorithm
Share the hundreds. We have 7 hundreds to share. Put 2 hundreds in each group. We used 6 hundreds and have 1 hundred left over.	• Write 2 above the hundreds; we put 2 hundreds in each group. • Write 6 below the hundreds; we used 6 of them. • Write 1 below the hundreds; we have 1 hundred left over.	$\begin{array}{l} \;\;2 \\ 3\overline{)735} \\ \;\;\underline{6} \\ \;\;1 \end{array}$
Trade the extras for tens. Trade the extra hundred for 10 tens & combine them with the 3 tens in 735.	• Write 3 beside the 1 extra hundred to show we have 13 tens to share.	$\begin{array}{l} \;\;2 \\ 3\overline{)735} \\ \;\;\underline{6} \\ \;\;13 \end{array}$
Share the tens. We have 13 tens to share. Put 4 tens in each group.	• Write 4 above the tens; we are put 4 tens in each group. • Write 12 below the 13; we used 12 of them. • Write 1 below the 12; we have 1 ten left over.	$\begin{array}{l} \;\;24 \\ 3\overline{)735} \\ \;\;\underline{6} \\ \;\;13 \\ \;\;\underline{12} \\ \;\;\;1 \end{array}$
Trade the extras for ones. Trade the extra ten for 10 ones & combine them with the 5 ones in 735.	• Write 5 beside the 1 to show that we have 15 ones to share.	$\begin{array}{l} \;\;24 \\ 3\overline{)735} \\ \;\;\underline{6} \\ \;\;13 \\ \;\;\underline{12} \\ \;\;\;15 \end{array}$

Share the ones. Put 5 ones in each group. There are no extras	• Write 5 above the ones; we put 5 ones in each group. • Write 15 below the 15; we used all 15 of them.	$\begin{array}{r} 245 \\ 3\overline{)735} \\ \underline{6} \\ 13 \\ \underline{12} \\ 15 \\ \underline{15} \end{array}$

To help assess if students have developed these connections, they can be asked to explain the meanings of each of symbols in the algorithm. For example,

$$\begin{array}{r} 245 \\ 3\overline{)735} \\ \underline{6} \\ 13 \\ \underline{12} \\ 15 \\ \underline{15} \end{array}$$

Why did you write 4 above the 3 in 735?
(4 is the number of tens that were placed in each group. It was placed above the tens because we were sharing tens.)

Why did you write 12 below the 13?
(12 is the number of tens that were used when we shared them.)

Why did you write 1 below the 12?
(1 is the number of tens that were left over after sharing.)

Why did you write a 5 beside that 1?
(After we traded that extra ten for 10 ones and combined it with the 5 ones we started with in 735, there were 15 ones to share.

Students need to make sense of similar step-by-step actions for mental skills. But for mental algorithms there are no connections to symbolic algorithms.

What Is The Current Status in classrooms?

In contrast to the understandings described above, most fourth-grade students

- do not have deep understandings of division other than repeated subtraction and some, often superficial, connections to multiplication, and they
- have not made the connection between step-by-step actions on models and the corresponding thinking needed to make sense of a standard long division algorithm.

Most curriculum materials in grade four do not allow enough time for students to

- understand division concepts, nor to
- make sense of the connections between actions and recording a standard algorithm, although most show those connections at the beginning of a unit.

The first of these concerns simply means that students require a much longer time to make sense of division concepts—when to use division, ways to represent division, and ways to reason with division. These understandings can be addressed by using an ongoing supplementary curriculum as described in *Meaningful Distributive Instruction: Developing Number Sense*. This instructional approach will provide repeated opportunities for students to make sense of the models and thinking strategies related to division.

While the second concern is more easily addressed, sometimes educators wonder why this is even a concern. Since students have been inventing their own procedures to solve problems throughout the number sense lessons, they typically use common mental computational strategies often using left to right procedures for addition, subtraction, and multiplication. Are they likely to discover the thinking that is used with a standard algorithm? The answer is, rarely. So even after extended instruction to develop number sense, almost all students can benefit from specific instruction to help them make sense of a standard algorithm. The good news, ...helping students make sense of using models to solve computation problems takes very little instructional time. Once students can solve these computation problems with models and they can explain their solution, they are ready to learn how to record the symbols corresponding with those step-by-step actions.

How Can We Improve Achievement With Step-by-Step Symbolic Procedures?

First, we can supplement our regular curriculum with five-minute student-centered lessons to help students develop a deep understanding of division, as described in *Meaningful Distributive Instruction: Developing Number Sense*. This will help students make sense of the conceptual base needed meaningful division skills.

Second, and this is the key to improving proficiency with symbolic procedures or skills, we can have students use a model to solve one division computation problem each day for about two weeks prior to learning a standard long division algorithm. This will help students make sense of the step-by-step actions to solve then explain computation problems, without using symbolic procedures yet (refer to pages 2-3 of this document).

The thinking they use with these models <u>must</u> correspond with the thinking they will use with the symbolic procedures in the standard algorithm. Otherwise, the actions that are recorded will not match the symbolic algorithm. When that happens, the students are forced to learn two separate algorithms, one for the models and one for the symbols. In those cases, the models are a hindrance, not an aid, to learning the symbolic procedure.

These problems take only two to three minutes each day. After about two weeks, a total of about 20-30 minutes of instruction, almost all students will understand how to use the model to solve computation problems and they will be able to explain their actions.

Then, it is relatively easy to help students learn to record those step-by-step actions. Since the step-by-step actions correspond to the thinking used with the symbolic procedure or algorithm, recording those actions leads directly to that algorithm. Learning how to record these actions usually takes most students no more than two lessons. For most textbooks this often corresponds to the first two lessons in a unit on the skill.

How Can We Organize The Curriculum To Develop Proficiency With Skills?

1. Supplement our regular curriculum with five-minute student-centered lessons to help students develop a deep understanding of a skill, that is, number sense.

2. Concretely provoke the thinking that will help students make sense of a skill. Then help students connect those actions to a standard algorithm.
    a. Have students use a model to solve one computation problem for a skill each day for about two weeks prior to learning a standard algorithm for that skill.
    b. Help students connect the step-by-step actions with models to the symbols that are recorded in an algorithm.

3. Practice the skill to develop accuracy and fluency, but only <u>after</u> students
    a. can solve problems involving the skill with a model,
    b. can explain their solution, and
    c. can record the step-by-step actions in a standard algorithm.

The supplementary number sense lessons should be used daily throughout the school year. These brief daily lessons give students the opportunity to learn when the skill can be used and when it should not be used; these lessons give students the opportunity to make sense of multiple representations and multiple reasoning strategies; these lessons give students the opportunity to develop number sense and a deep understanding of the topic. These same lessons also provide daily opportunities for challenge to learn new representations, new reasoning strategies, and to use them in new contexts and situations so that all students are expanding their understandings each day.

These conceptual previews for skills will only be used for the four or five key skills for a grade level. When students are expected to master a skill, these lessons provide much better opportunities for students to develop proficiency than current curricular materials.

Practice should be delayed until after students have made sense of the step-by-step actions and how to record those actions in the form of an algorithm. Two or three problems about three times a week should enable the students to develop accuracy and fluency within a few weeks.

Guidelines For Conceptual Previews For Skills

- Select a model that can be used to encourage the thinking that matches the algorithm
- Ask students to solve a computation problem using that model
- Guide the students to use thinking that is consistent with the thinking needed for the symbolic procedures use with the algorithm
- Ask the students to explain their thinking

Lessons to encourage the thinking needed for the algorithm

Typically these lessons are structured as below (prior to the lesson, select a model to illustrate the thinking)

1. A computational problem is posed (10 seconds)
2. Students are asked to solve that computational problem using the model (30—40 seconds)
3. The teacher guides students to use thinking that is consistent with the algorithm with questions (1 minute 30 seconds)
4. A student is asked to explain the step-by-step actions with the model and confirm their answer (30 seconds)
5. The teacher summarizes the step-by-step actions (10 seconds)

1. ***A Computational Problem Is Posed (10 seconds)***
   Any computational problem can be posed by showing a number with the model. Then simply tell students their job is to add (subtract, multiply, or divide) and tell them the second number. No symbols need to be used, but the problem may be written symbolically if needed. Just do not use the symbolic procedure yet, that is, do not yet solve the problem with symbols.

2. ***Students Are Asked To Solve That Computational Problem Using The Model (30—40 seconds)***
   Ask students to use the model that is shown to solve the computation problem.

3. ***The Teacher Guides Students To Use Thinking That Is Consistent With The Algorithm (1 minute 30 seconds)***
   Use questions to guide the students to use thinking that is consistent with the algorithm. For example when dividing the teacher might ask, "What does it mean to divide by 3? (make 3 equal groups or make groups of 3) Today we are going to make 3 equal groups. How can we share the hundreds? How many hundreds should we put in each group? How many hundreds did we just use? What can we do with this left over hundred? ..."

4. ***A Student Is Asked To Explain The Step-By-Step Actions With The Model And Confirm Their Answer (30 seconds)***
   After solving the problem using step-by-step actions with the model, ask one student to explain that procedure. The teacher may want to demonstrate again while the student is explaining.

5. ***The Teacher Summarizes The Step-By-Step Actions (10 seconds)***
   After the explanation, the teacher provides a recap of the actions.

Sample Conceptual Preview For Division

Teacher	"Today, let's divide 735 by 3. We are going to use these base ten blocks to help us solve this problem. What does it mean to divide by 3? (make 3 equal groups or make groups of 3) Today we are going to make 3 equal groups. How many hundreds should we put in each group?
Jina	"2 hundreds."
Teacher	"How did you know that?"
Jina	"If you put 2 hundreds in each group, there's not enough to go around again."
Teacher	"How many hundreds did we use?"

Marissa	"6."
Teacher	"How many hundreds are left over?
Kaitlyn	"1."
Teacher	"Can we put that extra hundred in one of the groups?"
Jacque	"No, because then it wouldn't be even."
Teacher	"What can we do with this extra hundred so we can share it?"
Dorothy	"We can trade it for 10 tens."
Teacher	"Ok, then how many tens do we have to share?"
Peter	"13 in all. The 10 you just got from the trade and the 3 you started with."
Teacher	"How many tens should we put in each group?"
Carlos	"4."
Teacher	"How many tens did we use?"
Juan	"12, but there's 1 left over."
Teacher	"What should we do with that extra ten?"
April	"Trade it for 10 ones."
Teacher	"How many ones do we have to share?"
Marissa	"15."
Teacher	"How many ones should we put in each group?"
Juan	"5."
Teacher	"How many ones did we use?"
Antonio	"All 15 of them."
Teacher	"Ok, we just split 735 it into 3 equal group. What is 735 divided by 3?"
Kaitlyn	"145, because each pile has 145 in it."
Teacher	"Yes. We divided by using the following actions share the hundreds trade the extras for tens share the tens trade the extras for ones share the ones. That process will help us divide any numbers."

Summary

To understand a skill, students need to have a deep understanding of the concepts that form the basis of the skill. Even that may not be sufficient for them to have made sense of the step-by-step actions that are needed to solve computational problems using a model for that skill. A few very brief lessons to help them make sense of the step-by-step actions used with a model to solve the computation problem prepares them for learning how to record the symbolic algorithm. Once they can solve and explain the computation using models and they have learned to record the actions symbolically, with some practice they can develop better accuracy and fluency. It should be noted that when students have this deep understanding, little practice is needed.

This instruction is not much different from many reform materials, with one exception. They do not ask the students to solve one computation problem each day using models and with guided instruction to ensure they are making sense of the step-by-step actions. These very brief lessons may only take 3 or 4 minutes. Over two weeks, only about 30 minutes of instruction is provided. But, that 30 minutes pays huge dividends in terms of computational proficiency. Students with that deep understanding

- learn the symbolic algorithm more quickly,
- develop accuracy more quickly,
- develop fluency more quickly, and,
- retain the skill better, even after vacations.

Number sense lessons are essential for students to develop flexibility in using a skill. But, conceptual previews to help students make sense of specific thinking that is consistent with a standard algorithm are needed to develop better accuracy and fluency.

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Web Bytes

Iowa Council of Teachers of Mathematics
This site offers information about the organization as well as links to resources for teachers. Information about grant opportunities and deadlines to apply can also be found here. www.iowamath.org

STEM Collaborative
Make the math connection with four interactive online learning adventures for middle school classrooms. Students will enjoy these brain-boosting explorations that use digital media in innovative and creative ways. Virtual environments, simulations, videos and interactive math activities challenge and motivate students to actively engage in learning. Choose one, two or all of these great web resources; then watch your students’ knowledge soar. www.stemcollaborative.org

Shodor
Online interactive tools to create such things as stem-and-leaf plots, function machines, fractals, image tool, clocks and many, many more. www.shodor.org

National Council of Teachers of Mathematics
An enormous amount of resources for teachers to use in planning lessons. From classroom aides to Washington legislation, it is all at www.nctm.org