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A survey of butterfly diagrams for knots and links

Mark Ronnenberg
University of Northern Iowa

A SURVEY OF BUTTERFLY DIAGRAMS FOR KNOTS AND LINKS

An Abstract of a Thesis
Submitted
in Partial Fulfillment
of the Requirement for the Degree
Master of Arts

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ABSTRACT

A "butterfly diagram" is a representation of a knot as a kind of graph on the sphere. This generalization of Thurston's construction of the Borromean rings was introduced by Hilden, Montesinos, Tejada, and Toro to study the bridge number of knots. In this paper, we study various properties of butterfly diagrams for knots and links. We prove basic some combinatorial results about butterflies and explore properties of butterflies for classes of links, especially torus links. The Wirtinger presentation for the knot group will be adapted to butterfly diagrams, and we translate the Reidemeister moves for knot diagrams into so-called "butterfly moves." The main results of this paper are proofs for the classifications of 1- and 2-bridge links using butterflies. In particular, we prove that a link has bridge number equal to two if and only if it is a rational link. Our proof of this result requires the use of an object which we call a *weave*. We prove that a weave is equivalent to a rational tangle, and vice-versa. We conclude with a brief discussion of some open questions involving butterfly diagrams.

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Has been approved as meeting the thesis requirement for the

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Date	Dr. Theron Hitchman, Chair, Thesis Committee
Date	Dr. Olena Ostapyuk, Thesis Committee Member
Date	Dr. Bill Wood, Thesis Committee Member
Date	Dr. Kavita Dhanwada, Dean, Graduate College

To my parents, Douglas and Mary Ronnenberg. Without your unwavering support, I would never have gotten this far.

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CHAPTER 1

INTRODUCTION

In 2012, Hilden, Montesinos, Tejada and Toro (HMTT) introduced the concept of a *butterfly* as a way to represent knots and links [3]. Their idea is to generalize Thurston's construction of the Borromean Rings [10]. Thurston's construction is achieved selecting six line segments, one on each face of a cube, which cut each face in half. We then glue the faces of the cube together in a particular way. The result of the gluing is a copy of the 3-dimensional sphere S^3, with a link embedded within it.

Notice that the cube is homeomorphic to S^2. Thus HMTT construct their butterflies out of graphs embedded on S^2, which divide S^2 into polygonal faces. We select special arcs on each of these faces, which we will call *trunks*. When we glue the faces in a particular way, "folding" along the trunks, the result is a copy of S^3 with a link embedded within it.

HMTT developed the concept of butterflies in hopes of generalizing the classification of 2-bridge links to 3-bridge links. They lay the groundwork for this endeavor in [4]. We believe that to gain insight into this problem, it will be useful to reconstruct the classification of 2-bridge links using butterflies. This is the main thrust of this paper.

In Chapter 2 we review basic notions about knots and links. In Chapter 3, we will define precisely what a butterfly is, as defined by HMTT. We will alter their definition slightly for this paper. This altered definition will simplify some of the arguments we make throughout this paper. Toward the end of Chapter 3, we will prove some basic results about butterflies, including a method for determining the crossing number of a knot diagram from a butterfly diagram.

Chapter 4 will introduce some examples of butterflies, including a treatment of butterflies for torus links. In Chapter 5, we will discuss knot groups as they relate to butterflies. In particular, we will adapt the Wirtinger presentation for knot groups to butterflies, and derive a simplified method from this adaptation. We will use this simplified method to derive the well-known presentation of the knot group for torus links using butterflies.

An open problem, posed by HMTT, is to find a complete set of "butterfly moves" to transform a given butterfly diagram into an equivalent one. In Chapter 6, we develop butterfly moves which are analogous to the traditional Reidemeister moves for knot diagrams. At the end of the chapter, we prove that these butterfly moves, along with the trunk-reducing move introduced by HMTT, is enough to determine equivalence of butterfly diagrams.

The remaining chapters focus on the classification of links by bridge number. In particular, we reconstruct the classifications of 1- and 2-bridge links using butterflies. In Chapter 7, we show that every 1-butterfly corresponds to the unknot. Chapter 8 begins with a review of the theory of rational tangles. We then introduce a new object, which we call a *weave*. At the end of the chapter, we will prove the every weave is a rational tangle, and vice-versa.

The main result of this paper is proved in Chapter 9. We prove that any 2-butterfly corresponds to a rational link, and that any rational link has a 2-butterfly. Thus we construct a new proof that any 2-bridge link is a rational link, and vice-versa. We conclude this paper with a brief discussion of further questions in Chapter 10.

CHAPTER 2

KNOTS AND LINKS

This chapter will introduce basic ideas and notations needed throughout the rest of this paper. For comprehensive discussions of these topics and more, see [1].

A knot can be considered as an embedding of a circle S^1 into S^3. Here, we let S^3 the 3-dimensional sphere, or the one point compactification of $\mathbb{R}^3$. By an embedding, we mean a map $f : S^1 \to f(S^1) \subset S^3$, which is a homeomorphism. We will consider a knot as an equivalence class of embeddings of a circle. The equivalence relation between knots will be *ambient isotopy.*

Definition 2.1. *(Isotopy) Two embeddings $f_0, f_1 : S^1 \to S^3$ are said to be* isotopic *if there exists an embedding $F : S^1 \times I \to S^3 \times I$ such that $F(x,t) = (f(x,t),t)$, $x \in S^1$, $t \in I = [0,1]$, and $f(x,0) = f_0(x)$, $f(x,1) = f_1(x)$.*

The mapping F is called a level-preserving isotopy connecting f_0 to f_1*.*

It can be shown that any two embeddings $S^1 \to S^3$ are isotopic [1]. Thus the notion of isotopy is not strong enough to differentiate between different knots.

Definition 2.2. *(Ambient Isotopy) Two embeddings $f_0, f_1 : S^1 \to S^3$ are said to be* ambient isotopic *if there is a level-preserving isotopy $H : S^3 \times I \to S^3 \times I$, with $H(x,t) = (h_t(x),t)$, such that $f_1 = h_1 f_0$ and f_0 is the identity map on S^3. The mapping H is called an* ambient isotopy*.*

The notion of ambient isotopy is closer to what we seek. However, some strange cases are allowed by our definition of embedding, known as *wild knots* [1]. We do not want to consider wild knots in our study. We will only consider *tame knots*, that is, knots which are ambient isotopic to a simple closed polygon in S^3. To ensure that we will only have to deal with tame knots, we will restrict our maps to the piecewise linear category. Thus we restrict our definitions of embeddings and isotopies to be piecewise linear.

We can now state our definition for a knot. We shall consider a knot to be a piecewise linear ambient isotopy equivalence class of piecewise linear embeddings $S^1 \to S^3$.

Definition 2.3. *(Knot Equivalence) Two piecewise linear knots are* equivalent *if they are piecewise linear ambient isotopic.*

A piecewise linear embedding of a finite disjoint union of circles into S^3 is called a *link*. If a link is an embedding of n circles, then it is an *n-component link*. A knot is a link with one component. Often we will use the terms "knot" and "link" interchangeably. When we really mean knot, and not link, or vice-versa, we shall say so. From now on we will assume that we are working within the piecewise linear category. Thus we will omit the phrase "piecewise linear."

It will often be convenient to consider projections of knots onto a plane in $\mathbb{R}^3$. Instead of considering a knot as an embedding in S^3, we may consider it in $\mathbb{R}^3$ via stereographic projection. Thus a knot K is a simple closed polygon in $\mathbb{R}^3$. Let $p : \mathbb{R}^3 \to E$ be an orthogonal projection of $\mathbb{R}^3$ onto the plane E. A point $P \in p(K) \subset E$ whose preimage $p^{-1}(P)$ contains more than one point of K is called a *multiple point*.

Definition 2.4. *(Regular Projection) A projection p of a knot K is called* regular *if*

(i) There are only finitely many multiple points, and each multiple point is a double point, that is, it has exactly two points in its preimage.

(ii) No vertex of K is mapped onto a multiple point.

For example, Figure 2.1 is a regular projection of a trefoil knot.

A regular projection is not enough to uniquely determine a knot. We need to include more information. Imagine looking down onto a knot from the projection point for a regular projection. Some of the arcs cross over other arcs. The arcs which go over are called *overpasses*, and the arcs which go under are called *underpasses*. If in a regular projection we indicate which arc is the overpass and which arc is the underpass at each double point, by breaking the underpass, then the knot can be recovered. We will refer to

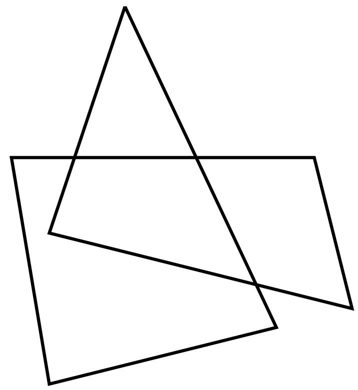


Figure 2.1: Regular projection of a trefoil

such a regular projection as a *knot diagram*. If orientation is also indicated, we will call such a projection an *oriented knot diagram*. In a knot diagram, double points will be called *crossings*. Figure 2.2 is a knot diagram for the trefoil, obtained from the regular projection in Figure 2.1.

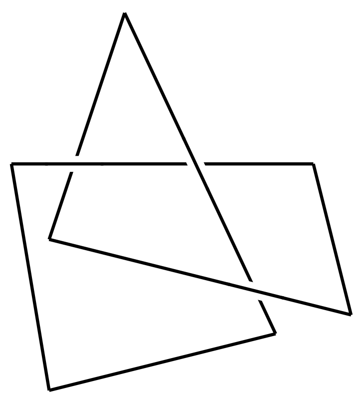


Figure 2.2: Knot diagram of a trefoil

Definition 2.5. *(Equivalence of Knot Diagrams) We say that two knot digrams are* equivalent *if they represent the same knot.*

We note that if two knot diagrams are related via a *planar isotopy*, that is, a continuous deformation in E which does not change any of the crossings of a diagram, then they are equivalent. This allows to draw knot diagrams with smooth curves, such as in Figure 2.3. This knot diagram corresponds to a trefoil, and is equivalent to the diagram in Figure 2.2. However, two equivalent knot diagrams may not in general be related by planar isotopy.

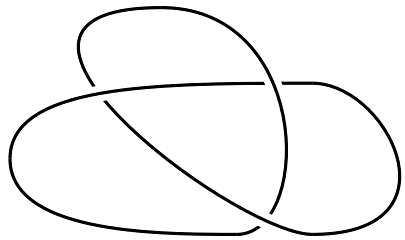
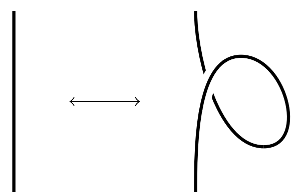


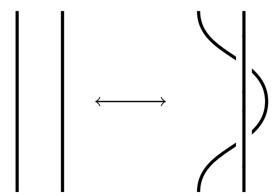
Figure 2.3: Knot diagram of a trefoil, which is equivalent to the diagram in Figure 2.2.

By a theorem of Reidemeister, there is another way to determine when two knot diagrams are equivalent. Consider the *Reidemeister moves* depicted in Figure 2.4. These are simple moves performed on a knot diagram which do not alter the topology of the corresponding knot. By Reidemeister's Theorem, the Reidemeister moves are enough to determine, up to ambient isotopy, whether two knot diagrams are equivalent.

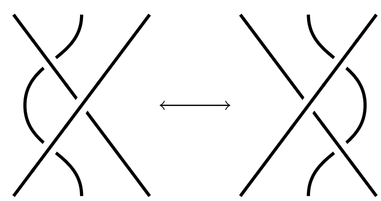
Theorem 2.6. *(Reidemeister's Theorem) Let D and D' be two knot diagrams. Then D and D' are equivalent if and only if they are related be a finite sequence of Reidemeister moves and planar isotopies.*



(a) Type I move



(b) Type II move



(c) Type III move

Figure 2.4: Reidemeister moves

CHAPTER 3

BUTTERFLY DIAGRAMS FOR LINKS

A *butterfly* diagram is a generalization of Thurston's construction of the Borromean Rings [10]. Butterflies were introduced in [3] by Hilden, Montesinos, Tejada, and Toro (HMTT) to study bridge numbers of links. The authors aimed to classify 3-bridge links, which is an open problem. For more on their efforts, see [4].

In the following discussion, and throughout the rest of the paper, we will use *k-cell* to denote an open k-dimensional disk. So, for example, a 1-cell is an open line segment, a 2-cell is an open disk, and a 3-cell is an open 3-ball.

Following HMTT, we will now lay the foundation for a definition of a butterfly. Let R be a connected graph embedded in $S^2 = \partial \mathbf{B}^3$, where $\mathbf{B}^3$ is a closed 3-cell, so that $S^2 - R$ is a disjoint union of open 2-cells. We will let C denote each open 2-cell generically.

For any $n \in \mathbb{N}$, let P_{2n} be the regular polygon that is the convex hull of the $2n^{th}$ roots of unity in the complex plane $\mathbb{C}$. We define a parametrization of C to be a function f from P_{2n} to the closure $\bar{C}$ of C, with the following properties:

1. The restriction of f to the interior of P_{2n} is a homeomorphism from the interior of P_{2n} to C.

2. The restriction of f to an edge of P_{2n} is a piecewise linear homeomorphism from that edge to an edge in the graph R.

3. The map induced by f which takes the set of edges of ∂P_{2n} to the set of edges of ∂C is at most 2 to 1.

Complex conjugation, $z \to \bar{z}$, restricted to P_{2n} or to the boundary of P_{2n}, defines an involution and an equivalence relation on the edges and vertices of P_{2n}, in which z_1 is equivalent to z_2 if and only if $\bar{z_1} = z_2$. This equivalence relation induces an equivalence relation on the edges and vertices of $\bar{C}$, and on the points of C. The equivalence relation on $\bar{C}$ induces an equivalence relation on $S^2 = \partial \mathbf{B}^3$. We denote the equivalence relation on a 2-cell C by $\sim$. We denote the equivalence relation on $S^2 = \mathbf{B}^3$, which is generated by the relations $\sim$, by $\simeq$. Then for all x and y in S^2, $x \simeq y$ if and only if there exists a finite sequence $x = x_1, \ldots, x_l = y$ with $x_i \sim x_{i+1}$ for $i = 1, \ldots, l-1$.

Each P_{2n} contains the line segment $[-1,1]$, which is the fixed point set of $z \to \bar{z}$ restricted to P_{2n}. The image of this line segment, $f_C([-1,1])$, is called the *trunk* t of C. A pair, (C,t), is called a *butterfly with trunk* t. The *wings* W and W' are just $f_C(P_{2n}\cap$ *upper half plane*) and $f_C(P_{2n}\cap$ *lower half plane*), and $W \cap W' = t$. We denote by T the collection of all trunks t over all C.

Let us denote by $M(R,T)$ the space $\mathbf{B}^3/\simeq$ with the topology of the identification map $p : \mathbf{B}^3 \to M(R,T)$, where $\simeq$ is the minimal equivalence relation generated by the equivalence relation $\simeq$ defined on S^2.

Equivalence classes of points of C, under $\sim$, contain two points except for those points in $f([-1,1])$, where there is only one point. If x is a vertex of R, its equivalence class under the equivalence relation $\simeq$, on $S^2 = \mathbf{B}^3$, is composed entirely of vertices of R. We classify the vertices as follows: A member of $R \cap T$ will be called an *A-vertex.* A member of $p^{-1}(p(v))$, $v \in R \cap T$, which is not an A-vertex will be called an *E-vertex.* A vertex of R which is neither an A-vertex nor an E-vertex will be called a B-vertex iff $p^{-1}(p(v))$ contains at least one non-bivalent vertex of R.

With these definitions in hand, we are ready to define an m-butterfly as HMTT do in [3].

Definition 3.1. *(HMTT m-butterfly) Let R and T be as above. For $m \geq 1$, an* m-butterfly *is a 3-ball $\boldsymbol{B}^3$ with m butterflies (C_i, t_i), $i = 1, \ldots, m$, on its boundary $S^2 = \partial \boldsymbol{B}^3$, such that R has only A-vertices, E-vertices, and B-vertices, each A-vertex and each E-vertex is bivalent in R, and T has m components.*

Now, for our purposes we wish to alter this definition slightly. First, we remove the condition that a B-vertex is such that $p^{-1}(p(v))$ contains a vertex which is non-bivalent. Instead, we will allow a B-vertex to simply be a vertex which is neither an A-vertex nor an

E-vertex. This will be useful to us later, in particular when we classify 1-butterflies.

Next, we want the graph R to be bipartite, with one part being the collection of B-vertices and the other part being the collection of A- and E-vertices. This restricts the butterfly graphs we are interested in to ones which can be obtained from the link-butterfly algorithm and applications of trunk-reducing moves, as we will describe later. We now make our definition of an m-butterfly, which we will use from now on.

Definition 3.2. *(m-butterfly) Let R and T be as above. For $m \geq 1$, an* m-butterfly *is a 3-ball $\boldsymbol{B}^3$ with m butterflies (C_i, t_i), $i = 1, \ldots, m$ on its boundary $S^2 = \partial \boldsymbol{B}^3$, such that (i) R has only A-vertices, E-vertices, and B-vertices, (ii) R is bipartite, with one part being A- and E-vertices, and the other part being B-vertices, (iii) the A- and E-vertices are bivalent in R, and (iv) T has m components.*

An m-butterfly can be represented by a plane graph, denoted by a pair (R, T), such that conditions (i), (ii), (iii), and (iv) are satisfied. We call such a plane graph a *butterfly diagram*. In Figure 3.1, we present a 3-butterfly diagram for the trefoil from Figure 2.3. Throughout this paper, thin black lines will represent the butterfly graph, thick black lines will represent trunks, stars will represent B-vertices, A-vertices are assumed at the ends of trunks, and E-vertices will be represented by disks. Note that in the 3-butterfly in Figure 3.1 there are no E-vertices.

The following theorem is important for understanding the relationship between butterflies and links embedded in 3-space.

Theorem 3.3. *([3]) For an m-butterfly (R, T), the space $M(R, T)$ is homeomorphic to S^3 and $p(T)$ is a knot or a link, where $p : \boldsymbol{B}^3 \to M(R, T)$ is the identification map.*

Often it will be necessary for us to consider distances between vertices on the boundary of a particular butterfly. However, we cannot simply use the standard graph metric. Instead, we want to think of measuring distance "along the boundary of the butterfly." To illustrate what we mean, we can imagine that the interior of a given butterfly is a body of water, and that its boundary is a coast. We can then liken measuring

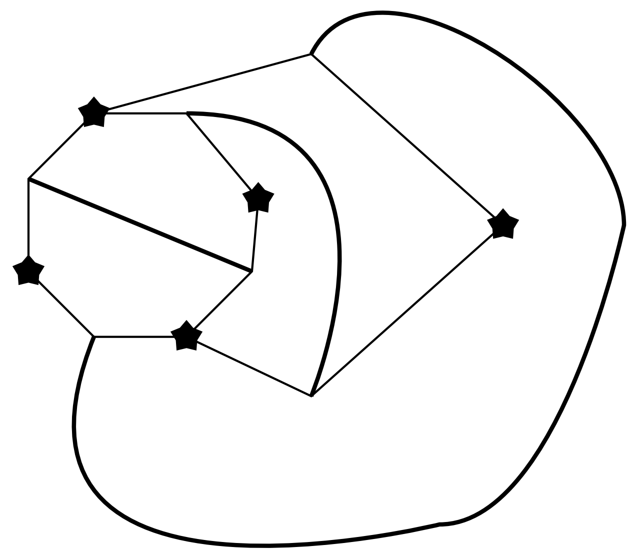


Figure 3.1: A 3-butterfly for the trefoil.

distance from two points on the boundary of the butterfly to measuring the length of the shoreline between them, which we do by counting graph edges. Consider, for example, the butterfly in Figure 3.2. Sticking with our water and coast analogy, we can measure the distance from α to β, traveling counterclockwise. If we imagine we are swimming from α to β, while remaining very close to the shoreline the entire time, the we must travel around the two edges which contain γ as an endpoint. In fact, in counting edges to measure distance, we will count these two edges twice. Thus the distance measured from α to β, traveling counterclockwise, is six. In fact, if we measure the distance from α to β clockwise, the distance is also six. In general, we should expect to obtain different values depending on whether we measured distance clockwise or counterclockwise.

Measuring distance in this way has the peculiar feature that we do not always obtain a value of zero when measuring distance from a vertex to itself. This is because in some cases it matters which "side" of a vertex we begin measuring from. For example, consider the vertex γ in Figure 3.2. We can measure the distance from γ to itself, beginning on the "trunk side," and traveling clockwise, to be equal to two. Note that we count the edge connecting γ to the univalent B-vertex twice.

When we measure distance as described above, we refer to it as *butterfly boundary distance.*

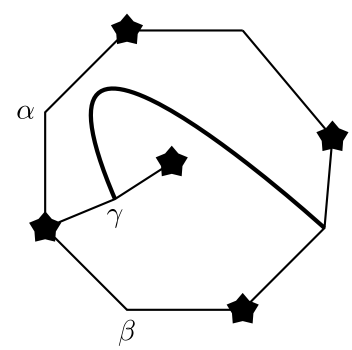


Figure 3.2: A butterfly with edges which must be double-counted.

Definition 3.4. *(t-equivalence) Let t be the trunk of a butterfly $\mathcal{B}$. Let a be an A-vertex of t. If two vertices α and α' of $\mathcal{B}$ are equidistant from a, on the "trunk side" of a, in butterfly boundary distance, then we say that α and α' are t-*equivalent.

***Remark* 3.5.** Note that if α and α' are t-equivalent, then $\alpha \sim \alpha'$, where $\sim$ is the equivalence relation on the butterfly containing t, as defined previously in this chapter.

We will now describe a practical algorithm for obtaining a knot diagram from a given butterfly diagram. This algorithm is referred to as the *butterfly-link* algorithm [3]. Consider an m-butterfly (R, T) which represents the knot K. For each $t \in T$ we perform the following steps. Identify t-equivalent pairs of A- or E-vertices on the boundary of the butterfly containing t. For each pair, draw an arc connecting the vertices which crosses t exactly once and never leaves the butterfly containing t. These are the arcs of the link which are obtained from the quotient map $p : \mathbf{B}^3 \to M(R,T)$. Consider a vertical line segment in P_{2n}, which connects two vertices of ∂P_{2n}. Then each arc we draw corresponds to the image of such a vertical line under p. These arcs will pass under the trunks in the resulting link diagram. The newly constructed arcs, together with the trunks, will make up a knot diagram for K. See [3] for details. Figure 3.3 shows the results of the butterfly-link algorithm on the 3-butterfly in Figure 3.1. The dotted lines represents the arcs drawn in the algorithm, which are under-arcs in the resulting diagram. The resulting diagram is equivalent to the trefoil diagram from Figure 2.3.

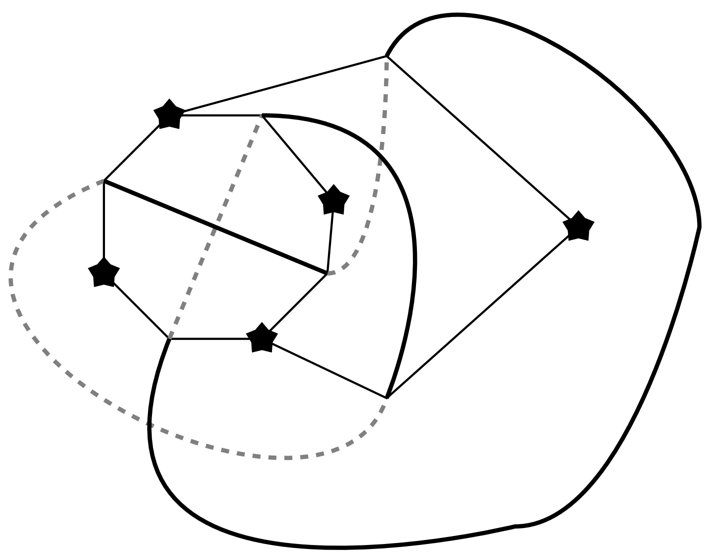


Figure 3.3: Butterfly-link algorithm on a 3-butterfly for the trefoil.

It is also possible to obtain a butterfly diagram from a given knot diagram. This process shall be called the *link-butterfly* algorithm. The link-butterfly algorithm is described in proving the following theorem, due to HMTT.

Theorem 3.6. *(Link-Butterfly Algorithm [3]) Every knot or link can be represented by an*

m-butterfly diagram, for some $m > 0$. Moreover, the m-butterfly can be chosen with no E-vertices.

Proof. Let L be a link, and D_L a diagram of L containing m arcs. Let $T = \{t_1, \ldots, t_m\}$ be the collection of disjoint arcs in D_L. Note that D_L divides the plane into components. In each bounded component, select a point b_i, $i \geq 1$. For the unbounded component, select a point $b_0 = \infty$. The $b_i's$ will be the B-vertices in the resulting butterfly diagram. The boundary points of the arcs t_i will be A-vertices of the diagram. Each A-vertex belongs to the boundary of two components. The B-vertices from these components are the A-vertex's *neighboring B-vertices.*

To construct an m-butterfly diagram (R, T), join each A-vertex to its neighboring B-vertices by arcs lying in the regions containing them. In this way we obtain a set of arcs R, and we assume that these arcs have mutually disjoint interiors amongst themselves and the arcs in T. Then (R, T) is an m-butterfly diagram. Note that the A-vertices are bivalent, and that there are no E-vertices. By applying the butterfly-link algorithm, we obtain a knot diagram equivalent to D_L. □

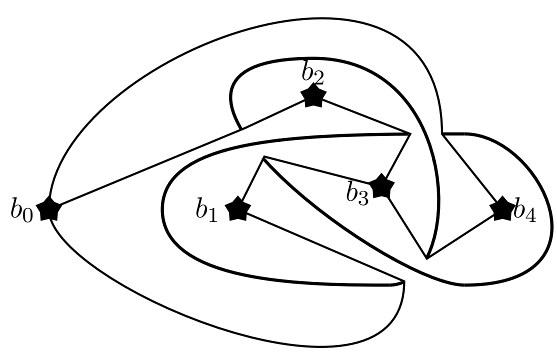


Figure 3.4: Applying the link-butterfly algorithm to a trefoil.

Consider a knot diagram D. An arc of D which is an overpass (that is, an arc which has an under-crossing) is called a *bridge* of D. We define a *maximal bridge* of D to be a bridge which is not contained in another bridge which has more under-crossings. Note that if an arc has no under-crossings at all, then we do not consider it as a bridge (nor as a maximal bridge). The *bridge number* of D is defined to be the number of maximal bridges in D. We can define bridge number for links.

Definition 3.7. *(Bridge Number) Let L be a link. We define the* bridge number *of L to be the least bridge number over all diagrams of L. We denote the bridge number of L by* $b(L)$.

In a similar vein, HMTT define what they call the *butterfly number* of a link.

Definition 3.8. *(Butterfly number) Let L be a link. The least m such that L has an m-butterfly diagram is called the* butterfly number of L. *We denote the butterfly number of L by* $m(L)$.

Theorem 3.9. *([3]) For any link L, the butterfly number of L is equal to the bridge number of L.*

As a consequence of Theorem 3.9, we can obtain results about the bridge number of a link by studying its butterfly number. This fact is of great importance when we study 1- and 2-bridge links, and is a motivating factor for HMTT's study of 3-butterflies in an attempt to classify 3-bridge links.

In their proof of Theorem 3.9, HMTT utilize what they call a *trunk-reducing move.* Let L be a link and $D = (R, T)$ an m-butterfly diagram of L obtained from the link-butterfly algorithm. Recall that D has no E-vertices. In fact, D will consist of only two types of butterflies. The butterflies which come from overpasses will have more than two A-vertices, while the trunks coming from underpasses will have exactly two A-vertices. The latter type of butterfly will be called a *simple butterfly.* Simple butterflies have four sides, see Figure 3.5. A trunk-reducing move allows us to replace a simple butterfly in D with an E-vertex. We have the following theorem.

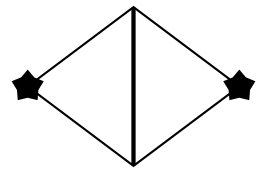


Figure 3.5: Simple butterfly.

Theorem 3.10. *([3]) A trunk-reducing move converts an m-butterfly diagram of a link L into an $(m-1)$-butterfly diagram of the same link L. The new diagram gets a new E-vertex in place of a simple butterfly.*

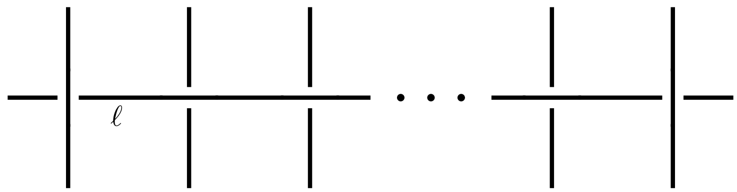
For details and proofs of Theorem 3.9 and Theorem 3.10, see [3]. A trunk-reducing move turns a given butterfly diagram into a different diagram of the same link. The notion of a link having more than one butterfly diagram is very important, and is analogous to the fact that a link has infinitely many knot diagrams. From this notion we can define *equivalence of butterfly diagrams.*

Definition 3.11. *(Equivalence of butterfly diagrams) Two butterfly diagrams are said to be* equivalent *if they represent the same link.*

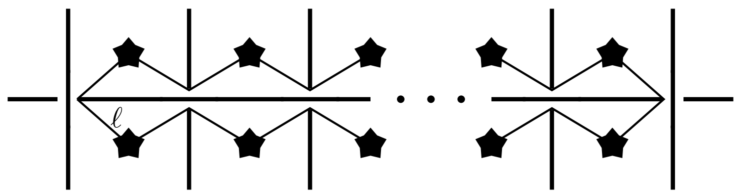
With the basic theory of butterflies laid out, we proceed to prove some simple combinatorial theorems about butterfly diagrams.

Theorem 3.12. *Let L be a link, and D a knot diagram of L, and $\mathcal{B}$ a butterfly diagram for L obtained from D by the link-butterfly algorithm. Let ℓ be an overpass of D. Then ℓ has n underpasses if and only if the butterfly corresponding to ℓ has $4(n+1)$ sides.*

Proof. First suppose ℓ has n underpasses. Apply the link-butterfly algorithm on D to obtain $\mathcal{B}$. In Figure 3.6 we show the application of the algorithm locally about ℓ. Recall that in measuring butterfly boundary distance, we do so by traveling along the boundary of a butterfly in some consistent orientation, and that we may have to double-count some edges. With this in mind, it can be shown that each underpass of ℓ will contribute four edges, including possible double-counting, to the butterfly of ℓ. Hence the underpasses contribute $4n$ edges. Also, the two arcs passing over ℓ will contribute two edges a piece to the butterfly for ℓ. Therefore the total number of edges of the butterfly of ℓ is $4n + 4 = 4(n + 1)$.



(a) n strands pass under the overpass ℓ.



(b) Butterfly corresponding to ℓ.

Figure 3.6: Constructing a butterfly on ℓ.

Now suppose the butterfly corresponding to ℓ in $\mathcal{B}$ has $4(n + 1)$ sides. The trunk splits the boundary of the butterfly in half, with n A-vertices on each half. Thus there are n pairs of A-vertices on the boundary of the butterfly which are identified across the trunk ℓ. In applying the butterfly-link algorithm, these n pairs will give us n underpasses of ℓ. □

Consider an oriented knot diagram D. If, as we trace out the diagram following its orientation, we alternate between overpasses and underpasses, then D is called an *alternating diagram*. If a link L has an alternating diagram, then we call L an *alternating link*. As a corollary to Theorem 3.12, we have the following.

Corollary 3.13. *Let L be an alternating link. Then L has a butterfly diagram in which each butterfly is an 8-gon.*

Proof. Since L is alternating, it has an alternating knot diagram. In this diagram, each overpass has exactly one underpass. Apply the link-butterfly algorithm. By Theorem 3.12, each butterfly will have $4(1+1) = 8$ sides. □

Let D be a knot diagram. We define the *crossing number* of D to be the number of crossings in D. Similarly to our definition of the bridge number of a link, we can define the crossing number of a link as follows.

Definition 3.14. *(Crossing Number) Let L be a link. We define the* crossing number *of L to be the least crossing number among all knot diagrams of L. We denote the crossing number of L by $cr(L)$.*

Our next theorem gives us a way to determine crossing numbers from butterflies.

Theorem 3.15. *Let L be a link and $\mathcal{B}$ a butterfly diagram of L. Let a and e be the number of A-vertices and E-vertices of $\mathcal{B}$, respectively. Then the crossing number of the link diagram corresponding to $\mathcal{B}$ is $\frac{a}{2} + e$.*

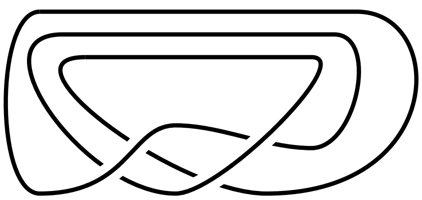
Proof. Expand each E-vertex of $\mathcal{B}$ into a simple butterfly via inverse trunk-reducing moves. Note that each simple butterfly contains exactly two A-vertices. Let n be the number of A-vertices in the resulting butterfly diagram. Let D be the link diagram corresponding to $\mathcal{B}$ via the butterfly-link algorithm. The identification of two A-vertices across a trunk corresponds to a crossing in D. Thus D has $\frac{n}{2}$ crossings. Now we return to the original butterfly diagram $\mathcal{B}$ by performing trunk-reducing moves on each of the simple butterflies we obtained earlier. Since there were e E-vertices in $\mathcal{B}$, we will perform e trunk-reducing moves. Each move eliminates two A-vertices. So the number of A-vertices in $\mathcal{B}$ is given by $a = n - 2e = 2(\frac{n}{2} - e)$. It follows that $\frac{n}{2} = \frac{a}{2} + e$. Since neither a trunk-reducing move nor its inverse alters the link diagram D, we conclude that the crossing number of D is given by $\frac{a}{2} + e$. □

As an immediate corollary, it is possible to compute the crossing number of a link from its butterfly diagrams.

Corollary 3.16. *Let L be a link. For a butterfly diagram of L, let a and e be the number of A-vertices and E-vertices, respectively. Then $cr(L)$ is the least value of $\frac{a}{2} + e$ taken over all butterfly diagrams of L.*

CHAPTER 4
BUTTERFLIES FOR CLASSES OF LINKS

In this section we present descriptions for butterflies of some important classes of links. We begin with a study of torus links. A torus link is a link which can be embedded on a torus without self-intersection. A familiar example is the trefoil knot of Figure 2.3. In particular, we will now describe a construction for a (p, q)-torus link, where p and q are integers and $p > 0$. Construct p horizontal parallel strands. For $q > 0$, the bottom most q strands are wrapped up and over the remaining strands, one at a time. For $q < 0$, the upper most q strands are wrapped down and over the remaining strands, one at a time. Once the wrapping is complete, close off the strands so that no new crossings are created. See Figure 4.1 for some examples.


$(3, 2)$

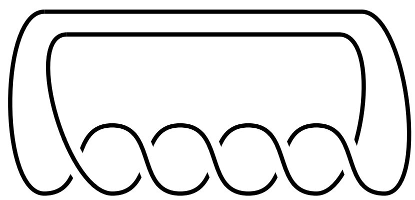
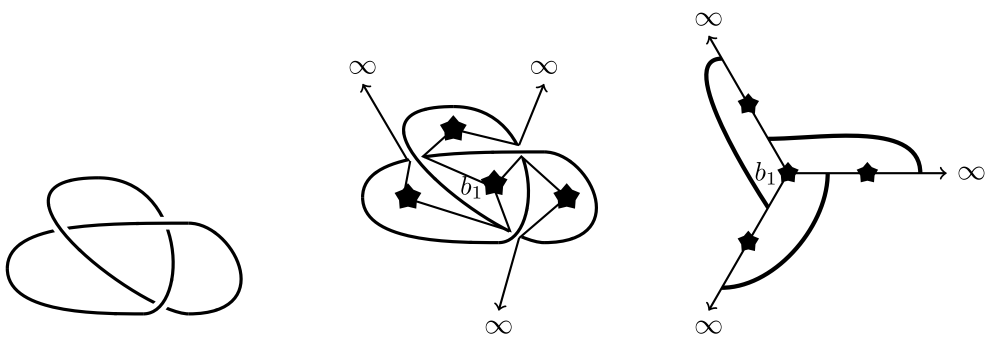

$(2, -5)$

Figure 4.1: Some examples of torus link diagrams

From this canonical diagram, we can construct a canonical butterfly diagram for a (p, q)-torus link. Our canonical butterfly diagram is obtained by applying the link-butterfly algorithm to a canonical knot diagram. In applying the algorithm, the B-vertex ∞ will be actually be taken as the north pole on $S^2 = \partial\mathbf{B}^3$. In projecting S^2 down to the plane, the B-vertex ∞ will really be the point at infinity of the plane. The butterfly diagram obtained from the algorithm will likely contain simple butterflies. We will replace these with E-vertices via trunk-reducing moves. The resulting butterfly diagram will consist of $|q|$ arcs, with endpoints at two B-vertices, one of which is $b_0 = \infty$

(since it lies in the unbounded component of the projection plane), the other we call b_1. These two B-vertices will each have degree $|q|$. Each arc of the graph will contain exactly two A-vertices, and $p-2$ E-vertices. The diagram will contain $2q + q(p-2) = pq$ total A- and E-vertices. For $q > 0$, the trunks will, moving from b_1 toward ∞, start on an A-vertex nearest to b_1, and end on the A-vertex on the adjacent arc which is nearest to $b_0 = \infty$ (moving counterclockwise about b_1). See Figure 4.2 and Figure 4.3.



(a) (b) (c)

Figure 4.2: Obtaining a canonical butterfly diagram for the $(2, 3)$-torus knot from its canonical knot diagram.

With our canonical diagrams in hand, we cite a few useful results without proof. The interested reader may find details in [7].

Theorem 4.1. *Let L_1 be a (p, q)-torus link.*

(i) Suppose $q > 0$. Let L_2 be a (q, p)-torus link. Then L_1 is equivalent to L_2.

(ii) Let L_2 be a $(p, -q)$-torus link. Then L_2 is the mirror image of L_1.

Our canonical butterfly diagram for torus links allows us to give a new proof for a basic result about torus links. First we prove the following lemma.

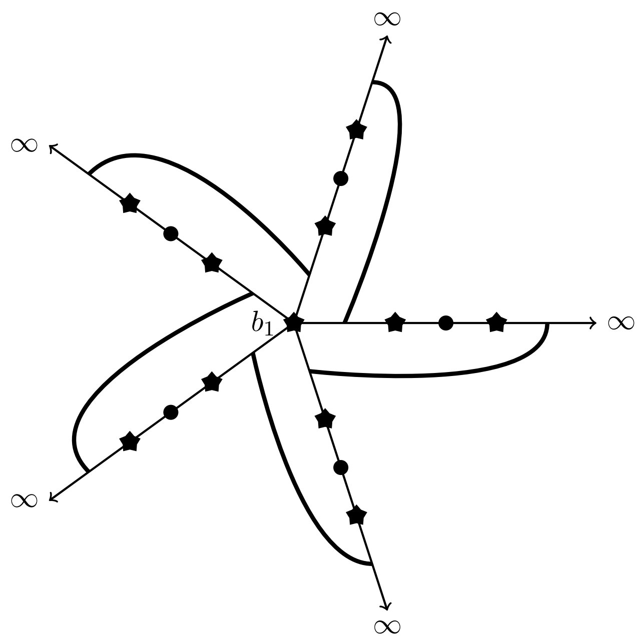


Figure 4.3: Canonical butterfly diagram for a (3,-5)-torus link.

Lemma 4.2. *Let L be a (p, q)-torus link and $\mathcal{B}_L$ a canonical butterfly diagram for L. Let a be an A-vertex of $\mathcal{B}_L$. Then the equivalence class of a, under the equivalence relation $\simeq$ on $S^2 = \boldsymbol{B}^3$, contains exactly p elements.*

Proof. Without loss of generality, we may assume $q > 0$. For the case $q < 0$, a similar argument will apply, the only difference being that we label trunks and vertices in a counterclockwise fashion.

We label the trunks of $\mathcal{B}_L$ as t_0, t_1, …, t_{q-1}, going clockwise about the B-vertex b_1. Next we orient each trunk of $\mathcal{B}_L$, so that the corresponding link diagram is an oriented diagram. In particular, we will orient each trunk as "moving away from" b_1. For each trunk t_i, we denote by a_i and a_i' the incoming and outgoing A-vertices of t_i, respectively. In Figure 5.6, the omitted sections of each ray, represented by three dots, are the segments containing the $p-2$ E-vertices. We note that these sections contain $2(p-2)$ edges.

Fix $i \in \{0, \ldots, p-1\}$. We will determine the size of the equivalence class of a_i'. As a result of the symmetric properties of $\mathcal{B}_L$, this will be sufficient to determine the size of the equivalence class of any A-vertex of $\mathcal{B}_L$.

Note that the butterfly boundary distance of a'_i to b_1 is $2 + 2(p-1) + 1 = 2p - 1$. Also note that a'_i is t_{i+1}-equivalent to some vertex e_1, which is either an E-vertex or an A-vertex. Since the butterfly boundary distance from b_1 to a'_i is $2p-1$, and the butterfly boundary distance from b_1 to a_{i+1} is 1, we obtain that the butterfly boundary distance from b_1 to e_1 must be $2p - 1 - 2 = 2p - 3$.

Similarly, for some j, we have e_j is t_{i+j+1}-equivalent to e_{j+1}, and that the butterfly boundary distance from b_1 to e_{j+1} is $2p - 1 - 2(j+1)$, where the trunk indices are taken modulo q. Now, note that the equivalence class of a'_i must contain another A-vertex. Then for some ℓ we obtain $e_\ell = a_j$, for some j. In particular, note that the butterfly boundary distance from b_1 to a_j is 1. Thus we want to find ℓ such that $2p - 1 - 2\ell = 1$. But then we see that $p = \ell$. So the sequence $a'_i = e_0, e_1, e_2, \ldots, e_{p-2}, e_{p-1} = a_j$ is such that the corresponding sequence of butterfly boundary distances to b_1 is $2p-1$, $2p-3$, $2p-5$, …, 3, 1. Thus the equivalence class of a'_i contains exactly p elements. □

Theorem 4.3. *Let L be a (p, q)-torus link. Let $d = gcd(p, q)$. Then L has d link components.*

Proof. Let $\mathcal{B}_L$ be the canonical butterfly diagram for L. Without loss of generality, suppose $q > 0$, since a similar argument applies if $q < 0$. Note that by Lemma 4.2, each equivalence class of A- or E-vertices under $\simeq$ contains exactly p elements.

We will count the components of L from $\mathcal{B}_L$. Let a_0 be an A-vertex of $\mathcal{B}$, which is nearer to $b_0 = \infty$ than to b_1 in butterfly boundary distance. Starting at the arc containing a_0, and traveling clockwise about b_1, label the arcs in ascending order modulo q. Then the arc containing a_0 is labeled "0". To trace out the component of L corresponding to a_0's equivalence class, we do the following. Trace out the equivalence class of a_0 clockwise from a_0. We will eventually reach an A-vertex. If it is on the trunk containing a_0, we are done. If not, then we move to the A-vertex on the other end of the trunk and trace out its equivalence class in the same way. Eventually, we will end up at the trunk containing a_0.

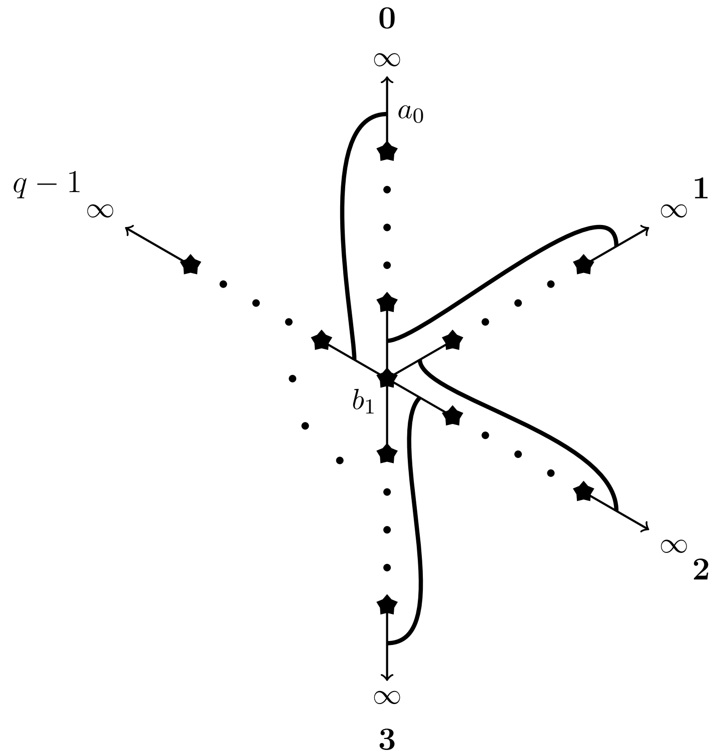


Figure 4.4: Canonical butterfly diagarm for a (p, q)-torus link, with graph arcs labeled.

When this happens, we will have traced out the component corresponding to a_0.

Now, we would like to know how many trunks are in each component. Again, by symmetry of $\mathcal{B}_L$, each component should contain the same number of trunks. In tracing a_0's component, the process terminates when we arrive at an A-vertex lying on the arc labeled "0". Since each equivalence class of A- and E-vertices contains p elements, we can find a positive integer ℓ such that $p\ell \equiv 0 \pmod q$. In particular, let ℓ be least such that this property holds. In other words, let $p\ell = lcm(p, q)$. Thus there are ℓ trunks in each component of L. There are q total trunks, so there are q/ℓ components of L. Now recall that $p\ell = lcm(p, q) = \frac{pq}{d}$. It follows that $\ell = \frac{q}{d}$, so $q/\ell = d$. Therefore L has d components.

□

The next class of knots we consider is the *twist knots*. Let n be a positive integer. A *twist knot of n half-twists*, which we denote by T_n, is obtained by twisting two parallel strands n times and closing them up so that the resulting knot is alternating. For example, a knot diagram of T_2 is depicted in Figure 4.5.

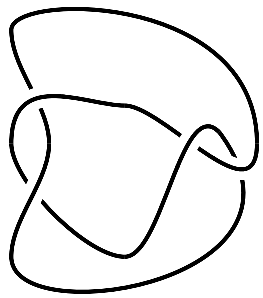
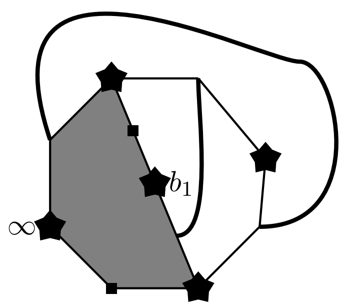
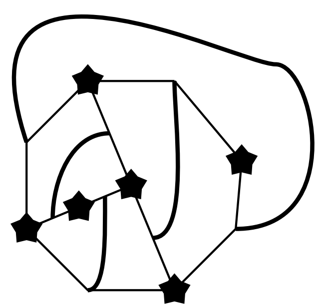


Figure 4.5: The twist knot T_2.



(a) General twist knot T_n. The squares represent A-vertices.



(b) The twist knot T_2.

Figure 4.6: Canonical butterfly diagrams for twist knots.

By applying the link-algorithm to a twist knot T_n, we can obtain a canonical butterfly diagram for twist knots. We present the canonical diagram in Figure 4.6a. The shaded region of Figure 4.6a will contain n 8-sided butterflies "stacked on top of each other." They are to be constructed as follows. Construct $n-1$ paths connecting the

B-vertices $b_0 = \infty$ and b_1 which intersect only at their endpoints. Each of these paths will contain four edges, including two A-vertices and one B-vertex. As a result, the B-vertices ∞ and b_1 will each have degree $n+1$. There is only one way to place the trunks in the resulting polygonal faces so that we obtain a proper $(n+2)$-butterfly diagram. The n 8-sided butterflies in the shaded region correspond to the n twists in the knot. For an example of a canonical butterfly diagram for T_2, see Figure 4.6b. We encourage the reader to apply the butterfly-link algorithm to Figure 4.6b and to compare the result with the knot diagram of T_2 in Figure 4.5.

CHAPTER 5

KNOT GROUPS AND BUTTERFLIES

Recall that knots are embedded in S^3 (or $\mathbb{R}^3$). It turns out that one can learn a lot about a knot by studying its complement in S^3. The famous Gordon-Luecke Theorem [2] states that if two knots have complements which are orientation-preserving homeomorphic, then the knots are isotopic, hence equivalent. This results does not hold for links in general, so in this section we will restrict our attention to knots. A powerful tool for studying knot complements is the *fundamental group* of the complement. The fundamental group of a knot complement will be referred to as the *knot group* of a knot.

It is known that homeomorphic spaces have isomorphic fundamental groups. Thus equivalent knots have isomorphic knot groups. Knot groups therefore are useful in distinguishing distinct knots. In this section we will develop methods for computing knot groups from butterflies based on the well-known *Wirtinger presentation.* The following discussion of the Wirtinger presentation is based heavily on [1].

The Wirtinger presentation is a way to present a knot group via generators and relations. We will describe the general method of the Wirtinger presentation, and then we will develop the method for butterflies. To obtain a Wirtinger presentation for a given knot, we will consider consider a knot diagram. We must put an orientation on the diagram. Each strand of the diagram is assigned a label s_i. These will be the generators of the knot group. The relations are obtained from the crossings of the diagram. Beginning with the under-arc which is leaving the crossing, travel counterclockwise on a small circle about the crossing, reading off the labels of the strands. The strands which are leaving the crossing are assigned an exponent of -1, whereas the strands that are entering the crossing are assigned an exponent of 1. In this way each crossing gives us a word in the generators of the knot group, which is taken as a relation.

This method is summarized in the following theorem.

Theorem 5.1. *(Wirtinger Presentation) Let s_i, $i = 1, 2, \ldots, n$ be the strands of an oriented knot diagram of a knot K. Then the knot group of K admits the following (Wirtinger) presentaion,*

$$\mathfrak{G}(K) = < s_1, \ldots, s_n | r_1, \ldots, r_n >,$$

where $r_i = s_i^{-1} s_j^{-\epsilon_i} s_k s_j^{\epsilon_i}$, as in Figure 5.1.

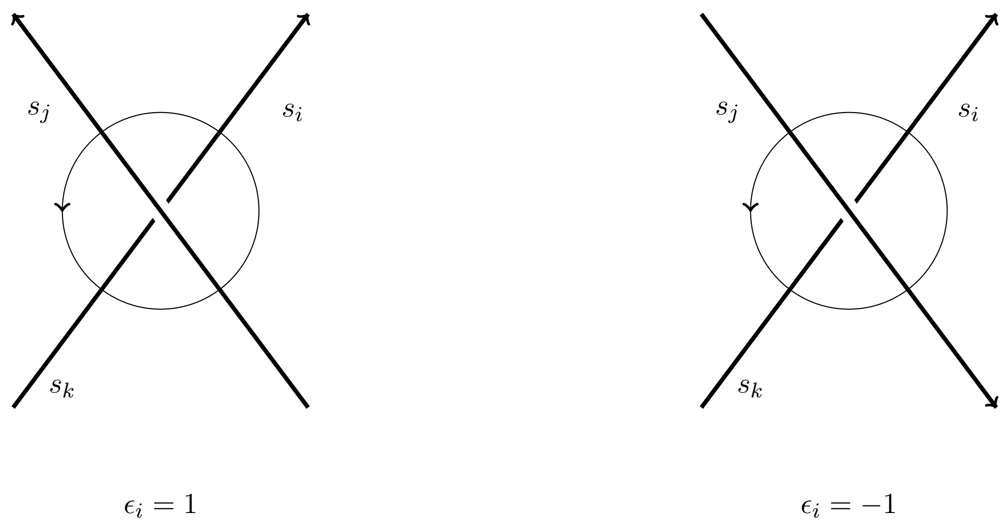


Figure 5.1: Obtaining relations for the Wirtinger presentation.

As a corollary, we have the following result, which is useful for simplifying a Wirtinger presentation.

Corollary 5.2. *Let K be a knot and $\mathfrak{G}(K) = < s_1, \ldots, s_n \mid r_1, \ldots, r_n >$ be a Wirtinger presentation for the knot group of K. Then each relation r_i is a consequence of the other relations r_j, $j \neq i$.*

***Example* 5.3.** Let us compute a Wirtinger presentation for the trefoil of Figure 2.3. Orient the knot diagram and label its strands and crossings as in Figure 5.2.

At each crossing, we perform the procedure outlined above for computing relations of the knot group. At crossing α, we obtain the relation $s_3^{-1} s_1 s_2 s_1^{-1}$. At β we get

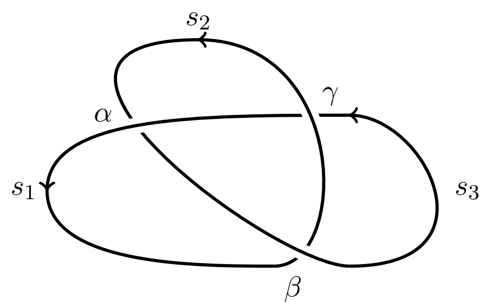


Figure 5.2: Oriented knot diagram for a trefoil. Crossings and strands are labeled for the Wirtinger presenation.

$s_2^{-1}s_3s_1s_3^{-1}$, and at γ we get $s_1^{-1}s_2s_3s_2^{-1}$. Then we obtain a Wirtinger presentation for the knot group, $\mathfrak{G}(K) =< s_1,\ s_2,\ s_3 \mid s_3^{-1}s_1s_2s_1^{-1},\ s_2^{-1}s_3s_1s_3^{-1},\ s_1^{-1}s_2s_3s_2^{-1} >$.

We will now simplify this presentation, utilizing Tietze transformations [9]. Tietze transformations allow us to eliminate generators and relations from the presentation of $\mathfrak{G}(K)$. By Corollary 5.2, we know any one of the relations of $\mathfrak{G}(K)$ is a consequence of the others. We will eliminate $s_1^{-1}s_2s_3s_2^{-1}$. Note that we could eliminate any of the three.

Now we can use $s_2^{-1}s_3s_1s_3^{-1}$ to solve for $s_2 = s_3s_1s_3^{-1}$. Doing this eliminates the relation $s_2^{-1}s_3s_1s_3^{-1}$ from the group presentation. We can then substitute this identity for s_2 into the relation $s_3^{-1}s_1s_2s_1^{-1}$ to obtain $s_3^{-1}s_1s_3s_1s_3^{-1}s_1^{-1}$. This can be rewritten as $s_1s_3s_1s_3^{-1}s_1^{-1}s_3^{-1}$. Thus we obtain a Wirtinger presentation for the trefoil as $\mathfrak{G}(K) =< s_1, s_3 \mid s_1s_3s_1s_3^{-1}s_1^{-1}s_3^{-1} >$.

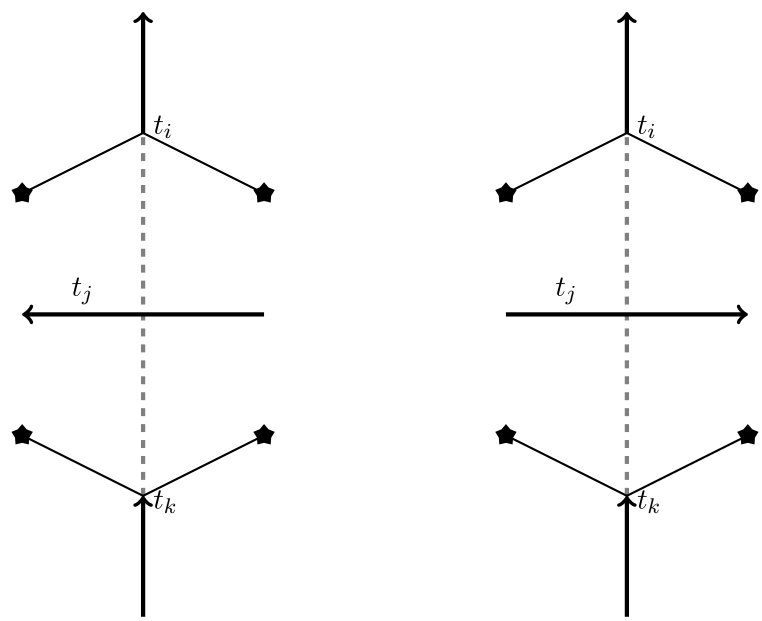
Hinting towards a result that we shall prove at the end of this chapter, we can use this presentation to obtain another, "nicer" presentation. Let $x = s_1s_3$, and $y = s_3s_1s_3$. Note that $y^{-1} = s_3^{-1}s_1^{-1}s_3^{-1}$. Then we have

$$x^3y^{-2} = (s_1s_3s_1s_3s_1s_3)\left(s_3^{-1}s_1^{-1}s_3^{-1}s_3^{-1}s_1^{-1}s_3^{-1}\right) = s_1s_3s_1s_3^{-1}s_1^{-1}s_3^{-1},$$

which is precisely the relation we obtained for our simplified Wirtinger presentation. Thus we obtain a new presentation for the knot group of the trefoil, $\mathfrak{G}(K) =< x, y \mid x^3y^{-2} >$.

We will now adapt the Wirtinger presentation for butterfly diagrams. Let K be a knot and $\mathcal{B}_K$ a butterfly diagram for K. We assume that $\mathcal{B}_K$ contains no E-vertices, since we can always use inverse trunk-reducing moves to obtain an equivalent butterfly diagrams with E-vertices replaced by simple butterflies. Let t_1, t_2, …, t_m be the trunks of $\mathcal{B}_K$. The trunks of $\mathcal{B}_K$ will correspond to generators of $\mathfrak{G}(K)$. Assign a consistent orientation to the trunks of $\mathcal{B}_K$. That is, assign orientations to the trunks in such a way that if we apply the butterfly-link algorithm on $\mathcal{B}_K$, the resulting knot diagram is oriented. To obtain relations for $\mathfrak{G}(K)$, we will perform the following procedure for each pair of t_j-equivalent A-vertices.

Let a_i and a_k be A-vertices of $\mathcal{B}_K$ which are t_j-equivalent. Suppose that a_i is an endpoint of the trunk t_i and a_k is an endpoint of the trunk t_k. Without loss of generality, suppose t_i is oriented so that it is leaving the butterfly containing t_j. Then t_k must be oriented so that it is entering the butterfly containing t_j. We then obtain a relation $r_i = t_i^{-1} t_j^{=\epsilon_i} t_k t_j^{\epsilon_i}$, where ϵ_i is determined via Figure 5.3.



$\epsilon_i = 1$ $\epsilon_i = -1$

Figure 5.3: Obtaining relations for the Wirtinger presentation from a butterfly diagram.

Since the identification of A-vertices across a trunk corresponds to a crossing in a knot diagram, and each trunk corresponds to a strand, it is easy to see that this process is equivalent to the standard method for obtaining a Wirtinger presentation. We would like to simplify the process by eliminating unnecessary generators from the presentation. It turns out we can eliminate the generators corresponding to simple butterflies (and E-vertices).

Theorem 5.4. *Let K be a knot and $\mathcal{B}_K$ an oriented butterfly diagram of K with no E-vertices. Let t_1, …, t_m be the trunks of $\mathcal{B}_K$ which are not contained in simple butterflies. Then the knot group of K admits the presentation*

$$\mathfrak{G}(K) =< t_1, \ldots, t_m | R_1, \ldots, R_m >,$$

where $R_i = \left(t_i t_{j_1}^{\epsilon_{j_1}} \ldots t_{j_n}^{\epsilon_{j_n}}\right) t_k^{-1} \left(t_{j_n}^{-\epsilon_{j_n}} \ldots t_{j_1}^{-\epsilon_{j_1}}\right)$, *as in Figure 5.4, and with ϵ_{j_l} determined as in Figure 5.3 for $l = 1$, …, n.*

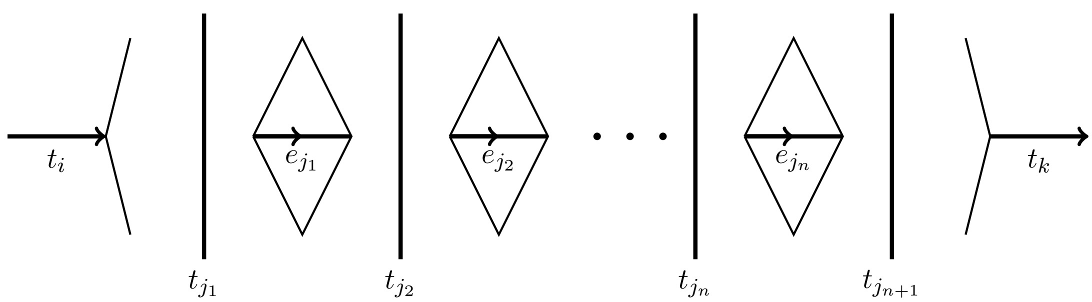


Figure 5.4: Obtaining a relation for a presentation of $\mathfrak{G}(K)$ from the equivalence class of a_i under $\simeq$.

Proof. Let e_1, …, e_ℓ be the trunks of $\mathcal{B}_K$ which are contained in simple butterflies. By performing trunk-reducing moves to each e_l we obtain a new butterfly diagram $\mathcal{B}'_K$ which is equivalent to $\mathcal{B}_K$ and which contains no simple butterflies. Let e'_1, …, e'_ℓ be the E-vertices of $\mathcal{B}'_K$, with e'_l obtained by performing a trunk-reducing move on the simple butterfly of $\mathcal{B}_K$ containing e_l.

Consider the trunk t_i, which does not belong to a simple butterfly. Let a_i be the "outgoing" A-vertex of t_i, in terms of t_i's orientation. We want to determine which vertices belong to the equivalence class under $\simeq$ of a_i. To do this we note t_l-equivalent pairs of vertices, starting with a_i, and following the implied orientation of the corresponding knot diagram. The final pair of t_l-equivalent vertices will contain an A-vertex, a_k, which is the endpoint of some trunk t_k which does not belong to a simple butterfly. a_k is the "incoming" A-vertex of t_k. We let $A_i = (a_i, e'_{j_1}, e'_{j_2}, \ldots, e'_{j_n}, a_k)$, a $(n+2)$-tuple of A- and E-vertices of $\mathcal{B}'_K$ which make up the equivalence class of a_i under $\simeq$. We let $T_i = (t_{j_1}, \ldots, t_{j_{n+1}})$ be the $(n+1)$-tuple of trunks of $\mathcal{B}'_K$ such that a_i is t_{j_1}-equivalent to t_{j_1}, e'_{j_l} is $t_{j_{l+1}}$-equivalent to $e'_{j_{l+1}}$ for $1 \leq l \leq n-1$, and e'_{j_n} is $t_{j_{n+1}}$-equivalent to t_k.

Now we turn $\mathcal{B}'_K$ back into $\mathcal{B}_K$ via inverse trunk-reducing moves. We note that the trunks T_i can be recovered in $\mathcal{B}_K$, so we can label the non-simple trunks in the same way. Now by applying the Wirtinger presentation for butterflies, we obtain a presentation for $\mathfrak{G}(K)$ with generators $t_1, \ldots, t_m$ and $e_1, \ldots, e_\ell$, and a set of relations. For the trunks in T_i, we obtain the following relations: $t_i t_{j_1}^{\epsilon_{j_1}} e_{j_1}^{-1} t_{j_1}^{-\epsilon_{j_1}}$, $e_{j_1} t_{j_2}^{\epsilon_{j_1}} e_{j_2} t_{j_2}^{-\epsilon_{j_1}}$, $\ldots$, $e_{j_l} t_{j_{l+1}}^{\epsilon_{j_{l+1}}} e_{j_{l+1}}^{-1} t_{j_{l+1}}^{-\epsilon_{j_{l+1}}}$, $\ldots$, $e_{j_n} t_{j_{n+1}}^{\epsilon_{j_{n+1}}} t_k^{-1} t_{j_{n+1}}^{-\epsilon_{j_{n+1}}}$.

Now, using these relations, we can solve for the generators corresponding to simple butterflies. From the first relation we listed we obtain $e_{j_1} = t_{j_1}^{-\epsilon_{j_1}} t_i t_{j_1}^{\epsilon_{j_1}}$. Then for $2 \leq l \leq n-1$, we obtain $e_{j_{l+1}} = t_{j_{l+1}}^{-\epsilon_{j_{l+1}}} e_{j_l} t_{j_{l+1}}^{\epsilon_{j_{l+1}}}$. Note that in the process of solving these relations, we can eliminate the generators e_{j_l}, $l = 1, \ldots, n$ from the presentation of $\mathfrak{G}(K)$, along with the relations.

This leaves us with the generators corresponding to the non-simple trunks, and one relation, $e_{j_n} t_{j_{n+1}}^{\epsilon_{j_{n+1}}} t_k^{-1} t_{j_{n+1}}^{-\epsilon_{j_{n+1}}}$. We substitute the identities for the generators e_{j_l}, $l = 1$, $\ldots$, $n-1$, into this relation to obtain the relation

$$\left(t_{j_n}^{-\epsilon_{j_n}} t_{j_{n-1}}^{-\epsilon_{j_{n-1}}} \ldots t_{j_2}^{-\epsilon_{j_2}} t_{j_1}^{-\epsilon_{j_1}}\right) t_i \left(t_{j_1}^{\epsilon_{j_1}} t_{j_2}^{\epsilon_{j_2}} \ldots t_{j_{n-1}}^{\epsilon_{j_{n-1}}} t_{j_n}^{\epsilon_{j_n}} t_{j_{n+1}}^{\epsilon_{j_{n+1}}} t_k^{-1} t_{j_{n+1}}^{-\epsilon_{j_{n+1}}}\right).$$

We can rewrite this relation as

$$\left(t_i t_{j_1}^{\epsilon_{j_1}} t_{j_2}^{\epsilon_{j_2}} \dots t_{j_{n-1}}^{\epsilon_{j_{n-1}}} t_{j_n}^{\epsilon_{j_n}} t_{j_{n+1}}^{\epsilon_{j_{n+1}}}\right) t_k^{-1} \left(t_{j_{n+1}}^{-\epsilon_{j_{n+1}}} t_{j_n}^{-\epsilon_{j_n}} t_{j_{n-1}}^{-\epsilon_{j_{n-1}}} \dots t_{j_2}^{-\epsilon_{j_2}} t_{j_1}^{-\epsilon_{j_1}}\right),$$

as desired. □

***Example* 5.5.** As an application of Theorem 5.4, let us compute a presentation for the knot group of the (3,2)-torus knot. We orient and label the trunks of the butterfly diagram in Figure 5.5. The generators of the knot group will correspond to the trunks t_1 and t_2. We also label the E-vertices as e_1 and e_2, and we label the A-vertices as a_1, a_1', a_2, and a_2'. b_0 and ∞ are B-vertices as in the construction of the canonical butterfly diagram as described in Chapter 4.

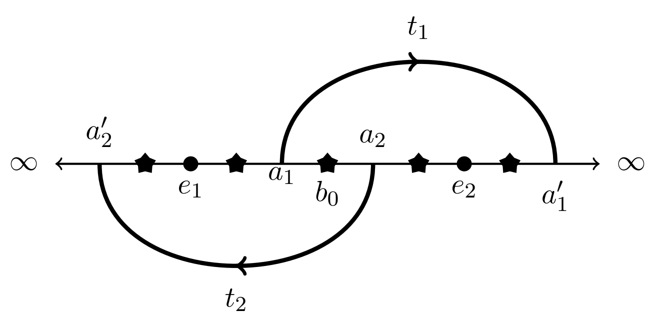


Figure 5.5: Oriented (3,2)-torus knot butterfly diagram, labeled for the Wirtinger presentation.

To determine the relations R_1 and R_2, we must first determine the equivalence classes, under the relation $\simeq$ on $S^2 = \mathbf{B}^3$, of the A- and E-vertices of the diagram. By following the orientation of t_1, we obtain $A_1 = (a_1', e_1, a_2)$ and $T_1 = (t_2, t_1)$, as in the proof of Theorem 5.4. That is, a_1' is t_2-equivalent to e_1, and e_1 is t_1-equivalent to a_2. Similarly, we have $A_2 = (a_2', e_2, a_1)$ and $T_2 = (t_1, t_2)$.

Thus we obtain the relations $R_1 = s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1}$ and $R_2 = s_2 s_1 s_2 s_1^{-1} s_2^{-1} s_1^{-1}$. Thus we obtain a Wirtinger presentation for $\mathfrak{G}(K)$ as $\mathfrak{G}(K) =< s_1, s_2 \mid R_1, R_2 >$. Note That $R_2 = R_1^{-1}$. Thus R_2 is redundant, so we can remove it from our presentation of

$\mathfrak{G}(K)$. Then we can express $\mathfrak{G}(K)$ with only one relation, R_1. We can, however, simplify the presentation even further. Let $x = s_1s_2$ and $y = s_2s_1s_2$. Then $x^3y^{-2} = R_1$. Thus we obtain $\mathfrak{G}(K) =< x, y \mid x^3y^{-2} >$, just as we did in Example 5.3.

Recall that by Theorem 4.1, we know that the (3,2)-torus knot and the (2,3)-torus knot are equivalent. So we should not be surprised that we obtained the same knot group presentation for both butterfly diagrams. What is interesting to note is that the (2,3)-torus knot admits a knot group presentation whose sole relation is x^2y^{-3}. We generalize this result in the following theorem.

Theorem 5.6. *Let K be a (p, q)-torus knot. Then $\mathfrak{G}(K)$ admits a presentation*

$$\mathfrak{G}(K) =< x, y \mid x^py^{-q} > .$$

Proof. We will use the canonical butterfly diagram for K. We will, without loss of generality, assume that $q > 0$. The proof for the $q < 0$ case is similar to the one we will describe.

The canonical butterfly diagram for K contains q trunks, $t_0, t_1, \ldots, t_{q-1}$. We will orient the trunks so that they are "going away from" the B-vertex b_0. See Figure 5.6. We will denote by a_i and a'_i the incoming and outgoing A-vertices of the trunk t_i, respectively.

To prove the theorem, we are going to apply Theorem 5.4. To do this, we must first determine the equivalence class of each a'_i under $\simeq$. Recall that by Lemma 4.2, each equivalence class of A- and E-vertices contains exactly p elements. Thus we obtain $A_i = (a'_i, e_1, e_2, \ldots, e_{p-2}, a_j)$, where each e_l is an E-vertex, for some integer $j = 0, 1, \ldots, q-1$.

Now we know that $T_i = (t_{i+1}, t_{i+2}, \ldots, t_{i+p-1})$. So a'_i is t_{i+1}-equivalent to e_1, e_1 is t_{i+2}-equivalent to e_2, and and finally e_{p-2} is t_{i+p-1}-equivalent to a_{i+p}. Since each trunk is oriented "outward" from b_0, we obtain the relation

$$R_i = t_it_{i+1}t_{i+2} \ldots t_{i+p-1}t_{i+p}^{-1}t_{i+p-1}^{-1} \ldots t_{i+2}^{-1}t_{i+1}^{-1}$$

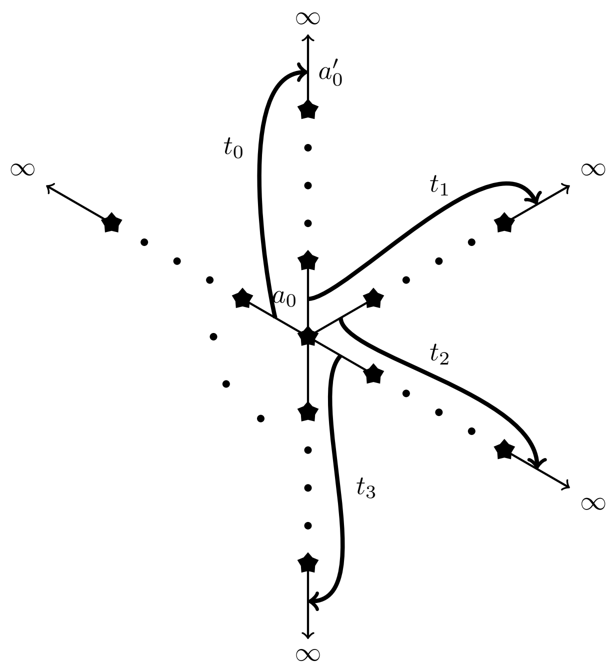


Figure 5.6: Oriented (p, q)-torus link butterfly diagram.

for each $i = 0, \ldots, q-1$, with the indices of the trunks taken modulo q.

To simplify our notation, we will let $X_i = t_i t_{i+1} t_{i+2} \ldots t_{i+p-1}$, for $i = 0, \ldots, q-1$, and the indices of trunks taken modulo q. Then we can rewrite our relations as $R_i = X_i X_{i+1}^{-1}$. We thus obtain a presentation of the knot group of K as $\mathfrak{G}(K) =< t_0, \ldots, t_{q-1} \mid R_0, \ldots, R_{q-1} >$. Our next goal is to simplify this presentation so as to match the presentation in the statement of the theorem.

Consider $X_0 X_{\ell_1} X_{\ell_2} \ldots X_{\ell_{q-1}} = (t_0 t_1 \ldots t_{q-1})^p$, where $\ell_j \equiv jp \pmod q$. Also note that since $R_i = X_i X_{i+1}^{-1}$, we obtain $X_i = X_{i+1}$. From this it follows that $X_1 = X_2 = \ldots = X_{q-1}$. Note that $R_{q-1} = X_{q-1} X_0^{-1}$. Then in $\mathfrak{G}(K)$, $X_{q-1} = X_0$. However, this has already been determined from the relations R_0, R_1, …, R_{q-2}. Thus R_{q-1} is redundant, so we can eliminate it from our presentation of $\mathfrak{G}(K)$.

Now note that, in particular, $X_1 = X_{\ell_j}$ for any j. Thus

$$(t_0 t_1 \dots t_{q-1})^p (X_1^{-1})^q = (X_0 X_{\ell_1} \dots X_{\ell_{q-1}})(X_{\ell_{q-1}}^{-1} \dots X_{\ell_1}^{-1} X_1^{-1}) = X_0 X_1^{-1}.$$

But $X_0 X_1^{-1}$ is a relation of the group, and is thus trivial. So it follows that $(t_0 \dots t_{q-1})^p (X_1^{-1})^q = 1$.

Let $x = t_0 \dots t_{q-1}$ and $y = X_1$. Then we obtain $x^p y^{-q} = 1$. Thus we can rewrite the knot group of K as $\mathfrak{G}(K) =< x, y \mid x^p y^{-q} >$. □

CHAPTER 6
BUTTERFLY MOVES AND REIDEMEISTER'S THEOREM

In Chapter 2, we saw that a given knot may have many knot diagrams. Recall that by Reidemeister's Theorem, any two diagrams for a given knot can be related by a finite sequence of Reidemeister moves and planar isotopies. It is also true that any given knot has many different butterfly diagrams. For example, consider the (2,3)- and (3,2)-torus knot butterfly diagrams in Figure 4.2 and Figure 6.1, respectively. By Theorem 4.1, we know that these diagrams are equivalent. This implies the problem of finding a set of *butterfly moves* to turn one butterfly diagram into an equivalent one. As we have seen, the trunk-reducing move and its inverse are such moves. In this section we will develop butterfly moves which correspond to the traditional Reidemeister moves for knot diagrams. We will prove that these moves, along with the trunk-reducing move, are all we need to obtain an analog to Reidemeister's Theorem for butterfly diagrams.

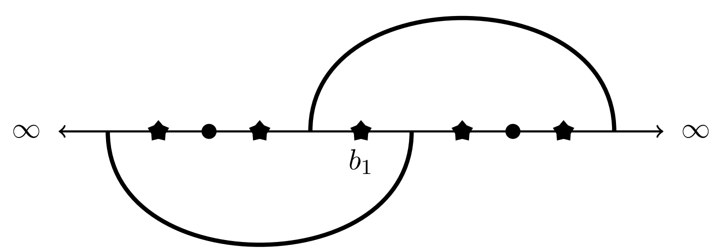


Figure 6.1: Canonical butterfly for a (3,2)-torus knot.

We shall consider each of the Reidemeister moves in turn, converting them into butterfly moves. Let K be a knot diagram throughout this section, and let $\mathcal{B}_K$ be a butterfly diagram corresponding to K. Throughout this section, we will assume that $\mathcal{B}_K$ contains no E-vertices. This will simplify our constructions. The omission of E-vertices will cause us no grief, for if a butterfly diagram contains E-vertices, we can convert them to simple butterflies via inverse trunk-reducing moves, perform our butterfly moves, and then convert the simple butterflies back into E-vertices.

First we consider the Type I move. Take a strand of K and apply a Type I move to it, as in Figure 6.2. Apply the link-butterfly algorithm to both diagrams in the figure; see Figure 6.3.

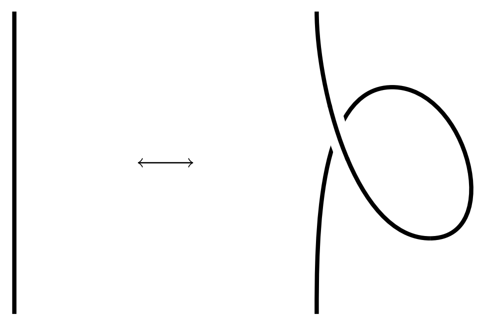


Figure 6.2: Type I Reidemeister move.

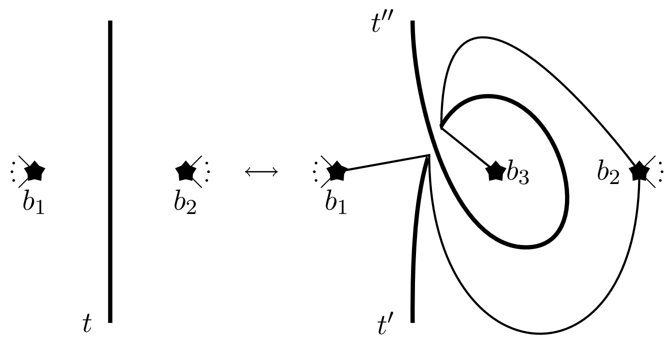


Figure 6.3: Type I butterfly move.

To perform a Type I butterfly move on $\mathcal{B}_K$, select two B-vertices of $\mathcal{B}_K$, b_1 and b_2, which are separated by a trunk t. Split this trunk into two pieces, t' and t'', and construct a path consisting of two edges connecting b_1 to b_2. These two edges will meet in an A-vertex, a_1. Now, create a new B-vertex b_3. Construct another path of two edges connecting b_3 to either one of b_1 or b_2. The choice of which B-vertex to connect b_3 to determines which side of the arc corresponding to t we will create a loop on in the corresponding knot diagram. Again the two edges meet in an A-vertex, a_2. Note that b_3 will be univalent. Finally, attach the loose ends of t' and t'' to the newly created A-vertices. We can either attach t' to a_1 and t'' to a_2, or we can attach t' to a_2 and t'' to a_1. This choice will determine whether the loop we create in the corresponding knot diagram goes over-under or under-over.

***Remark* 6.1.** Note that the B-vertices b_1 and b_2 must be t-equivalent. To see this, suppose b_1 and b_2 are B-vertices of $\mathcal{B}_K$ which are separated by a trunk, t, but are not t-equivalent. Let α be the minimum butterfly boundary distance from b_1 to an A-vertex of t, and let β be the minimum butterfly boundary distance from b_2 to an A-vertex of t. Since b_1 and b_2 are not t-equivalent, we can, without loss of generality, assume $\alpha > \beta$.

Now perform a Type I butterfly move for b_1 and b_2. This splits the butterfly containing t into two butterflies. In particular, we obtain a butterfly with $\alpha + \beta + 2$ sides. The new trunk t' divides the boundary of this butterfly into two paths, of lengths $\alpha + 1$ and $\beta + 1$, respectively. But we assumed $\alpha > \beta$, so $\alpha + 1 > \beta + 1$. This is a contradiction, since t' must split the boundary of the butterfly into two paths of equal length.

The Type I butterfly move is reversible, as suggested in Figure 6.3. This of course aligns with the fact that a standard Type I Reidemeister move is reversible.

To reverse a Type I butterfly move, we perform the following procedure. Locate a B-vertex b_3 which is univalent. Find the A-vertex a_1 which has the least butterfly boundary distance from b_3. Let t'' be the trunk connected to a_1. Note that a_1 is t''-equivalent to some A-vertex, a_2. This is because the butterfly boundary distance from a_1 to itself, via a path which goes through b_3, is 2. Traveling along a path of length 2, going the other way, we arrive at a_2. Let t' be the trunk connected to a_2. Recall that A-vertices are bivalent. We delete the two graph edges connected to a_1 and the two edges connected to a_2, along with the vertex b_3. We fuse the loose ends of t' and t'', to make a new trunk t.

Next we consider a Type II Reidemeister move. Take two strands of K and apply a Type II Reidemeister move, as in Figure 6.4. Apply the link-butterfly algorithm to both diagrams in the figure, as in Figure 6.5.

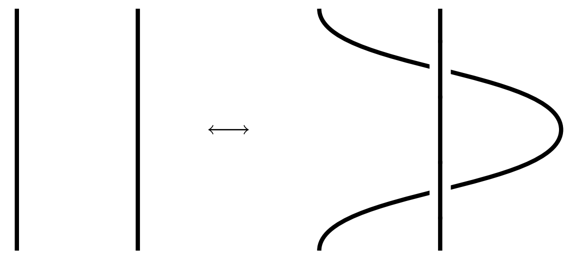


Figure 6.4: Type II Reidemeister move.

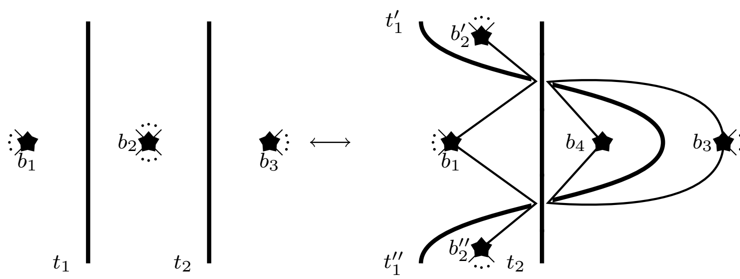


Figure 6.5: Type II butterfly move.

To perform a Type II butterfly move on $\mathcal{B}_K$, select three B-vertices, b_1, b_2, and b_3 as in Figure 6.5. We require that b_1 and b_2 are t_1-equivalent, and b_2 and b_3 are t_2-equivalent. In order to perform the move, we first construct a simple butterfly on b_3, in the process creating a new B-vertex b_4. Next, we split b_2 into two distinct B-vertices, b_2' and b_2''. We must take care in performing this step. We must connect the correct edges to b_2' and b_2''. To determine which edges to connect to b_2' and which edges to connect to b_2'', we do the following. Starting at b_3, travel along the boundary of the butterfly containing

b_2 and b_3 counterclockwise. Stop at the first edge which has b_2 as an endpoint. Call this edge e_1. From e_1, construct an arc counterclockwise about b_2, which terminates at the first edge connected to b_2 which lies in the butterfly containing both b_1 and b_2. Any edges which meet this arc will be attached to b_2'. Any remaining edges which are attached to b_2 will be attached to b_2'' after we perform the move. See Figure 6.6.

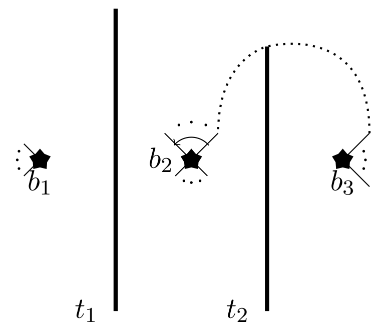


Figure 6.6: Determining which edges are connected to b_2' and b_2'', respectively.

Finally, to complete the move, construct two paths, each consisting of two edges, from b_1 to b_2 and from b_1 to b_2'. The edges of each of these paths will meet in an A-vertex. Split t_1 into two trunks, t_1' and t_1'', and attach them to the new A-vertices. This complete the Type II butterfly move.

***Remark* 6.2.** Note that b_1 and b_2 must be t_1-equivalent and b_2 and b_3 must be t_2-equivalent. To see this, first suppose b_1 and b_2 are not t_1-equivalent. Let α be the minimum butterfly boundary distance from b_1 to an A-vertex of t_1, and let β be the minimum butterfly boundary distance from b_2 to an A-vertex of t_1. Without loss of generality suppose $\beta > \alpha$, since that b_1 and b_2 are not identified t_1-equivalent. When we perform the Type II butterfly move, the butterfly containing t_1 is split into two butterflies, one of which has $\alpha + \beta + 1$ sides. The trunk of this butterfly splits the boundary of the butterfly into two paths of lengths $\alpha + 1$ and $\beta + 1$, respectively. It must be that $\alpha + 1 = \beta + 1$, but we assumed $\beta > \alpha$, so that $\beta + 1 > \alpha + 1$. Thus we have a contradiction. So it must be that $\alpha = \beta$.

Next we want to show that b_2 and b_3 must be t_2-equivalent. Let β be the minimum butterfly boundary distance from b_2 to an A-vertex of t_2, and let γ be the minimum butterfly boundary distance from b_3 to an A-vertex of t_2. Without loss of generality, suppose $\gamma > \beta$, so that b_2 and b_3 are not t_2-equivalent. Perform a Type II butterfly move, which creates a simple butterfly on b_3. We label the other B-vertex of this simple butterfly as b_4. In the resulting butterfly diagram, we label A-vertices as follows. The A-vertex between b_1 and b_2' is labeled as a_1, the A-vertex between b_2' and b_3 is labeled as a_2, and the A-vertex between b_3 and b_4 is labeled as a_3. For the butterfly move to correspond to a Type II Reidemeister move, we expect a_1 and a_3 to be t_2-equivalent. However, the butterfly boundary distance from a_1 to a_2 is $\beta + 1$, and the butterfly boundary distance from a_2 to a_3 is $\gamma + 1$. Since we assumed $\beta < \gamma$, this implies that a_1 and a_3 are not t_2-equivalent. Thus the butterfly move will not correspond to a Type II Reidemeister move. We conclude that b_2 and b_3 must be t_2-equivalent.

The Type II butterfly move is reversible, just as the Type I move is. To perform an inverse Type II butterfly move, we do the following. Select a simple butterfly of $\mathcal{B}_K$, with B-vertices b_3 and b_4, as in Figure 6.5. Locate B-vertices which are separated from b_3 and b_4 by a trunk, t_2, label them b_2', b_2'', and b_1, as in Figure 6.5. We require that b_3 is t_2-equivalent to both b_2' and b_2'', and that b_4 is t_2-equivalent to b_1. Two trunks, t_1' and t_1'', have A-vertices between b_2' and b_1, and b_2'' and b_1, respectively. To perform the inverse Type II move, delete the simple butterfly containing b_3 and b_4, leaving only b_3 in its place. Next delete the edges connecting b_1 to b_2' and b_1 to b_2'', and fuse the loose ends of t_1' and t_1'' to form a new trunk, t_1. Finally, combine b_2' and b_2'' into a single B-vertex, b_2.

Finally we consider Type III Reidemeister moves. Suppose K has an arrangement of strands as in Figure 6.7. Apply the link-butterfly algorithm to both diagrams in the figure, as in Figure 6.8.

To perform a Type III butterfly move on $\mathcal{B}_K$, we first locate a configuration as in Figure 6.8. The B-vertices b_1 and b_3 must be t_3-equivalent, b_2 and b_6 must be t_1-equivalent, and b_1 and b_4 must be t_1-equivalent. We simply replace the edges between

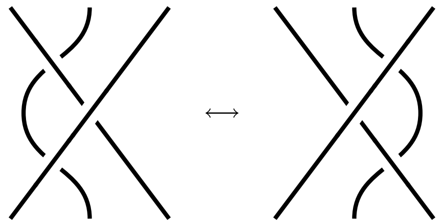


Figure 6.7: Type III Reidemeister move.

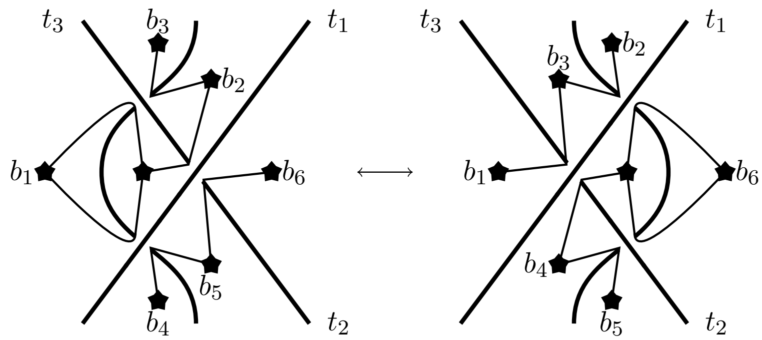
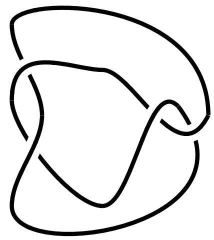


Figure 6.8: Type III butterfly move.

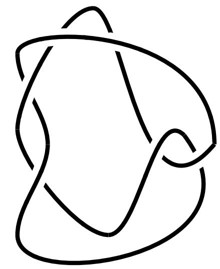
b_1, b_2, and b_3, and b_4, b_5, and b_6, as shown in the figure. Note that the simple butterfly is moved entirely from one side of t_1 to the other.

***Example* 6.3.** We now present an example which will apply each of the butterfly moves just described. We will transform a canonical butterfly diagram for the twist knot T_2 into an equivalent butterfly diagram via a sequence of butterfly moves. The sequence of butterfly moves is depicted in Figure 6.10. To help the reader see how the moves are applied, we provide the analogous sequence of Reidemeister moves, applied to a knot diagram of T_2, in Figure 6.9. The reader is encouraged to compare the diagrams in each figure to see how they relate. The reader may also find it useful to perform the butterfly-link algorithm to the butterfly diagrams in Figure 6.10, and to compare their results with the knot diagrams in Figure 6.9.

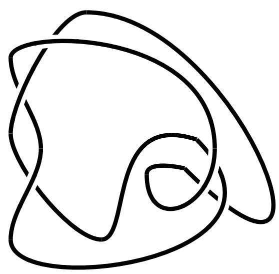
Furthermore, we have labeled B-vertices and trunks of the butterfly diagrams



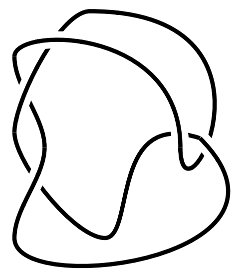
(a) The twist knot T_2



(b) Applying a Type II.



(c) Applying a Type III.

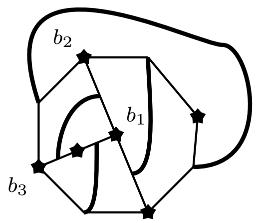


(d) Applying a Type I.

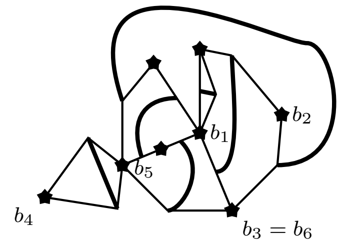
Figure 6.9: Performing a sequence of Reidemeister moves on T_2.

similarly to our discussions of the butterfly moves earlier in this chapter. So in Figure 6.10a, we have labeled B-vertices b_1, b_2, and b_3, which the relevant B-vertices for performing the desired Type II move. In Figure 6.10b, the B-vertices b_1, b_2, $b_3 = b_6$, b_4, and b_5 are the relevant B-vertices for performing the desired Type III move. We note that $b_3 = b_6$. As a result, by performing the Type III move, we obtain a butterfly diagram which contains a univalent B-vertex, which corresponds to b_2.

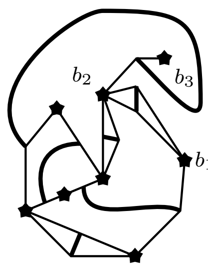
Finally, in Figure 6.10c, the B-vertices b_1, b_2, and b_3 are the relevant B-vertices for performing an inverse Type I move. Figure 6.10d depicts the end result of the sequence of



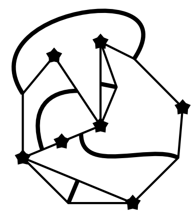
(a) Canonical butterfly for T_2, prepped for a Type II move.



(b) After applying a Type II move, and prepped for a Type III move.



(c) After applying a Type III move, and prepped for a Type I move.



(d) After applying a Type I move.

Figure 6.10: Performing a sequence of butterfly moves on a butterfly diagram for T_2, which is analogous to the moves in Figure 6.9.

butterfly moves.

Recall that two butterfly diagrams are said to be *equivalent* if they correspond to the same link. Thus far, we have defined four types of butterfly moves: three moves which are analogous to the Reidemeister moves for knot diagrams, and trunk-reducing moves. These moves transform a given butterfly diagram into an equivalent one. It is natural to ask if there are any other moves that we need to investigate in order to define equivalence for butterfly diagrams. It turns out that we do not.

Theorem 6.4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be butterfly diagrams for a link L. Then $\mathcal{B}_1$ is equivalent to*

$\mathcal{B}_2$ if and only if there is a finite sequence of butterfly moves which transforms $\mathcal{B}_1$ into $\mathcal{B}_2$.

Proof. First suppose there exists a finite sequence of butterfly moves which transforms $\mathcal{B}_1$ into $\mathcal{B}_2$. We know that trunk-reducing moves and their inverses do not alter the link diagram corresponding to a given butterfly diagram. Furthermore, Type I, II, and III butterfly moves are equivalent to Reidemeister moves on the corresponding link diagrams. Thus if two butterfly diagrams are related by a single Type I, II, or III butterfly move, then they must be equivalent. So it follows that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent.

Conversely, suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent. Let D_1 and D_2 be the link diagrams corresponding to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. We perform inverse trunk-reducing moves to $\mathcal{B}_1$ to obtain an equivalent butterfly diagram $\mathcal{B}_1'$ which does not contain any E-vertices. Not only are $\mathcal{B}_1$ and $\mathcal{B}_1'$ equivalent, but they have the exact same corresponding link diagram, D_1. Similarly, we can obtain a butterfly diagram $\mathcal{B}_2'$ which is equivalent to $\mathcal{B}_2$ and to which D_2 corresponds. Now since $\mathcal{B}_1$ and $\mathcal{B}_2$ where assumed to be equivalent, it must be that $\mathcal{B}_1'$ and $\mathcal{B}_2'$ are equivalent. In addition, it must be that D_1 and D_2 are equivalent as link diagrams. Thus by Theorem 2.6, there exists a finite sequence of Reidemeister moves and planar isotopies which transform D_1 into D_2. Now, planar isotopies on a link diagram do not alter corresponding butterfly diagrams, so we omit them from consideration. Each of the Reidemeister moves in the sequence taking D_1 to D_2 has an equivalent butterfly move. Thus we can apply these butterfly moves in the proper order, beginning with $\mathcal{B}_1'$, and obtaining $\mathcal{B}_2'$. Finally, perform inverse trunk-reducing moves to $\mathcal{B}_2'$ to obtain $\mathcal{B}_2$. Thus we have constructed a sequence of butterfly moves transforming $\mathcal{B}_1$ into $\mathcal{B}_2$. $\square$

Theorem 6.4 defines equivalence of butterfly diagrams in a way that is analogous to Theorem 2.6 for link diagrams. We take caution by noting that our constructions of butterfly moves are specific to our restricted definition of butterfly diagram. Thus while for our definition of butterfly diagram the moves we have described are enough to define equivalence, this may not be the case for the more general definition of butterfly diagram initially proposed by HMTT.

CHAPTER 7

CLASSIFICATION OF 1-BUTTERFLIES

The next few chapters will focus on the classification of links by their butterfly number. Recall that Theorem 3.9 states that the butterfly number of a link is equivalent to its bridge number. Thus classifying links by their butterfly number is equivalent to classifying them by their bridge number. The latter classification is well known for 1- and 2-bridge links. The classification of 3-bridge links is still an open problem. It is our goal to recreate the classifications of 1- and 2-bridge links via their butterfly diagrams. Thus we will classify 1- and 2-butterflies.

We begin by studying 1-butterflies. It is well known that any 1-bridge link is the unknot. The same holds for butterflies.

Theorem 7.1. *Let L be a link. Then L is the unknot if and only if $m(L) = 1$.*

Before proving Theorem 7.1, we need to discuss some properties of 1-butterfly diagrams in general. Let $\mathcal{B} = (R, T)$ be a 1-butterfly. By definition, R is embedded on $S^2 = \partial \mathbf{B}^3$ so that $S^2 \setminus R$ is a single 2-cell (in other words, $S^2 \setminus R$ is homeomorphic to a disk). If R contained any cycles, then R would necessarily divide S^2 into more than one 2-cell. Thus for $\mathcal{B}$ to be a 1-butterfly, it must be that R is a finite tree.

A vertex of a tree which has degree equal to one is called a *leaf.* Since we will only consider finite trees, R will always have a leaf. Furthermore, since we require A- and E-vertices to be bivalent in R, it follows that if a vertex of R is a leaf, then it is a B-vertex.

Since R divides $S^2 = \partial \mathbf{B}^3$ into a single polygonal face, each edge of R has exactly two edges of ∂P_{2n} in its preimage under $f : P_{2n} \to C \subseteq S^2$. Thus in measuring butterfly boundary distance for $\mathcal{B}$, each edge is double-counted. As a consequence of Theorem 3.12, we have the following lemma.

Lemma 7.2. *Let $\mathcal{B}$ be a 1-butterfly diagram. Suppose $\mathcal{B}$'s only butterfly has N sides. Then N is divisible by 4.*

Proof. Let t be the unique trunk of $\mathcal{B}$. Let D be the knot diagram corresponding to $\mathcal{B}$, via the butterfly-link algorithm. Let m be the number of underpasses of the arc of D which corresponds to t. Then by Theorem 3.12, t belongs to a butterfly with $4(m+1)$ sides. But $\mathcal{B}$ has exactly one butterfly, of which t is the trunk. This butterfly has N sides. Thus $N = 4(m+1)$, so 4 divides N. □

Since each edge of R is double-counted, Lemma 7.2 implies that R must have an even number of edges when strictly considered as a graph, rather than the boundary of a butterfly.

Our next lemma states that no 1-butterfly can have 0 or 4 sides. For a 1-butterfly to be properly defined, it must have at least 8 sides.

Lemma 7.3. *Let $\mathcal{B} = (R, T)$ be a 1-butterfly diagram. Then $\mathcal{B}$'s only butterfly has $4(n+1)$ sides, with $n \geq 1$.*

Proof. Suppose $n = -1$, so that $\mathcal{B}$'s only butterfly has 0 sides. Then R has no edges. Note, however, that R is connected. Thus R consists of a single vertex, which must be a B-vertex. This is not a butterfly diagram, since there are no A-vertices, and hence no trunks.

Next suppose $n = 0$. Then $\mathcal{B}$'s only butterfly has 4 sides. This means that as a graph, R is a tree with exactly two edges. Since R is a tree with two edges, it must have two leaves, which are B-vertices, and one other vertex. This other vertex must be an A-vertex. However, since it is the only A-vertex, there is no way to construct a trunk which has distinct endpoints. See Figure 7.1. This violates our requirement that the trunks of $\mathcal{B}$ must not have coincident endpoints. So if $n = 0$, $\mathcal{B}$ is not a butterfly diagram.

Thus it must be that $n \geq 1$. □

Lemma 7.4. *Let $\mathcal{B} = (R, T)$ be a 1-butterfly diagram such that $\mathcal{B}$'s only butterfly has exactly 8 sides. Then R must be one of the graph in Figure 7.2, and $\mathcal{B}$ corresponds to the unknot.*

Proof. First we show by construction that R must match the graph of the

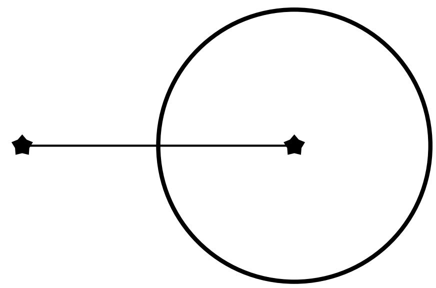


Figure 7.1: A 1-butterfly diagram cannot have four sides.

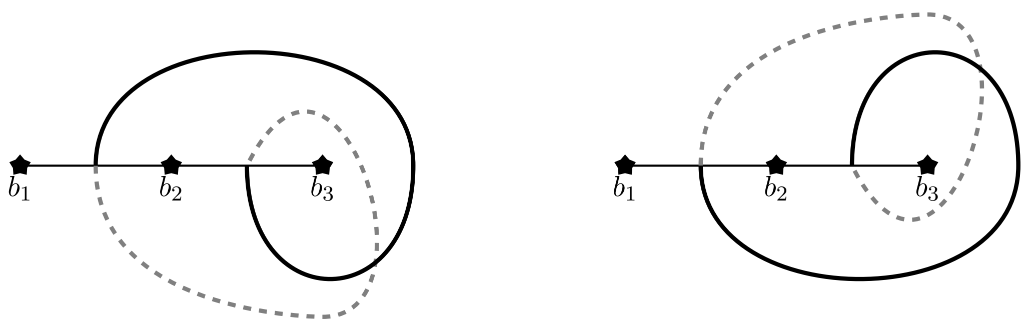


Figure 7.2: 8-sided 1-butterfly diagrams. The corresponding knot diagrams are mirror images of each other, and are the unknot.

butterfly in Figure 7.2. Note that since the butterfly has 8 sides, R must have exactly four graph edges. Consider a leaf of R, which we know is a B-vertex. Label this B-vertex b_1. Since b_1 is a leaf, it is univalent, and thus only has one edge, h_1, connected to it. Since R is bipartite, we know that the other vertex on h_1 must be either an A- or E-vertex. Let x_1 be this vertex. Thus b_1 is connected to x_1 by h_1. Now A- and E-vertices are bivalent in R, so x_1 has another edge, h_2, connected to it, and the other endpoint of h_2 must be a B-vertex. Label this vertex b_2.

Now we have accounted for two of R's four graph edges, so we must account for two more. Note that it cannot be that b_2 has degree three. If it did, then there would be two edges attached to b_2 distinct from h_2. The endpoints of these edges would have to be univalent in R. We know that any univalent vertex of R must be a B-vertex. Thus b_2 would be connected to two B-vertices, contradicting the fact that R is bipartite.

Thus b_2 must be bivalent in R. Let h_3 be the edge connected to b_2 which is distinct from h_2. We know that the other endpoint of h_3 must be either an A- or E-vertex. Let x_2 be this vertex. Again, since A- and E-vertices of R must be bivalent, there is another edge h_4 attached to x_2. The other endpoint of h_4 must be a B-vertex, b_3.

Thus we have constructed all four graph edges of R. Now, note that for us to be able to properly place a trunk in $\mathcal{B}$, R must contain exactly two A-vertices. Since x_1 and x_2 are the only vertices of R which are not B-vertices, it follows that x_1 and x_2 must be A-vertices.

Thus we obtain the graph of the butterfly diagrams in Figure 7.2. To see that both of these diagrams correspond to the unknot, we simply apply the butterfly-link algorithm to both of them and note that the resulting knot diagrams are clearly trivial. □

Now we are ready to prove Theorem 7.1. First we suppose $m(L) = 1$. Our goal is to show that L is the unknot. Let $\mathcal{B}_L$ be a 1-butterfly diagram for L, whose only butterfly has $4(n+1)$ sides. We proceed by induction on n.

For our base case we consider $n = 1$. Note that by Lemma 7.3, we need not consider the cases $n = -1$ or $n = 0$. Since $n = 1$, the butterfly in $\mathcal{B}_L$ must have eight sides, and so by Lemma 7.4, we know that L is the unknot.

Now suppose that for some positive integer $n > 1$ that any 1-butterfly with exactly $4(m+1)$ sides is trivial if $m < n$. We want to show that a 1-butterfly with exactly $4(n+1)$ sides is trivial. So suppose $\mathcal{B}_L$ consists of a butterfly with $4(n+1)$ sides. Since $\mathcal{B}_L$ is a 1-butterfly diagram, its graph is a finite tree. Thus its graph has a leaf, which we know is a B-vertex of degree equal to one. Let b_1 be this B-vertex. Since b_1 is univalent, it is connected to exactly one graph edge, h_1. Let x be the other vertex on h_1. The vertex x is either an A- or an E-vertex. So x is bivalent. Let h_2 be the edge attached to x which is distinct from h_1, and let b_2 be the B-vertex on h_2's other end. We now have two separate cases to consider: x is an A-vertex and x is an E-vertex.

First suppose that x is an A-vertex. Let t be the unique trunk of $\mathcal{B}_L$. Then t has an endpoint on x. Let y be the vertex which is t-equivalent to x. We may assume that y is an E-vertex, since if y is an A-vertex, it can be shown that $\mathcal{B}_L$ has 8 sides, contradicting our hypothesis of $n > 1$. Perform an inverse trunk-reducing move on y, creating a simple butterfly in its place. Let y' be the A-vertex of this simple butterfly which is t-equivalent to x, and let h_3 and h_4 be the graph edges with endpoints on y'. Now, we can perform a Type I butterfly move which will remove the edges h_1, h_2, h_3, and h_4, including the vertices b_1, x, and y, as in Figure 7.4. In doing so we obtain a 1-butterfly diagram which is equivalent to $\mathcal{B}_L$ and which has $4n$ sides. Thus by the induction hypothesis, L is the unknot.

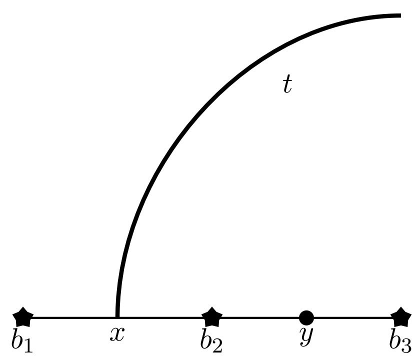


Figure 7.3: x is an A-vertex and y is an E-vertex.

Now suppose that x is an E-vertex. Again let t be the unique trunk of $\mathcal{B}_L$ and let y and z be the vertices which are t-equivalent to x. We next consider cases: y and z are both A-vertices, exactly one of y and z is an A-vertex, and both y and z are E-vertices.

If both y and z are A-vertices, then x is the only E-vertex of $\mathcal{B}_L$. Then $\mathcal{B}_L$ must be the butterfly in Figure 7.5. Note that, for example, y is connected to a leaf of R. Thus by our previous argument, we can perform an inverse trunk-reducing move on x, and then apply a Type I butterfly move to $\mathcal{B}_L$ to obtain an equivalent butterfly diagram with $4n$ sides. Thus by the induction hypothesis, $\mathcal{B}_L$ corresponds to the unknot.

Now we assume that at least one of y and z is an E-vertex. Perform

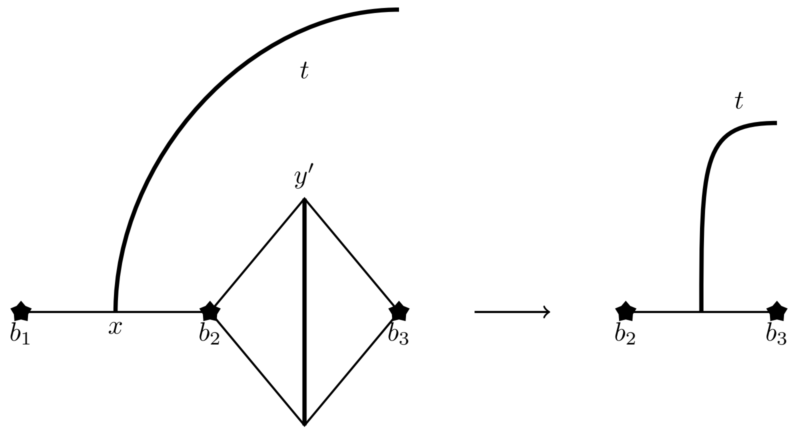


Figure 7.4: After inverse trunk-reducing move is applied to y, and then applying a Type I butterfly move.

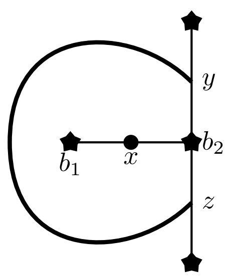


Figure 7.5: $\mathcal{B}_L$ if both y and z are A-vertices.

trunk-reducing moves on x and, as necessary, on y and z. We label the A-vertices of the simple butterfly obtained from x as x' and x'', as in Figure 7.6. We label the A-vertices in place of y and z as y' and z', respectively. We also let t' and t'' be the trunks on y' and z'', respectively. Note that the butterfly boundary distance from x' to x'' is two. Thus the butterfly boundary distance from y' to z' must also be two, since y' is t-equivalent to x', and z'' is t-equivalent to x''. Let b_3' and b_3'' be the B-vertices which are t-equivalent to b_2, and let b_4 be the B-vertex which is t-equivalent to b_1. Note that b_4 is t'-equivalent to b_3' and is t''-equivalent to b_3''. Thus we can perform a Type II butterfly move, which will remove the simple butterfly containing x' and x'', remove the edges connecting b_4 to b_3'

and b_4 to b_3'', combine b_3' and b_3'' into a single B-vertex, and will combine the trunks t' and t'' into a single trunk s. See Figure 7.6.

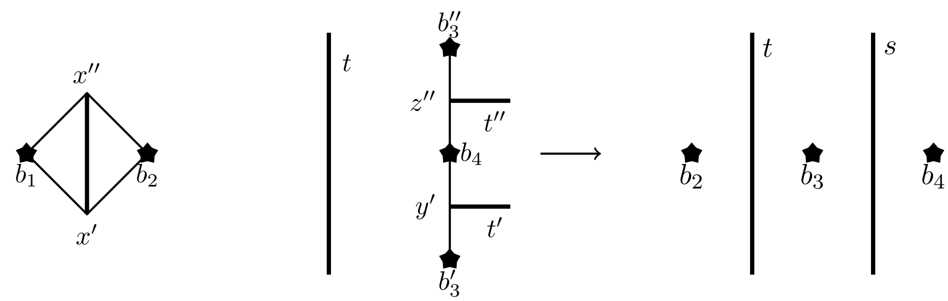


Figure 7.6: After applying inverse trunk-reducing moves to all E-vertices of $\mathcal{B}_L$, and the result of a Type II butterfly move.

Next, we claim that by performing a Type II butterfly move, we create at most one simple butterfly. To show this, we consider two cases: exactly one of y_1 and y_2 is an E-vertex, and both y_1 and y_2 are E-vertices. First, we suppose, without loss of generality, that y_1 is an E-vertex, and that y_2 is an A-vertex. An argument similar to the one we present holds if the roles of y_1 and y_2 are reversed.

Since y is an E-vertex, we perform an inverse trunk-reducing move to create a simple butterfly in its place. Notice that since z is an A-vertex, $t'' = t$. When we perform a Type II move to remove the simple butterfly obtained from x, we also remove two edges from the simple butterfly on y'. The result is as in Figure 7.7. Notice that the trunk s, obtained from combining t and t', actually coincides with t. As a result, no new butterflies are created at all, so the result of performing a Type II move is still a 1-butterfly diagram. Next suppose that both y_1 and y_2 are E-vertices. Then we perform inverse trunk-reducing moves on both y_1 and y_2, creating simple butterflies in their places. In this case, when we perform a Type II move to remove the simple butterfly on x, we create a simple butterfly. Note that the trunk of this simple butterfly is s, which we obtain by combining the trunks t' and t''. See Figure 7.8. Notice that we do not create any other new butterflies. Also note that the butterfly diagram we obtain has 8 fewer sides. Thus performing a Type II

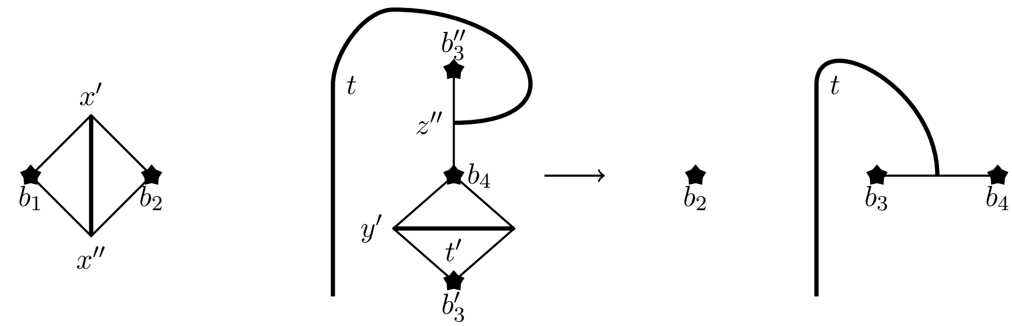


Figure 7.7: After applying inverse trunk-reducing moves to all E-vertices of $\mathcal{B}_L$.

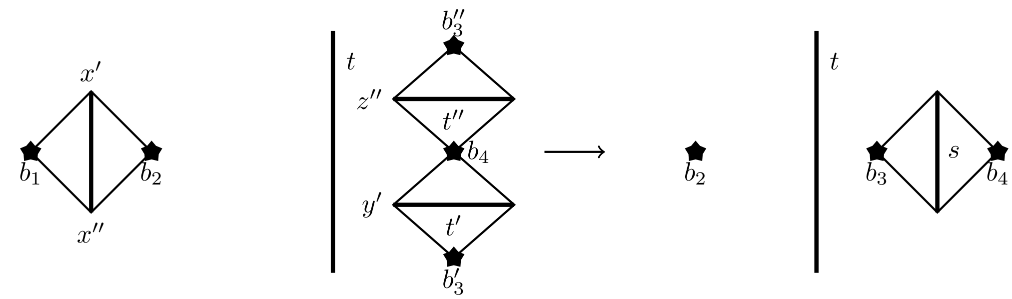


Figure 7.8: After applying inverse trunk-reducing moves to all E-vertices of $\mathcal{B}_L$.

move creates at most one simple butterfly. So we can, if necessary, perform a trunk-reducing move on this simple butterfly to obtain a 1-butterfly which is equivalent to $\mathcal{B}_L$, and which has $4(n-1)$ sides. Then by the induction hypothesis, L is the unknot.

We have seen that for any knot L, if $m(L) = 1$, then L is the unknot. To complete the proof of Theorem 7.1, we must prove the converse. That is, if L is the unknot, then $m(L) = 1$. However, this is trivial in light of Lemma 7.4, which tells us that the 1-butterflies of Figure 7.2 correspond to the unknot. Thus our proof of Theorem 7.1 is complete.

Corollary 7.5. *Any $(p, 1)$- or $(p, -1)$-torus link is trivial.*

Proof. Let p be a positive integer. Note that by Theorem 4.1, the $(p, -1)$-torus link is the mirror image of the $(p, 1)$-torus link. Thus if we can show that the $(p, 1)$-torus link is trivial, it will immediately follow that the $(p, -1)$-torus link is trivial.

So let L be a $(p, 1)$-torus link. Its canonical butterfly diagram is a 1-butterfly. Thus $m(L) = 1$. By Theorem 7.1, L must be the unknot. $\square$

CHAPTER 8
RATIONAL TANGLES AND WEAVES

In the previous chapter, we achieved a classification of 1-butterflies: they all correspond to the unknot. In this chapter and the following, we will describe a classification for links with butterfly number equal to 2. This is, of course, equivalent to classifying 2-bridge links. Such a classification is well-known, and it is our goal to reconstruct this classification using butterflies.

Before we can discuss the classification of 2-bridge links and 2-butterflies, we must first discuss *rational tangles.*

Definition 8.1. *A* tangle diagram *is a link diagram contained in a disk that consists of two arcs whose four endpoints are fixed along the boundary circle of the disk. We will often omit the boundary circle in our figures. We assume that the four fixed endpoints are located at the NW, NE, SE, and SW compass points on the boundary of the disk. We will often refer to the arcs within the disk as a* tangle*.*

Definition 8.2. *We say that two tangle diagrams are* equivalent *if one can be transformed into the other by a finite sequence of planar isotopies and Reidemeister moves restricted to the interior of the disk, and which keep the four endpoints of the arcs fixed. We use $\cong$ to denote equivalence of two tangles.*

Definition 8.3. *Let T, T_1, and T_2 be tangles. The* tangle sum *of T_1 and T_2, denoted $T_1 + T_2$, is the result of horizontally arranging T_1 and T_2 and connecting their adjacent arc endpoints, as in Figure 8.1. The* tangle product *of T_1 and T_2, denoted $T_1 \star T_2$ is the the result of vertically stacking T_1 and T_2 and connecting their adjacent arc endpoints, as in Figure 8.2. The* rotation of a tangle *T ,denoted T^r, is the tangle obtained by rotating T by 90 degrees clockwise. The* mirror of a tangle *T, denote $-T$, is the tangle resulting from switching all of the crossings of T. Finally, we define the* inverse of a tangle *T to be $-T^r$, denoted by T^i.*

(a) T_1 and T_2.   (b) $T_1 + T_2$.

Figure 8.1: Tangle sum.

(a) T_1 and T_2.   (b) $T_1 \star T_2$.

Figure 8.2: Tangle product.

We will next want to define what we call a *rational tangle*, which is a special type of tangle. We will build rational tangles out of *integer* and *reciprocal* tangles. We denote integer tangles by $[n]$, where $n \in \mathbb{Z}$, and reciprocal tangles by $\frac{1}{[n]}$. The integer tangle $[n]$ is obtained by taking two parallel horizontal arcs and twisting them $|n|$ times. The reciprocal tangle $\frac{1}{[n]}$ is obtained by taking two parallel vertical arcs and twisting them $|n|$ times. Some examples are depicted in Figure 8.3 and in Figure 8.4.

Definition 8.4. *A* rational tangle *is a tangle that can be constructed as follows. We declare all integer and reciprocal tangles to be rational tangles. Given a rational tangle T_k, the tangle T_{k+1} that is obtained from T_k by either (1) adding an integer tangle to the right,*

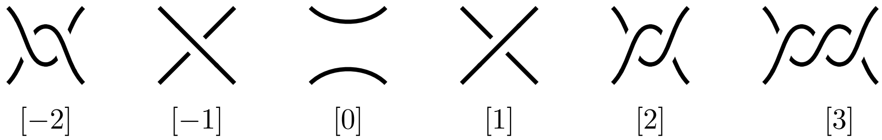


Figure 8.3: Examples of integer tangles.

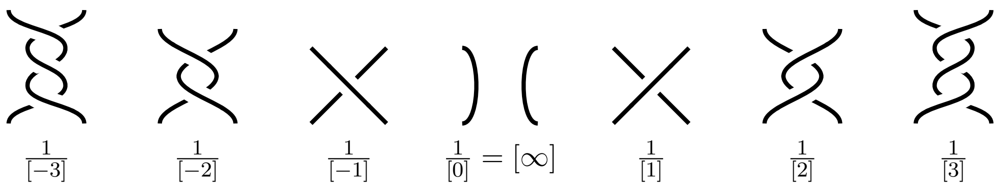


Figure 8.4: Examples of reciprocal tangles.

or (2) multiplying by a reciprocal tangle to the bottom, is also a rational tangle. A diagram of a rational tangle that depicts the finite sequence in the inductive construction of the tangle is called a twist diagram *of the rational tangle.*

***Remark* 8.5.** Note that a rational tangle is built inductively, starting from [0] or $[\infty]$, by adding integer tangles on the right, or multiplying by reciprocal tangles on the bottom. Since the very first crossing of a rational tangle can equally be seen as horizontal or as vertical, we will always assume that the first crossing is horizontal, as in [5]. That is, we always build our rational tangles by twisting from the integer tangle [0].

The following lemma gives us some identities for handling sums and products of integer and reciprocal tangles.

Lemma 8.6. *Let n and m be integers. Then*

(i) $[n] + [m] \cong [n+m]$

(ii) $\frac{1}{[n]} \star \frac{1}{[m]} \cong \frac{1}{[n+m]}$

A key fact for us is that each rational tangle has an associated fraction (this is part of why we use the name *rational* tangle). It will be especially useful to express such a fraction as a continued fraction, and to relate this to a canonical form for rational tangles. We will state several results, with proof omitted, which will lay out the relationship between rational tangles and their fractions. For details, see [5]. We note that for a rational tangle T, we may denote T^i by $\frac{1}{T}$.

Theorem 8.7. *(The Product-to-Inverse Equivalence) Let T be a rational tangle and $n \in \mathbb{Z}$. Then*

$$T \star \frac{1}{[n]} \cong \frac{1}{[n] + \frac{1}{T}} \text{ and } \frac{1}{[n]} \star T \cong \frac{1}{\frac{1}{T} + [a]}.$$

Definition 8.8. *Let T be a rational tangle. A* continued fraction form *of T is given by*

$$T = [a_n] + \frac{1}{[a_{n-1}] + ... + \frac{1}{[a_2] + \frac{1}{[a_1]}}},$$

where each a_i is nonzero, except possibly for a_n. We write $T = [[a_1], \ldots, [a_n]]$ to denote a continued fraction form of T.

Definition 8.9. *Let T be a rational tangle, with continued fraction form*

$$[a_n] + \frac{1}{[a_{n-1}] + ... + \frac{1}{[a_2] + \frac{1}{[a_1]}}}$$

We define the fraction of T, *denoted by $F(T)$, to be*

$$F(T) = a_n + \frac{1}{a_{n-1} + ... + \frac{1}{a_2 + \frac{1}{a_1}}}.$$

Theorem 8.10. *Every rational tangle has a diagram in continued fraction form.*

Lemma 8.11. *Let T be a rational tangle in continued fraction form. Then $F(T)$ has the following properties.*

(i) $F(T + [k]) = F(T) + [k]$ *for any* $k \in \mathbb{Z}$.

(ii) $F(\frac{1}{T}) = \frac{1}{F(T)}$.

(iii) $F(T * \frac{1}{[k]}) = \frac{1}{k+\frac{1}{F(T)}}$ *for any* $k \in \mathbb{Z}$.

Theorem 8.12. *(Conway's Theorem) Two rational tangles are equivalent if and only if their fractions are equal.*

It is useful to assume that a given rational tangle is in a specified form. Continued fraction form is one such form. However, we desire a form with even stronger conditions on the integers a_1, ..., a_n in the continued fraction form.

***Remark* 8.13.** Let T be a rational tangle in continued fraction form,

$$T = [a_n] + \frac{1}{[a_{n-1}] + \ldots + \frac{1}{[a_2]+\frac{1}{[a_1]}}}.$$

We can always assume that n is odd [5].

Definition 8.14. *Let* T *be a rational tangle in continued fraction form, with*

$$T = [a_n] + \frac{1}{[a_{n-1}] + \ldots + \frac{1}{[a_2]+\frac{1}{[a_1]}}}.$$

Then T *is said to be in* canonical form *if* T *is alternating and* n *is odd. Moreover,* T *is said to be* positive *or* negative *according to the sign of its terms.*

Theorem 8.15. *Every rational tangle has a diagram in canonical form.*

There are necessary and sufficient conditions for equivalence of rational links. We defer this discussion until Chapter 10.

Now that we have described rational tangles, we will describe a new object, which we call a *weave*. Let p and q be integers, with $p \geq 2$, $q \neq 0$, $|q| \leq p/2$, and $gcd(p, q) = 1$. Consider a circle S which is constructed as a graph with $2p$ vertices. There exist two distinct vertices of this graph whose distance in the graph metric is equal to p. Label these vertices NW and SE. Note that NW and SE define two "halves" of S, each with $p - 1$ vertices of the graph. We would like to label two more vertices as SW and SE, analogous to the boundary circle for a tangle diagram for a rational tangle. For $q > 0$, SW will be the vertex which is the endpoint of the counterclockwise path beginning at NW, whose length is q. The vertex labeled NE will be the vertex which is the endpoint of a counterclockwise path of length q beginning at SE. For $q < 0$, SW is the vertex which is the endpoint of a clockwise path of length $-q$ beginning at SE. The vertex which is the endpoint of a clockwise path of length $-q$ beginning at NW is labeled NE.

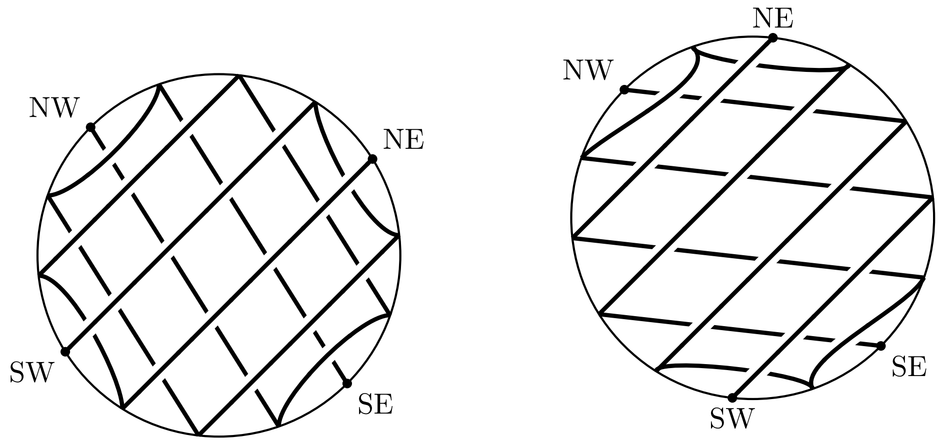
Next We will construct $p-1$ arcs, ℓ_i, $i = 1, \ldots, p-1$, within the closed disk bounded by S. We place these arcs so that each ℓ_i has an endpoint on a vertex in each half of S, each ℓ_i meets S only at its endpoints, and so that no two arcs cross each other. Note, in particular, that no arc should have an endpoint on NW or SE.

We will now construct $p-1$ more arcs, ℓ'_i, $i = 1, \ldots, p-1$, which we will declare to go "under" the arcs ℓ_i, as in a link diagram. For each a_i which is a vertex of the graph of S that is distinct from NW, SW, SE, and NE let b_i be the vertex whose graph distance from SW is equal to the graph distance of a_i from SW. Construct an arc ℓ'_i connecting a_i to b_i which only meets S at its endpoints. For any ℓ_j that may meet ℓ'_i, we will assume that ℓ'_i goes under ℓ_j. We also assume that if $i \neq j$, then ℓ'_i and ℓ'_j do not intersect. As a result, we will have constructed $p-1$ arcs.

The union of the arcs ℓ_i and ℓ'_i is called a (p, q)-*weave*. A diagram constructed as above is called a *weave diagram*. We may omit the boundary circle and labels NW, SW, SE, and NE when it is convenient. Similarly to how we treat a rational tangle as a subset of a link, we will treat a (p, q)-weave as a subset of a link. A weave diagram can then be considered as a subset of a link diagram, which we can manipulate with planar isotopies and Reidemeister moves. Often we will denote a (p, q)-weave by $w(p, q)$.

***Example* 8.16.** In Figure 8.5, we depict a $(7, 3)$-weave diagram and a $(7, -2)$-weave diagram. We invite the reader to try constructing each of these weaves on their own, and then compare their results with the figure.

***Remark* 8.17.** Let p and q be integers, with $p > 0$, $q \neq 0$, $|q| \leq p/2$, and $gcd(p, q) = 1$. Let $q' \in \mathbb{Z}$ such that $q \equiv q' \ (mod\ p)$. Let W_1 be a (p, q)-weave diagram. We will construct a (p, q')-weave diagram, even though it may be that $|q'| \geq p/2$.



(a) $(7, 3)$-weave.

(b) $(7, -2)$-weave.

Figure 8.5: Examples of weave diagrams.

Without loss of generality, suppose $q' \geq p/2$. A construction which is similar to the following can be described for the case $q' \leq -p/2$. Construct the graph of the boundary circle of the diagram as usual, and label the points NW and SE. Now, since $q \equiv q'$ $(mod$ $p)$, $q - q' = np$ for some $n \in \mathbb{Z}$. Note that since $p > 0$ and $q' > q$, it must be that $n < 0$. Thus we obtain $q - np = q'$, and note that $-pn > 0$. Now we want to label the vertices SW and NE. Suppose $q > 0$. We travel counterclockwise from NW q edges, arriving at a vertex we will label x. Now, from x, we travel counterclockwise $-np$ more edges, arriving at a vertex, y, after traveling a total of $q - np = q'$. Now, one trip around the boundary circle requires traveling over $2p$ edges. Thus if n is even, we see that $y = x$. Thus We label x as SW. We see that this is the same as the SW of W_1, and thus we have constructed a weave diagram which is identical to W_1. If n is odd, then our trip ends on the opposite half of the boundary circle. In this case we simply label y as NE instead, and again this NE coincides with the NE of W_1. Thus we have again constructed a weave diagram which is identical to W_1.

Similar arguments show that if $q < 0$ we will also construct a weave diagram which is the same as W_1. Thus a (p, q)- and (p, q')-weave diagram are identical. So we will always assume that $|q| \leq p/2$.

We will now describe a sequence of Reidemeister moves on a (p, q)-weave diagram which will reduce it into a different weave diagram, plus an additional crossing. The location of this additional crossing in relation to the new weave will depend on the value of q. Thus we will describe sequences for two cases: $q > 0$ and $q < 0$.

Let W be a (p, q)-weave. First suppose $q > 0$. Let a_1 be the vertex on the boundary circle that is adjacent to SE via a clockwise path, and let a_2 be the other vertex which is adjacent to SE. Let ℓ_1 denote the overstrand with endpoints at a_1 and a_2. Let ℓ_2 be the overstrand which is "to the left" of ℓ_1 in our weave diagram. Let ℓ_3 be the understrand with endpoint at SE, and let α denote the crossing of ℓ_1 and ℓ_3. Finally, let ℓ_4 be the overstrand with endpoint at NE, and let β denote the crossing of ℓ_3 and ℓ_4. See Figure 8.6a for a weave diagram of the $(5, 2)$-weave, with the indicated labeling. Note that for $w(5, 2)$, ℓ_2 and ℓ_4 are the same strand.

The first move we make is a Type III Reidemeister move which pulls ℓ_2 from the left side of α to the other side. This sets up a Type II Reidemeister move, in which we pull ℓ_1 completely out from under ℓ_2. In the case of the $(5, 2)$-weave, as in Figure 8.6c, this sets up another Type II Reidemeister move, which will pull ℓ_1 completely off of ℓ_3. Finally, we perform a Type I move which will remove the crossing α. This results in NW and SW being connected by a single arc.

In general, there may be more overstrands which we must pull ℓ_1 underneath to set up this Type II move. In particular, we perform the Type III and Type II moves as above for the $|q| - 1$ overstrands between ℓ_1 and the overstrand with endpoint at SW. After moving these overstrands, the crossing β is exposed. When we perform the above sequence of Reidemeister moves on a (p, q)-weave diagram, removing $|q| - 1$ overstrands from the diagram and exposing a crossing, we say we have performed an *un-weaving step.*

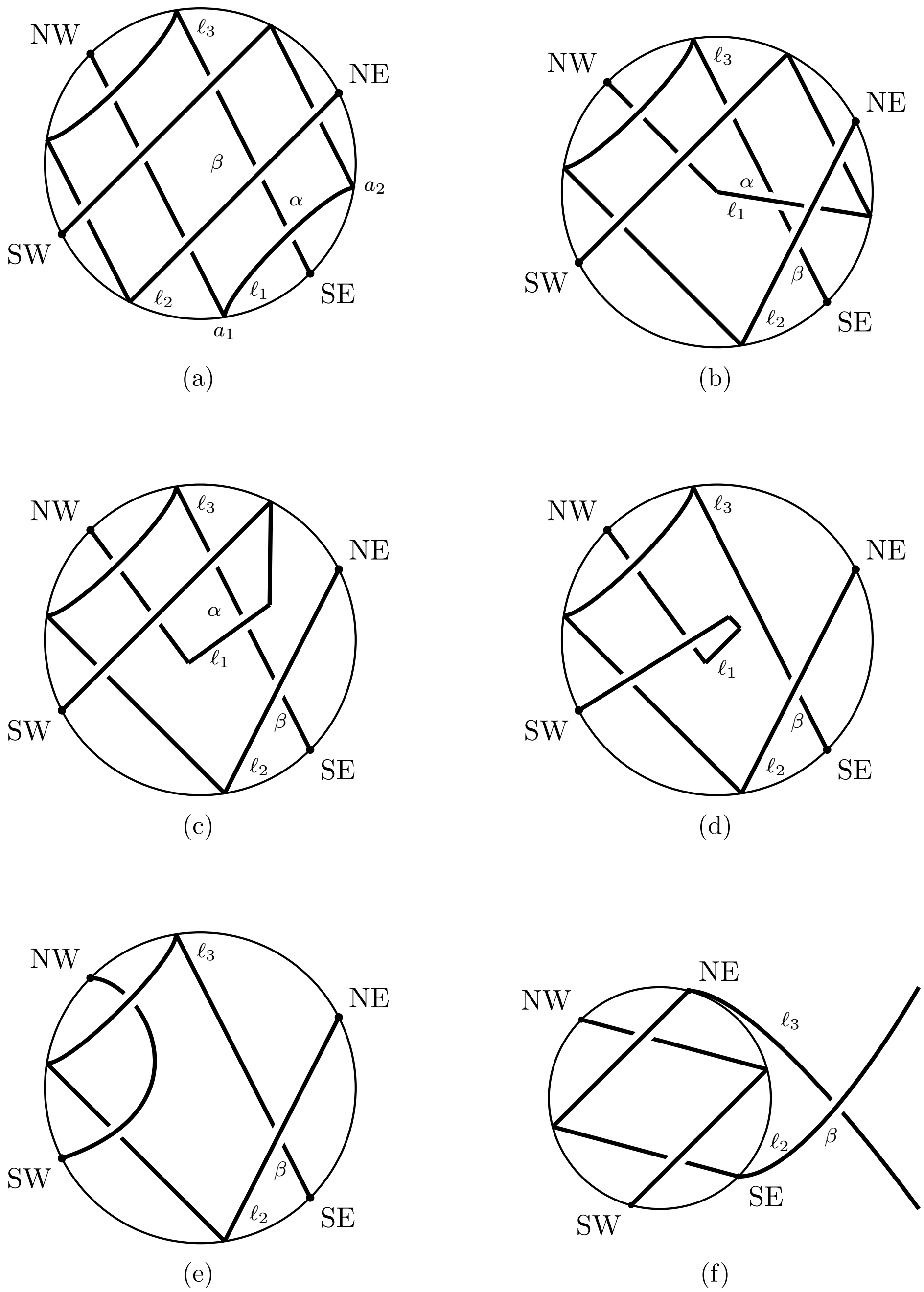


Figure 8.6: Performing an un-weaving step on a $(5, 2)$-weave.

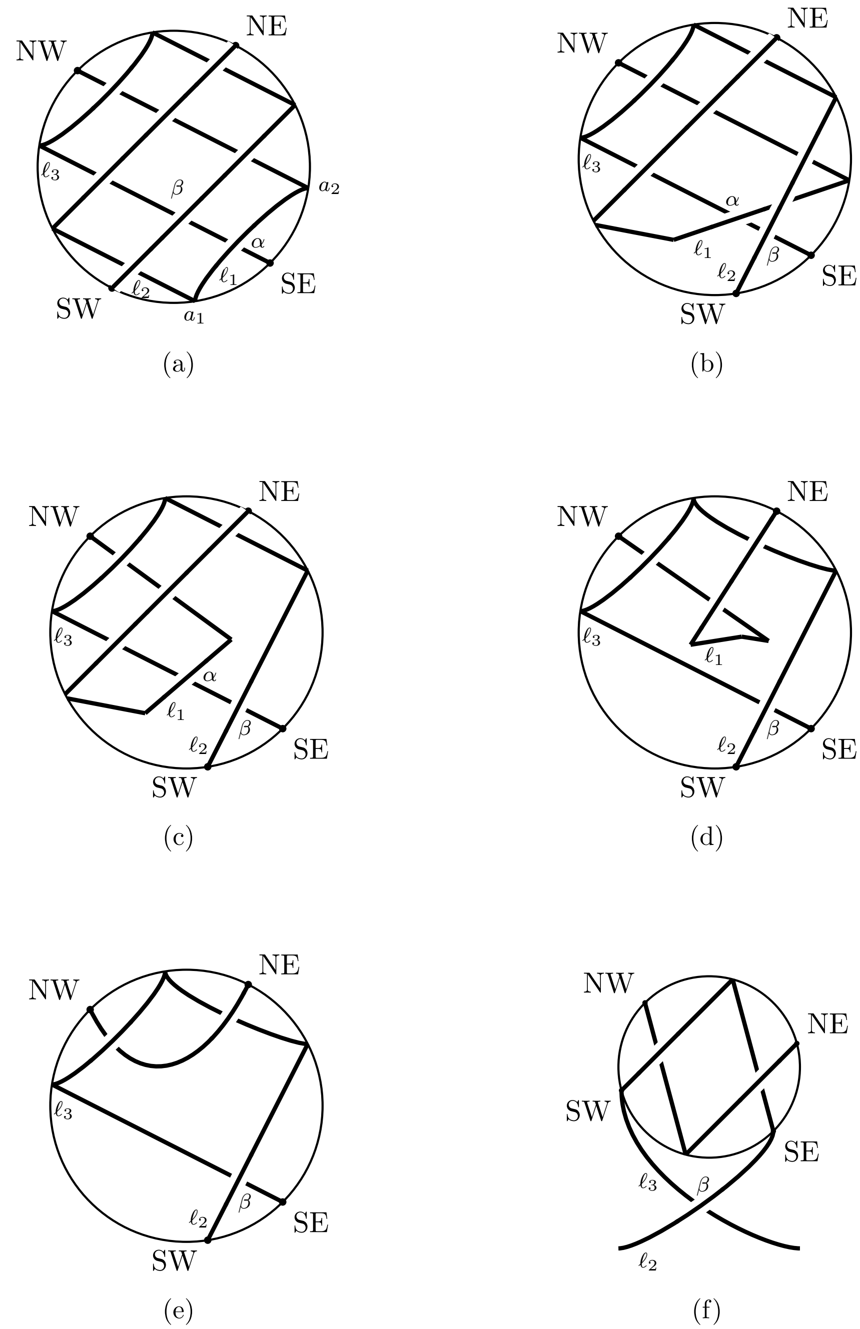


Figure 8.7: Performing an un-weaving step on a (5,-2)-weave.

In Figure 8.7, we show how to perform an un-weaving step on the weave $w(5,-2)$. Notice that instead of exposing a crossing to the right, we expose a crossing on the bottom. In general, this will be the case for any weave $w(p,q)$ with $q < 0$.

Notice that in Figure 8.6f, we have depicted the result of an un-weaving step on the $(5,2)$-weave as a $(3,-1)$-weave plus an integer tangle [1]. In particular, the result of an un-weaving step on any (p,q)-weave includes a weave. We would like to determine which weave we obtain via an un-weaving step. Suppose $q > 0$. The crossing we expose via an un-weaving step is the crossing of the understrand ℓ_3 with endpoint at SE, and the overstrand ℓ_2 with endpoint at NE. Let x be the endpoint of ℓ_3 which is not SE, and y be the endpoint of ℓ_2 which is not NE. Then after performing an un-weaving step, we see that x is the NE of the new weave, and y is the SE of the new weave. Now, there are $q-1$ overstrands between ℓ_2 and the overstrand with endpoint at x. Thus the distance from SE to NE in the new weave is q, since we travel along a path which hits exactly one endpoint of each of the $q-1$ overstrands, and ends at x. Also, NW and SW are unchanged when we perform an un-weaving step, so the distance from NW to SW is still q. Then, since we removed $q-1$ overstrands, the new weave is a $(p-q,q)$-weave. Note that we may consider the new weave as a $(p-q,q-p)$-weave if $q > \frac{p-q}{2}$.

In the case $q < 0$, we note that y will be on the opposite side of SE. After performing an un-weaving step, y will be the SE of the new weave. Also, x will be the SW of the new weave. There were $-q-1$ overstrands between SW and SE in $w(p,q)$. We removed these overstrands, plus the overstrand ℓ_2 from the original weave. Thus the new weave has $p+q-1$ overstrands. Notice that the offset does not change. So the new weave is $w(p+q,q)$, if $-\frac{p+q}{2} < q < 0$.

We summarize our results about un-weaving a (p,q)-weave in the following lemma. We recall, for emphasis, that any weave is a *tangle*.

Lemma 8.18. *Let W be a (p,q)-weave.*

(i) If $q > 0$, then performing an un-weaving step on W results in $W' + [1]$, where W' is a $(p-q,q)$-weave.

(ii) If $q < 0$, then performing an un-weaving step on W results in $W' \star \frac{1}{[1]}$, where W' is a $(p+q, q)$-weave.

We already know that any weave is a tangle. Our goal is to show that any weave is a *rational tangle*. We will proceed by induction on p. The following lemma will be useful to us.

Lemma 8.19. *Let W be a (p, q)-weave.*

(i) If $q = 1$, then W is the integer tangle $[p-1]$.

(ii) If $q = -1$, then W is the reciprocal tangle $\frac{1}{[p-1]}$.

Proof. We proceed by induction on p. Suppose $p = 2$. Note that by Remark 8.17, $w(2, 1) = w(2, -1)$. We depict $w(2, 1)$ in Figure 8.8. Notice that $w(2, 1)$ is clearly the integer tangle [1]. Note that [1] is equivalent to $\frac{1}{[1]}$.

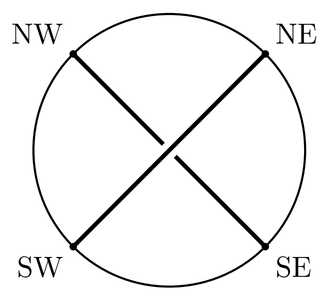


Figure 8.8: $(2, 1)$-weave, which is equivalent to the integer tangle [1].

Now suppose that for every positive integer $2 \leq m < p$, with $p > 2$, $w(m, 1)$ is equivalent to $[m-1]$, and $w(m, -1)$ is equivalent to $\frac{1}{[m-1]}$. We will now show that the claim holds for p. Consider the weave $w(p, 1)$. By Lemma 8.18, $w(p, 1)$ is equivalent to $w(p-1, 1) + [1]$. By the induction hypothesis, $w(p-1, 1)$ is equivalent to the rational tangle $[p-2]$. Then $w(p-1, 1) + [1]$ is equivalent to $[p-2] + [1] \cong [p-1]$ by Lemma 8.6.

Next consider the weave $w(p, -1)$. We apply Lemma 8.18, noting that $w(p, -1)$ is equivalent to $w(p-1, -1) \star \frac{1}{[1]}$. By the induction hypothesis, $w(p-1, -1)$ is equivalent to the reciprocal tangle $\frac{1}{[p-2]}$. Then $w(p-1,-1)$ is equivalent to $\frac{1}{[p-2]} \star \frac{1}{[1]} \cong \frac{1}{[p-1]}$ by Lemma 8.6.

□

Since any integer or reciprocal tangle is rational, Lemma 8.19 implies that any $(p, \pm 1)$-weave is a rational tangle. When $p = 2$, this will give us our base case for our induction. We now prove the general theorem.

Theorem 8.20. *Let W be a (p, q)-weave.*

(i) If $q > 0$, then W is equivalent to the rational tangle T with $F(T) = \frac{p-q}{q}$.

(ii) If $q < 0$, then W is equivalent to the rational tangle T with $F(T) = \frac{-q}{p+q}$.

Proof. We will proceed by induction on p. Note that the case $p = 2$ is a direct result of Lemma 8.19. Next we suppose that for some positive integer p, for all $2 \leq m < p$, (i) and (ii) hold for $w(m, q)$ for any integer q such that $gcd(m, q) = 1$. We will show that (i) and (ii) hold for $w(p, q)$ for any q with $gcd(p, q) = 1$.

To prove (i), suppose $q > 0$ and $gcd(p, q) = 1$. We consider two cases: $q \leq \frac{p-q}{2}$ and $q > \frac{p-q}{2}$. First suppose $q \leq \frac{p-q}{2}$. By Lemma 8.18, $w(p, q)$ is equivalent to $w(p-q, q) + [1]$. Note that $p - q < p$ and that $gcd(p-q, q) = 1$, so by the induction hypothesis, we have $F(w(p-q, q)) = \frac{(p-q)-q}{q} = \frac{p-2q}{q}$. Then by Lemma 8.11,

$$F(w(p-q, q) + [1]) = F(w(p-q, q)) + 1 = \frac{p-2q}{q} + 1 = \frac{p-q}{q}.$$

For the second case, suppose $q > \frac{p-q}{2}$. Again, we have $w(p, q)$ is equivalent to $w(p-q, q) + [1]$, but since $q > \frac{p-q}{2}$, we consider $w(p-q, q)$ as $w(p-q, q-(p-q)) = w(p-q, 2q-p)$. Note in particular that $2q - p < 0$, and that $gcd(p-q, 2q-p) = 1$. By the induction hypothesis, we obtain

$$F(w(p-q, 2q-p)) = \frac{p-2q}{(p-q)+(2q-p)} = \frac{p-2q}{q}.$$

Again by Lemma 8.11,

$$F(w(p-q, 2q-p) + [1]) = F(w(p-q, 2q-p)) + 1 = \frac{p-2q}{q} + 1 = \frac{p-q}{q}.$$

To prove (ii), suppose $q < 0$ and $gcd(p, q) = 1$. Again, we consider two cases: $q > \frac{p+q}{2}$ and $q \leq \frac{p+q}{2}$. First suppose $q > \frac{p+q}{2}$. Then by Lemma 8.18, $w(p, q)$ is equivalent to $w(p+q, q) \star \frac{1}{[1]}$. By the induction hypothesis, we obtain
$F(w(p+q, q)) = \frac{-q}{(p+q)+q} = \frac{-q}{p+2q}$. Then

$$F\left(w(p+q, q) \star \frac{1}{[1]}\right) = \frac{1}{1 + \frac{1}{F(w(p+q,q))}} = \frac{1}{1 + \frac{1}{\frac{-q}{p+2q}}} = \frac{1}{1 + \frac{p+2q}{-q}} = \frac{-q}{-q+p+2q} = \frac{-q}{p+q}.$$

Finally, suppose $q \leq \frac{p+q}{2}$. We know that $w(p, q)$ is equivalent to $w(p+q, q)$. But since $q \leq \frac{p+q}{2}$, we take $w(p+q, q)$ to be $w(p+q, q+(p+q)) = w(p+q, p+2q)$. Note that $p+q < p$, and $p+2q > 0$. By the induction hypothesis, we obtain
$F(w(p+q, p+2q)) = \frac{p+q-(p+2q)}{p+2q} = \frac{-q}{p+2q}$. Then we have

$$F\left(w(p+q, p+2q) \star \frac{1}{[1]}\right) = \frac{1}{1 + \frac{1}{F(w(p+q,p+2q))}} = \frac{1}{1 + \frac{1}{\frac{-q}{p+2q}}} = \frac{-q}{p+q}.$$

□

Theorem 8.20 tells us that the class of weaves is contained in the class of rational tangles. It turns out that a partial converse is also true. To prove this, we need to be able to turn a given rational tangle into a weave. Since the process we described for transforming a weave into a rational tangle involved only sequences of Reidemeister moves, it is reversible. Thus, given $w(p, q) + [1]$ or $w(p, q) \star \frac{1}{[1]}$, we can "weave in" the extra crossing given to obtain a weave. We illustrate this process for the weave $w(5, 2)$, in Figure 8.9, and for $w(5, -2)$ in Figure 8.10.

Lemma 8.21. *Let W be a (p, q)-weave.*

(i) If $q > 0$, then $W + [1]$ is equivalent to $w(p+q, q)$.

(ii) If $q > 0$*, then* $W \star \frac{1}{[1]}$ *is equivalent to* $w(2p-q, q-p)$*.*

(iii) If $q < 0$*, then* $W + [1]$ *is equivalent to* $w(2p+q, p+q)$*.*

(iv) If $q < 0$*, then* $W \star \frac{1}{[1]}$ *is equivalent to* $w(p-q, q)$*.*

Proof. To prove (i), suppose $q > 0$. Note that there are $q-1$ overpasses of W with endpoints between SE and NE on the boundary circle. By weaving in the integer tangle [1], which is attached to W at SE and NE, we will pull each of the $q-1$ overpasses through hte crossing at [1], by the sequence of Reidemeister moves outlined previously. This will create $q-1$ new overpasses, and an additional overpass from the over-crossing in the integer tangle [1]. Thus we have obtained a weave W' with a total of $p-1+(q-1)+1 = p+q-1$ overpasses. Then if $W' = w(p', q')$, we have $p' = p+q$. Notice that W' has $q-1$ the distance from SE to NE, around the boundary circle of W', is q. Then we see that $q' = q$. Thus $W' = w(p+q, q)$.

Next we prove (ii). We notice that there are $p-q-1$ overpasses of W with endpoints between SE and NE on the boundary circle. When we perform a weaving step, weaving in $\frac{1}{[1]}$, we pull each of these $p-q-1$ overpasses through the crossing of $\frac{1}{[1]}$. In doing so, we create $p-q-1$ new overpasses, plus an additional overpass from the over-crossing of $\frac{1}{[1]}$. Thus we create $p-q$ total new overpasses. So we have a new weave $W' = w(p', q')$ with a total of $p-1+p-q = 2p-q-1$ overpasses. So $p' = 2p-q$. Now, in W' there are $q+p-q-1 = p-1$ overpasses with endpoints between NW and SW. So we might expect that $q' = p$. However, we have $p > p - \frac{q}{2} = \frac{2p-q}{2}$. Thus we take $q' = p-(2p-q) = q-p$. Thus $W' = w(2p-q, q-p)$.

To see that (iii) holds, assume $q < 0$. In this case, there are $p+q-1$ overpasses of W with endpoints between SE and NE on the boundary circle. To weave in the integer tangle [1], we pull each of these overpasses through [1] by the same sequence of Reidemeister moves as we did in (i). We obtain a new weave W'. The weave W' has $p+q$ more overpasses than W. We obtain $p+q-1$ of these overpasses from the $p+q-1$ overpasses of W which we performed the moves on, and we obtain an additional overpass from the integer tangle [1]. Thus W' has $p-1+(p+q) = 2p+q-1$ total overpasses. If we suppose that $W' = w(p', q')$, then it follows that $p' = 2p+q$.

Recall that in W, the distance from NW to NE, around the boundary circle, was $-q$. In the weave W', the distance from NW to NE is $-q+p+q = p$. So we might expect that $q' = -p$, giving $W' = w(2p+q, -p)$. However, since $q < 0$, we have $-p < \frac{-(2p+q)}{2}$. Thus we take W' to be $w(2p+q, -p+2p+q) = w(2p+q, p+q)$, as desired.

Finally, we prove (iv). In this case, there are $-q-1$ overpasses with endpoints between SE and SW. When we perform a weaving step, we pull these $-q-1$ overpasses through the crossing of $\frac{1}{[1]}$, creating $-q-1$ new overpasses. The result of the weaving step is a new weave $W' = w(p', q')$ with a total of $(p-1)-q-1+1 = p-q-1$ overpasses. Thus $p' = p-q$. Now, the weaving step had no effect on the number of overpasses with endpoints between NW and NE, which is $-q$. Thus we see that $q' = q$. So $W' = w(p-q, q)$. □

We can generalize Lemma 8.21 for general positive integer tangles and reciprocal tangles.

Lemma 8.22. *Let W be a (p, q)-weave, and n a positive integer.*

(i) If $q > 0$, then $W + [n]$ is equivalent to $w(p+nq, q)$.

(ii) If $q > 0$, then $W \star \frac{1}{[n]}$ is equivalent to $w(p(n+1)-nq, q-p)$.

(iii) If $q < 0$, then $W + [n]$ is equivalent to $w(p(n+1)+nq, p+q)$.

(iv) If $q < 0$, then $W \star \frac{1}{[n]}$ is equivalent to $w(p-nq, q)$.

Proof. We can prove all four statements via induction on n. First suppose $n = 1$. Then all four statements are true as a direct consequence of Lemma 8.21. Now suppose that all four statements hold for some positive integer n. We will show that each statement holds for $n+1$.

First we consider (i). We have $w(p, q) + [n+1] \cong w(p, q) + [n] + [1] \cong w(p+nq, q)$, by the induction hypothesis. Then by Lemma 8.21, $w(p+nq, q) + [1] \cong w(p+(n+1)q, q)$.

For (ii), we have $w(p,q) \star \frac{1}{[n+1]} \cong w(p,q) \star \frac{1}{[1]} \star \frac{1}{[n]} \cong w(2p-q, q-p) \star \frac{1}{[n]}$ by Lemma 8.21. Then by the induction hypothesis, we have

$w(2p-q, q-p) \star \frac{1}{[n]} \cong w(2p-q-n(q-p), q-p) = w(p(n+2)-(n+1)q, q-p)$.

To prove (iii) and (iv), we assume $q < 0$. For (iii), we have

$w(p,q) + [n+1] \cong w(2p+q, p+q) + [n]$ by Lemma 8.21. Then we have

$w(2p+q, p+q) + [n] \cong w(2p+q+n(p+q), p+q) = w(p(n+2)+(n+1)q, p+q)$ by the induction hypothesis.

Finally, to prove (iv), we have $w(p,q) \star \frac{1}{[n+1]} \cong w(p-nq, q) \star \frac{1}{[1]}$ by the induction hypothesis. Then by Lemma 8.21, we have

$w(p-nq, q) \star \frac{1}{[1]} \cong w(p-nq-q, q) = w(p-(n+1)q, q)$.

Thus all four statements hold for $n+1$. We conclude by induction that all four statements hold for all positive integers. $\square$

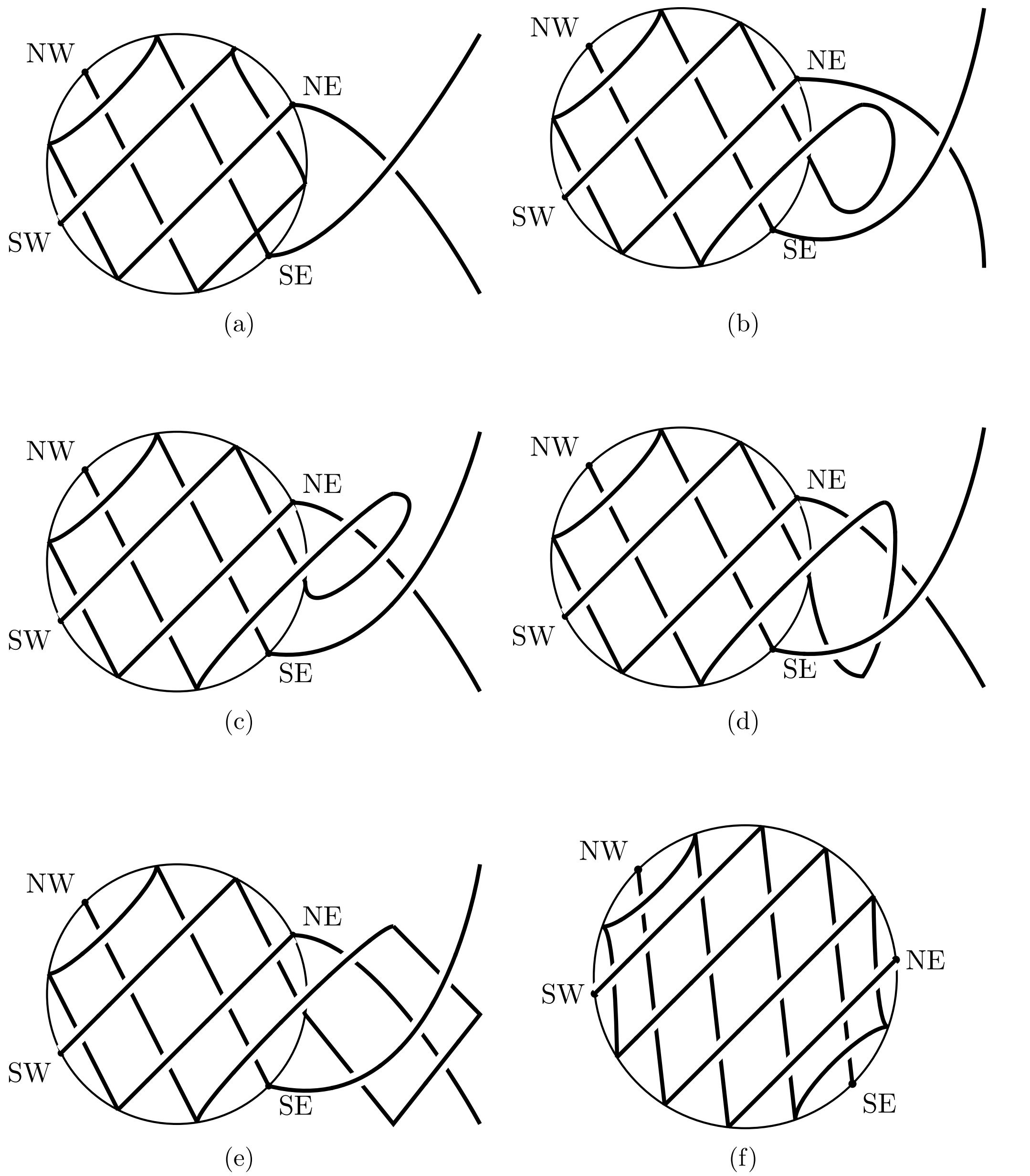


Figure 8.9: Performing a weaving step on a $(5, 2)$-weave, plus an additional crossing.

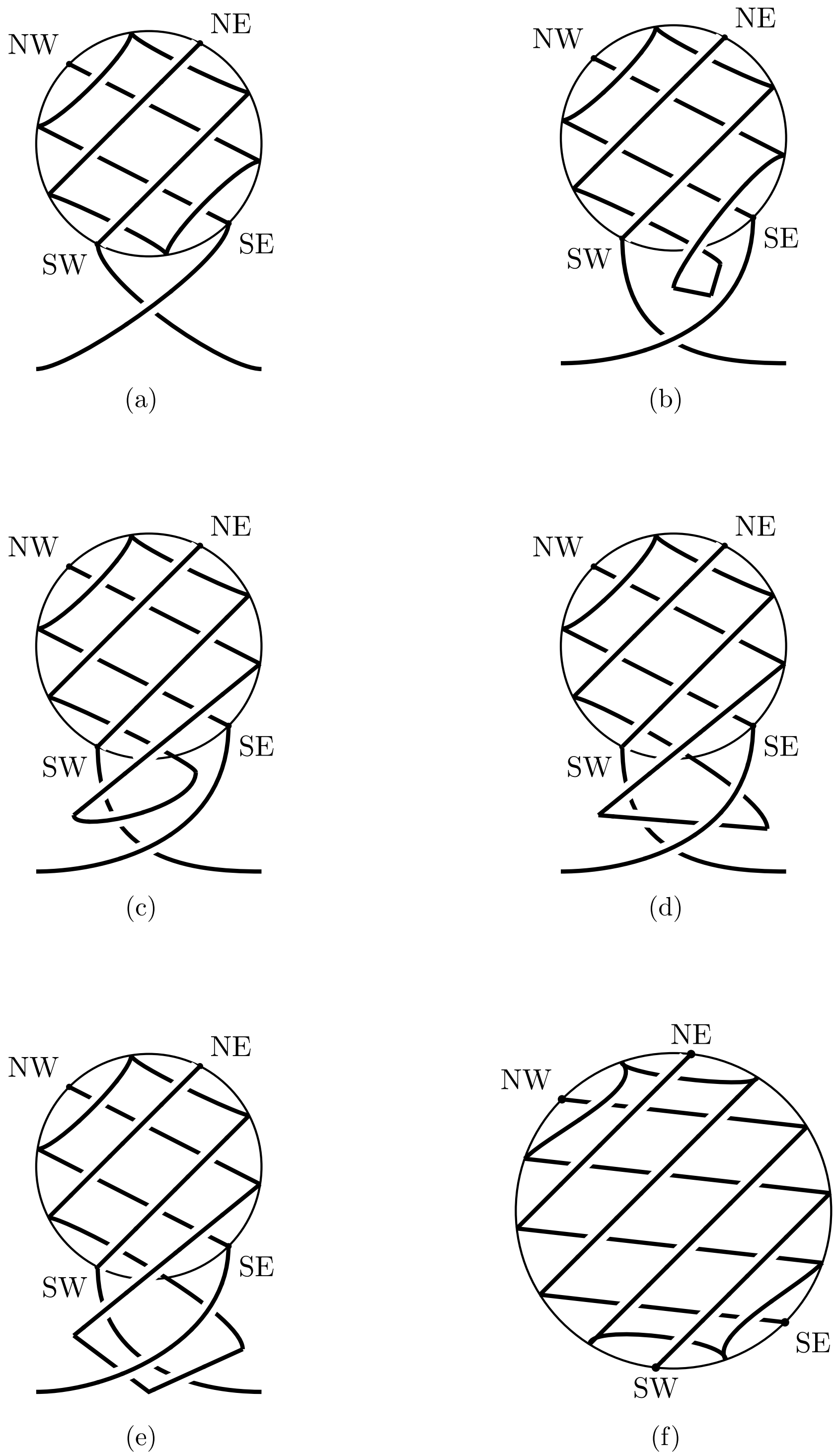


Figure 8.10: Performing a weaving step on a (5,-2)-weave.

Theorem 8.23. *Let T be a rational tangle in canonical form*

$$T = [a_{2n+1}] + \frac{1}{[a_{2n}] + \ldots + \frac{1}{[a_2] + \frac{1}{[a_1]}}},$$

and $F(T) = \frac{p}{q}$. Then T is equivalent to the weave $w(p+q, q)$.

Proof. We proceed by induction on n, or the number of terms in the continued fraction of T. First suppose $n = 0$. Then T has only one term in its continued fraction form, and is thus an integer tangle. Then $q = 1$, and by Lemma 8.19 T is equivalent to the weave $w(p+1, 1)$.

Now suppose that $n > 0$, and that the statement of the theorem holds for any rational tangle with $2m+1$ terms in continued fraction form, where $0 \leq m < n$.

Since T is in canonical form, we note that T is equivalent to $(T' \star \frac{1}{[a_{2n}]}) + [a_{2n+1}]$, where T' is a rational tangle, in canonical form, with

$$T' = [a_{2n-1}] + \frac{1}{[a_{2n-2}] + \ldots + \frac{1}{[a_2] + \frac{1}{[a_1]}}}.$$

Then choose integers p' and q' such that

$$\frac{p'}{q'} = a_{2n-1} + \frac{1}{a_{2n-2} + \ldots + \frac{1}{a_2 + \frac{1}{a_1}}}.$$

Since T' has only $2n-1$ terms in its continued fraction form, the induction hypothesis implies T' is equivalent to the weave $w(p'+q', q')$.

We notice that

$$\frac{p}{q} = a_{2n+1} + \frac{1}{a_{2n} + \frac{1}{a_{2n-1} + \frac{1}{a_{2n-2} + \ldots + \frac{1}{a_2 + \frac{1}{a_1}}}}} = a_{2n+1} + \frac{1}{a_{2n} + \frac{1}{\frac{p'}{q'}}}.$$

Then we obtain

$$a_{2n+1} + \frac{1}{a_{2n} + \frac{1}{\frac{p'}{q'}}} = a_{2n+1} + \frac{p'}{a_{2n}p' + q'} = \frac{a_{2n}a_{2n+1}p' + a_{2n+1}q' + p'}{a_{2n}p' + q'}.$$

We have

$$gcd(a_{2n}a_{2n+1}p' + a_{2n+1}q' + p', a_{2n}p' + q') = gcd(p', q') = 1,$$

so we obtain $p = a_{2n}a_{2n+1}p' + a_{2n+1}q' + p'$ and $q = a_{2n}p' + q'$.

We will now show that $(T' \star \frac{1}{[a_{2n}]}) + [a_{2n+1}]$ is equivalent to $w(p+q, q)$. First we consider $T' \star \frac{1}{[a_{2n}]}$. Recall that $T' \cong w(p'+q', q')$. Note that, since $q' < p'$, we have

$$\frac{p'+q'}{2} > \frac{q'}{2} + \frac{q'}{2} = q'.$$

Thus, by Lemma 8.22, part (ii),

$$w(p'+q', q') \star \frac{1}{[a_{2n}]} \cong w((p'+q')(a_{2n}+1) - a_{2n}q', q' - (p'+q')) = w(a_{2n}p' + p' + q', -p').$$

Now, since $a_{2n} \geq 1$, and $p' > q' > 0$, we have

$$\frac{a_{2n}p' + p' + q'}{2} > \frac{a_{2n}p' + p'}{2} \geq \frac{p' + p'}{2} = p'.$$

Thus $-p' < \frac{a_{2n}p'+p'+q'}{2}$. Then by Lemma 8.22, part (iii), we have

$$w(a_{2n}p'+p'q', -p')+[a_{2n+1}] \cong w((a_2p'+p'+q')(a_{2n+1}+1)+a_{2n+1}(-p'), -p'+a2n+1p'+p'+q').$$

We notice that

$$(a_2p' + p' + q')(a_{2n+1} + 1) + a_{2n+1}(-p') = a_{2n}a_{2n+1}p' + a_{2n+1}q' + p' + a_{2n}p' + q' = p + q,$$

and that

$$-p' + a_{2n+1}p' + p' + q' = a_{2n}p' + q' = q.$$

Thus it follows that

$$\left(T' \star \frac{1}{[a_{2n}]}\right) + [a_{2n+1}] \cong w\left(p+q,q\right).$$

□

CHAPTER 9
2-BUTTERFLIES AND RATIONAL LINKS

In this chapter, we will prove that the 2-bridge links coincide with a class of links known as the *rational links*. We give a proof that utilizes butterflies and weaves. In Chapter 8, we proved that every weave is a rational tangle, and that almost every rational tangle is a weave (the rational tangles [0] and [∞], for instance, cannot be realized as weaves as we have defined them). A rational link is a link obtained from a rational tangle in a particular way. We will utilize our previous results to relate a link obtained from a weave to a link obtained from a rational tangle, and vice-versa.

Definition 9.1. *Let T be a rational tangle. We call the link obtained by connecting NW to NE, and SW to SE, without creating new crossings, the* numerator closure *of T. We call the link obtained by connecting NW to SW, and NE to SE, without creating new crossings, the* denominator closure *of T. We denote the numerator and denominator closures of T by $N(T)$ and $D(T)$, respectively. See Figure 9.1.*

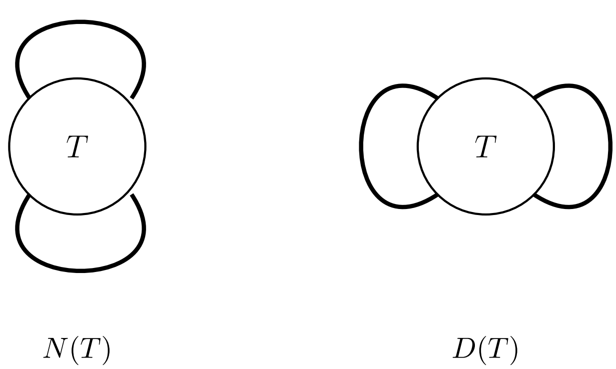


Figure 9.1: Numerator and denominator closures of a rational tangle T.

Definition 9.2. *Let L be a link. Suppose L is equivalent to the numerator closure of a rational tangle T, with continued fraction form*

$$T = [a_n] + \frac{1}{[a_{n-1}] + \ldots + \frac{1}{[a_2] + \frac{1}{[a_1]}}},$$

such that each a_i is nonzero. Then we call L a rational link*, and we denote L by its* Conway notation*, $[[a_1\ a_2\ \ldots\ a_n]]$.*

Note that, without loss of generality, we will only consider numerator closures of rational tangles, since $N(T) = D(T^r)$, and $D(T) = N(T^r)$ [6].

***Remark* 9.3.** Suppose L is a rational link with Conway notation $[[a_1\ \ldots\ a_n]]$. Note that in Definition 9.2, we require that each a_i is a nonzero integer. Since L is the numerator closure of a rational tangle T with

$$F(T) = \frac{p}{q} = a_n + \frac{1}{a_{n-1} + \ldots + \frac{1}{a_2 + \frac{1}{a_1}}},$$

it must be that $p > q$, if $\frac{p}{q}$ is in reduced form. This will be very important for us later on.

Furthermore, recall Remark 8.5 and Remark 8.13. In the Conway notation $[[a_1\ \ldots\ a_n]]$, the integers a_i alternate between corresponding to integer and reciprocal tangles. Since we can always assume that the first tangle in T is horizontal, and since we assume n is odd, the last tangle in T is horizontal. That is, both a_1 and a_n correspond to integer tangles.

Theorem 9.4. *Let $L_1 = N(T_1)$ and $L_2 = N(T_2)$ be rational links. If T_1 and T_2 have fractions which are equal, then L_1 and L_2 are equivalent.*

There are necessary and sufficient conditions for the equivalence of two rational links [6]. However, we defer discussion of this result until Chapter 10.

Now that we have defined what we mean by rational links, we discuss 2-butterflies. Recall that a 2-butterfly consists of a graph R, which divides $S^2 = \partial\mathbf{B}^3$ into two polygonal faces, each of which contains an arc t, which we call a trunk. Each polygonal face, with its trunk, is called a butterfly. We would like for our 2-butterfly diagrams to consist of a cycle graph. That is, we would like every vertex of the butterfly graph to be bivalent. Consider, for example, the 2-butterfly in Figure 9.2. This 2-butterfly contains a univalent B-vertex. We note that this diagram can be transformed into a 2-butterfly in which each vertex is bivalent via a Type I butterfly move.

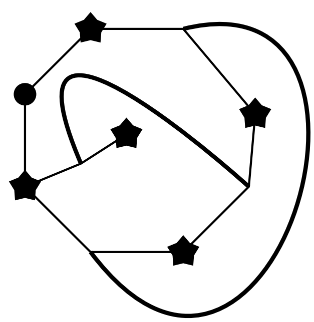


Figure 9.2: 2-butterfly with a univalent B-vertex.

In fact, we can show that any 2-butterfly which contains a univalent B-vertex can be transformed into a 2-butterfly in which every vertex is bivalent. The proof of this claim is very similar to the proof of Theorem 7.1. Let $\mathcal{B}$ be a 2-butterfly, and let b_1 be a univalent B-vertex of $\mathcal{B}$. Since b_1 is univalent, it is connected to exactly one other vertex, which we label x. Since the graph of $\mathcal{B}$ is bivalent, x is either an A-vertex or an E-vertex. If x is an A-vertex, we can perform a Type I butterfly move, removing b_1 and associated edges. If x is an E-vertex, we can perform a Type II butterfly move, removing b_1 and associated edges. By inducting on the length of the path from b_1 to the nearest non-univalent B-vertex, we can show that $\mathcal{B}$ is equivalent to a 2-butterfly diagram in which each vertex is bivalent. We leave it to the reader to compare to the proof of Theorem 7.1 and fill in details.

Lemma 9.5. *Let $\mathcal{B} = (R, T)$ be a 2-butterfly diagram. Then $\mathcal{B}$ is equivalent to a 2-butterfly diagram in which each vertex is bivalent.*

From now on, when we consider a 2-butterfly diagram, we will assume that each of its vertices is bivalent.

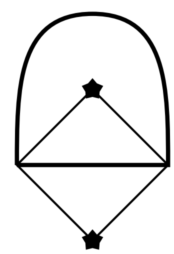


Figure 9.3: This is not a 2-butterfly, since the trunks have coincident endpoints.

Let $\mathcal{B} = (R, T)$ be a 2-butterfly diagram. Let n be the number of edges in R. By Lemma 7.2, we know that n is divisible by 4. Let $p \in \mathbb{Z}$ such that $n = 4p$. If $p = 1$, then $\mathcal{B}$ is as in Figure 9.3. Note that this fails to be a 2-butterfly, because the two trunks have coincident endpoints. Thus we have proved the following lemma.

Lemma 9.6. *Let $\mathcal{B} = (R, T)$ be a 2-butterfly diagram. Then R has $4p$ edges, where $p \geq 2$.*

The simplest 2-butterfly diagram is the case where $p = 2$, as in Figure 9.4. In this case, the butterfly graph is a cycle graph with 8 edges. The corresponding link of two components is known as the Hopf Link.

Definition 9.7. *Let $\mathcal{B} = (R, T)$ be a 2-butterfly diagram. Suppose $T = \{t_1, t_2\}$, such that t_1 is the trunk of the bounded component. Fix an A-vertex a_1 of t_1. Let $2q$ denote the minimum butterfly boundary distance from a_1 to an A-vertex of t_2, measured counterclockwise. We call q the* offset *of $\mathcal{B}$.*

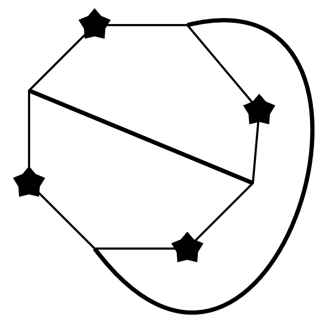


Figure 9.4: 2-butterfly diagram representing the Hopf Link.

Since the two trunks of a 2-butterfly diagram have distinct endpoints, the offset q is always defined. Also, since R is bipartite, we have $q \geq 1$. In the case of the 2-butterfly diagram of Figure 9.4, the offset is $q = 1$.

Note that to construct a 2-butterfly diagram, all we need to do is (1) construct a cycle graph with an acceptable number of edges, and (2) place the trunks. These two steps correspond to picking an integer p so our graph has $4p$ edges, and choosing the offset q. Thus we can associate to each 2-butterfly diagram an ordered pair (p, q).

Definition 9.8. *Let $\mathcal{B} = (R, T)$ be a 2-butterfly diagram. Suppose R contains $4p$ edges and the offset of $\mathcal{B}$ is q, with $0 < q < p$, and $gcd(p, q) = 1$. Then we call $\mathcal{B}$ a (p, q)* 2-butterfly diagram*.*

***Example* 9.9.** In Figure 9.5, we depict a $(7, 3)$ 2-butterfly diagram. Notice that each vertex is bivalent. Furthermore, the graph contains 14 B-vertices, 10 E-vertices, and 4 A-vertices. In general, a (p, q) 2-butterfly diagram will contain $2p$ B-vertices, $2p - 4$ E-vertices, and 4 A-vertices.

***Remark* 9.10.** In our definition of a (p, q) 2-butterfly, we require $gcd(p, q) = 1$. This

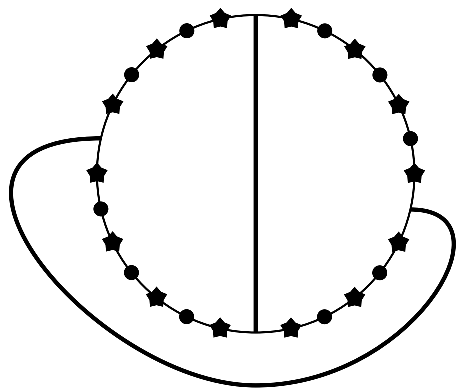


Figure 9.5: $(7,3)$ 2-butterfly diagram.

requirement ensures that a (p,q) 2-butterfly diagram really is a butterfly diagram. If $gcd(p,q) > 1$, one can show that the resulting diagram will have vertices which are not A-, E-, or B-vertices.

***Remark* 9.11.** Let $\mathcal{B}$ be a (p,q) 2-butterfly diagram. Let $\mathcal{B}'$ be a (p,q') 2-butterfly diagram, where $q' \equiv q \ (mod\ p)$. Then $\mathcal{B} = \mathcal{B}'$. To see this, first note that the graphs of $\mathcal{B}$ and $\mathcal{B}'$ are cycle graphs with $4p$ edges. Let t_1 be the trunk of the bounded component of the projection plane. Fix an A-vertex a_1 of t_1. We label the vertices of $\mathcal{B}$ modulo $2p$, with 0 at a_1, in a counterclockwise fashion. Thus each A-vertex of t_1 is labeled 0. See Figure 9.6. Now, traveling counterclockwise from a_1 along a path of length $2q$, we arrive at an A-vertex a_2 of the other trunk, t_2. The A-vertex a_2 is thus labeled $2q$. Similarly, the other A-vertex of t_2, a_2', is labeled $2q$ as well.

Now, since $q' \equiv q \ (mod\ p)$, $4q' \equiv 4q \ (mod\ 4p)$. Since $4np \equiv 0 \ (mod\ 4p)$, if we travel around the cycle graph $4np$ times, starting at a_2, we end up either at a_2 or a_2'. Thus $\mathcal{B}'$ will have A-vertices in the exact same location as $\mathcal{B}$, so $\mathcal{B} = \mathcal{B}'$. As a consequence, we are justified in requiring $0 < q < p$.

Theorem 9.12. *Let L be a link with a (p,q) 2-butterfly diagram $\mathcal{B}_L$.*

(i) L is a two-component link if and only if p is even.

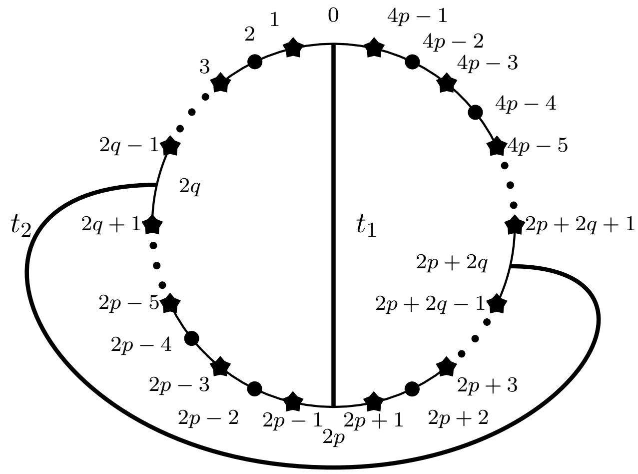


Figure 9.6: A (p, q) 2-butterfly with its vertices labeled modulo $4p$.

(ii) L is a knot if and only if p is odd.

Proof. First we will show that there are precisely two equivalence classes, under the map p, of A- and E-vertices of $\mathcal{B}_L$. Let t_1 and t_2 be the trunks of $\mathcal{B}_L$. Fix an A-vertex a_1 of t_1. We label the A- and E-vertices of $\mathcal{B}_L$ modulo $2p$, beginning with 0 at a_1, in ascending counterclockwise order. See Figure 9.7.

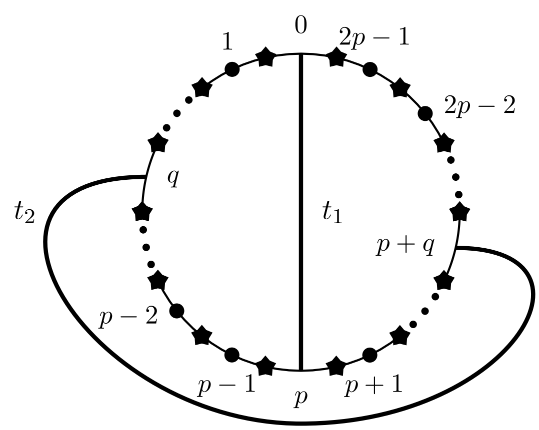


Figure 9.7: A (p, q) 2-butterfly, with A- and E-vertices labeled modulo $2p$.

We claim that the two equivalence classes of A- and E-vertices of $\mathcal{B}_L$, under the equivalence relation $\simeq$, are the sets of even and odd labeled vertices, respectively. Let A be the set of A- and E-vertices of $\mathcal{B}_L$. Suppose $x \simeq y$. Then x and y are t_1-equivalent, or x and y are t_2-equivalent. If x and y are t_1-equivalent, then they are equidistant, via butterfly boundary distance, from a_1. Let $2d$ be the common distance of x and y to a_1, which is even since the butterfly graph is bipartite. Without loss of generality, we assume that x is the endpoint of the counterclockwise path, starting at a_1, with length $2d$. We will assume y is the endpoint of such a path, going clockwise. Then x is labeled d and y is labeled $2p - d$. If d is odd, then so is $2p - d$. Thus x and y are both labeled as odd. Similarly, if d is even, then both x and y are labeled as even.

A similar argument will show that if x and y are t_2-equivalent, then they are both labeled even or are both labeled odd.

Now let $x \in A$ such that x is labeled as $2k$. Let $x' \in A$ such that x' is t_1-equivalent to x. Let $2d$ be the butterfly boundary distance of x to a_1. Then the butterfly boundary distance of x' to a_1 is also $2d$. Thus we can construct a path of length $4d$ which starts at x and ends x'. If this path goes clockwise, then $2k < p$, and $2k = d$. Then x' is labeled as $d - 2d = -d$, which is even. If the path goes counterclockwise, then $2k > p$, and $2k \equiv -d$ $(mod\ 2p)$. Then x' is labeled as $-d + 2d = d$, which is even. Thus each vertex in the equivalence class of x is labeled as even.

Similar arguments will prove the cases in which x is labeled as odd, and in which x' is t_2-equivalent to x. We conclude that the equivalence classes under $\simeq$ of vertices in A are precisely the sets of even and odd labeled A- and E-vertices of $\mathcal{B}_L$.

To prove (i), suppose p is even. Let a_1 be the A-vertex of t_1 which is labeled as 0 as above, and a_2 be the other A-vertex of t_1. Note that a_2 is labeled as p. Since p is even, a_1 and a_2 belong to the same equivalence class. Thus t_1 corresponds to a component of the link L. Then t_2 must also correspond to a component of L. Thus L is a two-component link.

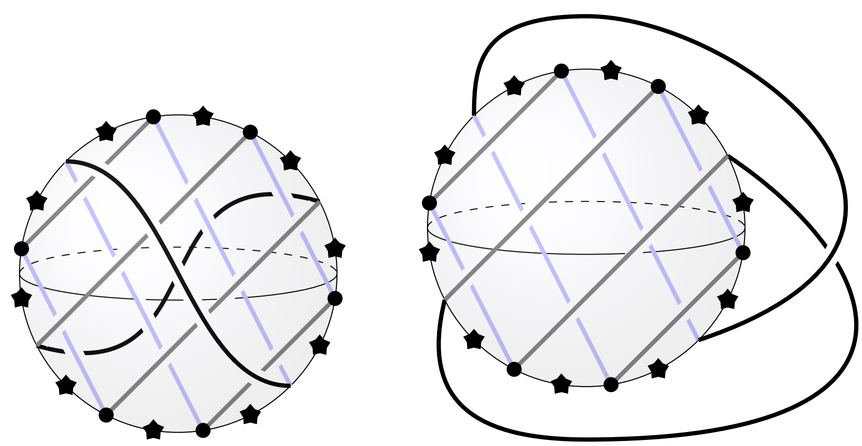
Now suppose L is a two-component link. Note that if we label the vertices as before, a_1 and a_2 are still labeled as 0 and p, respectively. Since L is a two-component link, t_1 corresponds to a component of L. Thus a_1 and a_2 must belong to the same equivalence class under $\simeq$. Since a_1 is labeled as 0, which is even, a_2 must also be labeled as even. Thus p is even.

To prove (ii), first suppose p is odd. Note that a_1 is labeled as 0, and thus even, and a_2 is labeled as p, which is odd. Thus a_1 and a_2 are not members of the same equivalence class under $\simeq$. Thus L must be a one-component link. In other words, L is a knot.

Conversely, suppose L is a knot. Then a_1 and a_2 do not belong to the same equivalence class under $\simeq$. Since a_1 is labeled as 0, which is even, and a_2 is labeled as p, it must be that p is odd.

□

Theorem 9.13. *Let $\mathcal{B} = (R, T)$ be a (p, q) 2-butterfly diagram. Then $\mathcal{B}$ corresponds to $N(w(p, k) + [1])$, where $-p/2 < k < p/2$ and $k \equiv q \pmod p$.*



(a) $(5, 2)$ 2-butterfly on a sphere. (b) Pulling the trunks off of the sphere.

Figure 9.8: Turning a 2-butterfly into a weave.

Proof. Recall that R is a cycle graph embedded on $S^2 = \partial\mathbf{B}^3$. The sphere is split into two hemispheres, which make up the two butterflies of $\mathcal{B}$. Let B_1 and B_2 be these two butterflies. When we fold the wings of B_1 and B_2, we obtain a link L which is embedded in S^3. In particular, we imagine the arcs of the link L which are obtained as under-arcs from the butterfly-link algorithm as embedded on the sphere S^2. We imagine that the over-arcs corresponding to the trunks of B_1 and B_2 come off of S^2, being contained within the unit ball.

We want to shift our perspective, from the center of the ball, to outside of it. We instead imagine t_1 and t_2 as being contained outside of the unit ball, with the under-arcs of L still being embedded on S^2. Note in particular that each hemisphere of S^2 contains exactly $p-1$ arcs of L. We arrange the structure so that, from our perspective, we are looking down directly at the center of B_1, with the graph R appearing as the equator of the sphere. See Figure 9.8a.

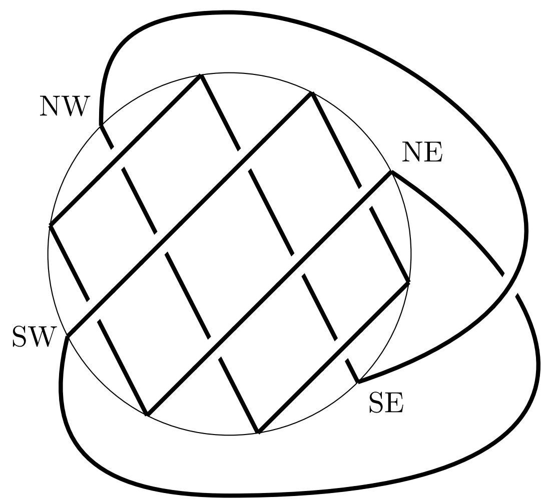


Figure 9.9: Flattening the sphere, giving the numerator closure of $w(5,2)+[1]$.

Next we pull the two trunks, t_1 and t_2, away from the sphere. From our persepctive, we want to pull them to the right. In doing so, we make so that t_1 crosses over t_2 from our point of view. See Figure 9.8b. Next we flatten S^2 down into a disk, which we imagine is embedded in a plane. The $p-1$ arcs from one hemisphere will now appear to pass over the $p-1$ strands of the other hemisphere. See Figure 9.9.

We can arrange this process so that t_1 has endpoints at NW and SE of the disk, and t_2 has endpoints at NE and SW of the disk. Then each of the $p-1$ under-arcs contained in the disk has endpoints which are equidistant from SW of the disk, since its endpoints come from E-vertices of $\mathcal{B}$ which are t_2-equivalent. Furthermore, since the offset of $\mathcal{B}$ is q, the distance around the boundary of the disk from NW to SW is q. Similarly, the distance from SE to NE is q. Then we see that the disk contains a (p, q)-weave. Thus the result of the process is $N(w(p, q) + [1])$. □

Theorem 9.14. *Let L be a link with a (p, q) 2-butterfly diagram $\mathcal{B}$. Then $L = N(T)$, where T is a rational tangle with fraction $F(T) = \frac{p}{q}$.*

Proof. By Theorem 9.13, we know that $\mathcal{B}$ corresponds to $N(w(p, k) + [1])$, where $-p/2 < k < p/2$ and $k \equiv q \ (mod\ p)$. If $k > 0$, then $k = q$. Thus $N(w(p, k) + [1]) = N(w(p, q) + [1])$. By Theorem 8.20, $w(p, q) + [1]$ is a rational tangle with fraction

$$F(w(p, q) + [1]) = F(w(p, q)) + 1 = \frac{p - q}{q} + 1 = \frac{p}{q}.$$

If $k < 0$, then $k = q - p$. Thus $N(w(p, k) + [1]) = N(w(p, q - p) + [1])$. Again by Theorem 8.20, $w(p, q - p) + [1]$ is a rational tangle with fraction

$$F(w(p, q - p) + [1]) = F(w(p, q - p)) + 1 = \frac{p - q}{p + (q - p)} + 1 = \frac{p - q}{q} + 1 = \frac{p}{q}.$$

□

We can prove a partial converse to Theorem 9.14. Its proof will rely on the fact that we can turn $N(w(p, q) + [1])$ into a (p, k) 2-butterfly diagram, where $k \equiv q \ (mod\ p)$, and $0 < k < p$. This transformation is obtained by reversing the process described in the proof of Theorem 9.13. We leave the details to the reader.

Theorem 9.15. *Let $L = N(T)$ be a rational link with $F(T) = \frac{p}{q}$. Then L has a (p, k) 2-butterfly diagram, where $k \equiv q \ (mod\ p)$, and $0 < k < p$.*

Proof. Let T be a rational tangle, in canonical form, with $F(T) = \frac{p}{q}$, such that

$L = N(T)$. We will consider two cases: $q < \frac{p}{2}$ and $q > \frac{p}{2}$.

First suppose $q < \frac{p}{2}$. Then by Theorem 8.23, T be equivalent to the weave $w(p+q, q)$. Thus L is equivalent to $N(w(p+q,q)$. Since $q < \frac{p}{2}$, we know $q < \frac{p+q}{2}$. Thus by Lemma 8.18, $N(w(p+q,q))$ is equivalent to $N(w(p,q)+[1])$, which can be transformed into a (p,q) 2-butterfly diagram, which corresponds to L.

Now suppose $q > \frac{p}{2}$. Again by Theorem 8.23, T is equivalent to the weave $w(p, q-p)$. Thus L is equivalent to $N(w(p+q,q))$. Since $\frac{p}{2} < q < p$, we know $\frac{p}{2} + q < 2q < p+q$, from which we obtain $\frac{p}{4} + \frac{q}{2} < q < \frac{p+q}{2}$. Thus by Lemma 8.18, $N(w(p+q,q))$ is equivalent to $N(w(p,q-p)+[1])$. Note that we consider the weave as $w(p,q-p)$ since $q > \frac{p}{2}$. We can transform $N(w(p,q-p)+[1])$ into a (p,q) 2-butterfly as a consequence of Remark 9.11. $\square$

CHAPTER 10

FUTURE QUESTIONS

In this chapter, we address a few directions in which future research can take. These include problems which we have thus far failed to resolve, in spite of much effort, and problems which we have considered only briefly.

As noted in Remark 9.11, a (p, q) 2-butterfly is the same as any (p, q') 2-butterfly, with $q \equiv q'$ $(mod\ p)$. This is related to a result which is known as Schubert's Theorem.

Theorem 10.1. *(Schubert's Theorem) Let* $L_1 = N(T_1)$ *and* $L_2 = N(T_2)$ *be rational links with* $F(T_1) = \frac{p}{q}$ *and* $F(T_2) = \frac{p'}{q'}$*. Then* L_1 *and* L_2 *are equivalent as links if and only if*

(i) $p = p'$

(ii) Either $q \equiv q'$ $(mod\ p)$ *or* $qq' \equiv 1$ $(mod\ p)$*.*

The original proof of this theorem is due to Schubert [8], who showed that the double-branched cover of a rational link is a lens space. He then used the classification of lens spaces to obtain the statement of Theorem 10.1. Kauffman and Lambropoulou give a combinatorial proof of Schubert's Theorem in [6].

We have attempted to construct a proof of Schubert's Theorem utilizing butterflies, thus far to no avail. We are able to show that if $p = p'$ and $q \equiv q'$ $(mod\ p)$, then a (p, q) 2-butterfly and a (p', q') 2-butterfly are the same (this is the content of Remark 9.11. We hope that obtaining a proof of Schubert's Theorem which utilizes butterflies may give us valuable insight into the classification of 3-bridge links.

The classification of 3-bridge links is an open problem. Hilden, Montesinos, Tejada, and Toro (HMTT) tackle this problem in [4], although they do not achieve a classification. They approach the problem using 3-butterfly diagrams. Similarly to how we could, without loss of generality, consider only 2-butterfly diagrams in which each vertex is bivalent, HMTT consider 3-butterfly diagrams in which there are no univalent vertices.

HMTT prove that each 3-butterfly diagram can be associated to a 3-tuple of rational numbers, $(p/n, q/m, s/l)$. They also prove some special cases of equivalence of 3-butterfly diagrams.

Theorem 10.2. *(i) The 3-butterfly $(p/n, q/m, s/l)$ is equivalent to the 3-butterflies $(q/m, s/l, p/n)$ and $(s/l, p/n, q/m)$.*

(ii) If $s > q$, then the 3-butterfly $(p/n, q/m, s/l)$ is equivalent to $(p/n', s/l', q/m')$, where $n' \equiv (n + s - q) \pmod{p}$, $m' \equiv (m + p - s) \pmod{q}$, and $l' \equiv (l + q - p) \pmod{s}$.

Notice that the statement of this theorem is similar to Shubert's Theorem. However, Theorem 10.2 does not provide both necessary and sufficient conditions for the equivalence of two 3-butterfly diagrams. As of yet, no such conditions are known.

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